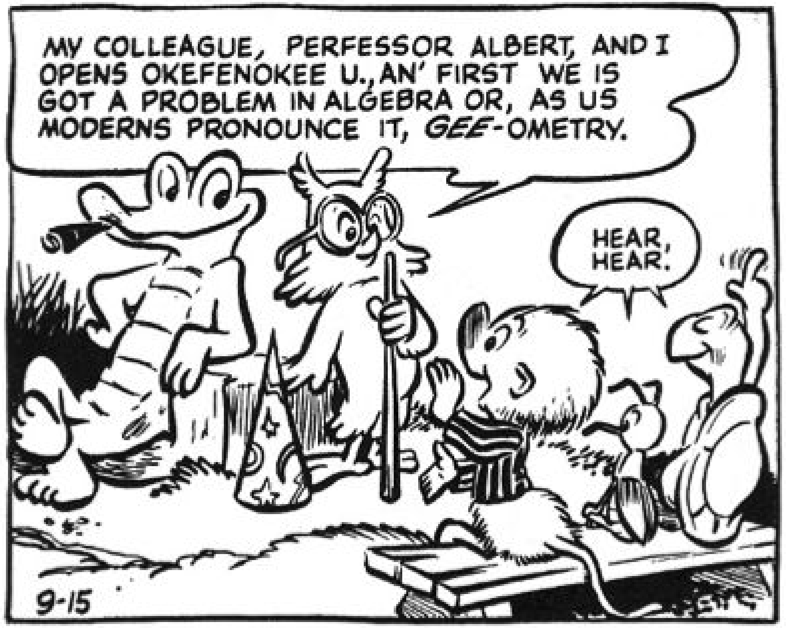
# Chp 3 Lee


MY COLLEAGUE, PERFESSOR ALBERT, AND I OPENS OKEFENOKEE U., AN' FIRST WE IS GOT A PROBLEM IN ALGEBRA OR, AS US MODERNS PRONOUNCE IT, *GEE*-OMETRY.
HEAR, HEAR.
9-15

$$
\begin{aligned}
\min \quad & c'x & & \\
& Ax & = & \ b\,; \\
& x & \geq & \ \mathbf{0}\,.
\end{aligned}
\tag{P}
$$

# Basic partition

A **basic partition** of $A \in \mathbb{R}^{m \times n}$ is a partition of $\{1, 2, \ldots, n\}$ into a pair of ordered sets $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_{n-m})$ so that $A_\beta := [A_{\beta_1}, A_{\beta_2}, \ldots, A_{\beta_m}]$ is an invertible $m \times m$ matrix.

$$
\begin{aligned}
A &:= \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 0 \\ 3/2 & 3/2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
b &:= (7, 9, 6, 33/10)',
\end{aligned}
$$

We associate a **basic solution** $\bar{x} \in \mathbb{R}^n$ with the basic partition via:

$$\begin{aligned} \bar{x}_\eta &:= \mathbf{0} && \in \mathbb{R}^{n-m}\,; \\ \bar{x}_\beta &:= A_\beta^{-1} b && \in \mathbb{R}^{m}\,. \end{aligned}$$

Note that every basic solution $\bar{x}$ satisfies $A\bar{x} = b$, because

$$A\bar{x} = \sum_{j=1}^{n} A_j \bar{x}_j = \sum_{j \in \beta} A_j \bar{x}_j + \sum_{j \in \eta} A_j \bar{x}_j = A_\beta \bar{x}_\beta + A_\eta \bar{x}_\eta = A_\beta \left( A_\beta^{-1} b \right) + A_\eta \, \mathbf{0} = b\,.$$

A basic solution $\bar{x}$ is a **basic feasible solution** if it is feasible for (P). That is, if $\bar{x}_\beta = A_\beta^{-1} b \geq \mathbf{0}$ .

we observe that the feasible region of (P) is the solution set, in $\mathbb{R}^n$ ,

$$\begin{aligned} x_\beta \quad + \quad A_\beta^{-1} A_\eta x_\eta \quad &= \quad A_\beta^{-1} b \,; \\ x_\beta \geq \mathbf{0} \quad , \quad x_\eta &\geq \mathbf{0} \,. \end{aligned}$$

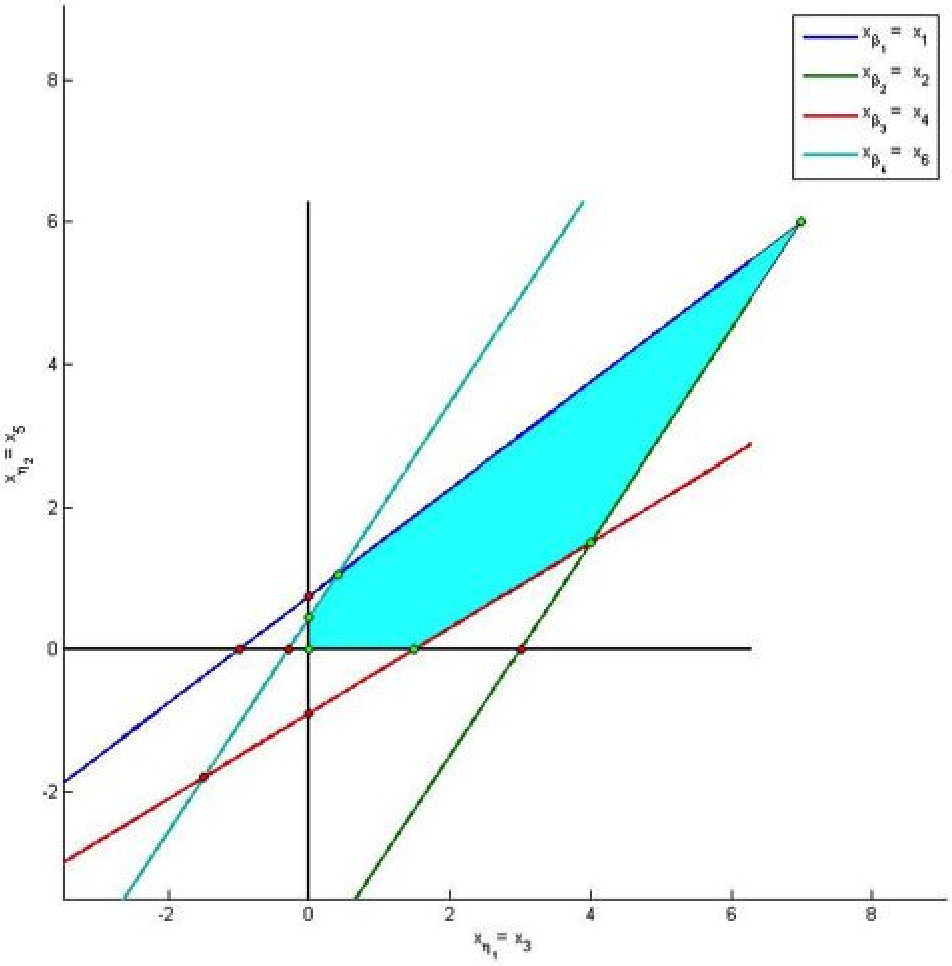
$$\begin{aligned} \left(A_\beta^{-1} A_\eta\right) x_\eta \quad &\leq \quad A_\beta^{-1} b \,; \\ x_\eta \quad &\geq \quad \mathbf{0} \,. \end{aligned}$$

Notice how we can view the $x_\beta$ variables as slack variables.

$$
\begin{aligned}
A &:= \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 0 \\ 3/2 & 3/2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
b &:= (7, 9, 6, 33/10)', \\
\beta &:= (\beta_1\,,\ \beta_2\,,\ \beta_3\,,\ \beta_4) \ = \ (1, 2, 4, 6)\,, \\
\eta &:= (\eta_1\,,\ \eta_2) \ = \ (3, 5)\,.
\end{aligned}
$$

$$
\begin{aligned}
A_\beta &= [A_{\beta_1}, A_{\beta_2}, A_{\beta_3}, A_{\beta_4}] = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 3/2 & 3/2 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \\
A_\eta &= [A_{\eta_1}, A_{\eta_2}] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
x_\beta &= (x_1, x_2, x_4, x_6)' \\
x_\eta &:= (x_3, x_5)'.
\end{aligned}
$$

$$
\begin{aligned}
A_\beta^{-1} A_\eta &= \begin{pmatrix} -1 & 4/3 \\ 1 & -2/3 \\ 2 & -10/3 \\ -1 & 2/3 \end{pmatrix}, \\
A_\beta^{-1} b &:= (1, 3, 3, 3/10)',
\end{aligned}
$$



$x_{\beta_1} = x_1$
$x_{\beta_2} = x_2$
$x_{\beta_3} = x_4$
$x_{\beta_4} = x_6$
$x_{\eta_2} = x_5$
$x_{\eta_1} = x_3$

A set $S \subset \mathbb{R}^n$ is a **convex set** if it contains the entire line segment between every pair of points in $S$ . That is,

$$\lambda x^1 + (1-\lambda)x^2 \in S \text{ , whenever } x^1, x^2 \in S \text{ and } 0 < \lambda < 1 \text{ .}$$

For a convex set $S \subset \mathbb{R}^n$ , a point $\hat{x} \in S$ is an **extreme point** of $S$ if it is not on the interior of any line segment wholly contained in $S$ . That is, if we *cannot* write

$$\hat{x} = \lambda x^1 + (1-\lambda)x^2 \text{ , with } x^1 \neq x^2 \in S \text{ and } 0 < \lambda < 1 \text{ .}$$

**Theorem 3.2**
Every basic feasible solution of standard-form (P) is an extreme point of its feasible region.

$$\begin{aligned} \bar{x}_\eta &:= 0 && \in \mathbb{R}^{n-m} \ ; \\ \bar{x}_\beta &:= A_\beta^{-1} b && \in \mathbb{R}^{m} \ . \end{aligned}$$

**Theorem 3.3**
Every extreme point of the feasible region of standard-form (P) is a basic solution.

**Corollary 3.4**
For a feasible point $\hat{x}$ of standard-form (P), $\hat{x}$ is extreme if and only if $\hat{x}$ is a basic solution.

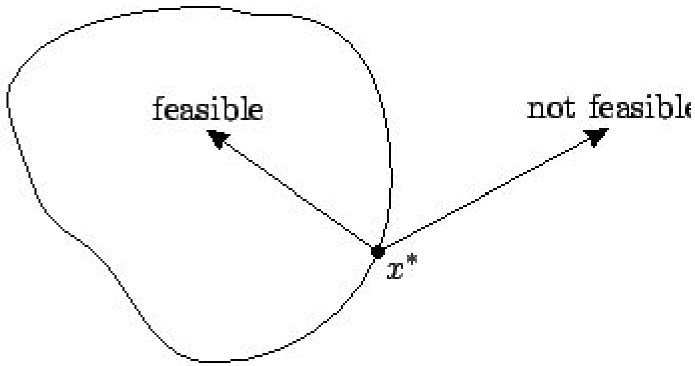



**Figure:** Feasible directions

For a point $\hat{x}$ in a convex set $S \subset \mathbb{R}^n$, a **feasible direction relative to the feasible solution** $\hat{x}$ is a $\hat{z} \in \mathbb{R}^n$ such that $\hat{x} + \epsilon\hat{z} \in S$, for sufficiently small positive $\epsilon \in \mathbb{R}$.

$$b = A(\hat{x} + \epsilon\hat{z}) = A\hat{x} + \epsilon A\hat{z} = b + \epsilon A\hat{z} ,$$

so we need $A\hat{z} = \mathbf{0}$. That is, $\hat{z}$ must be in the null space of $A$.

Focusing on the standard-form problem (P), we associate a **basic direction** $\bar{z} \in \mathbb{R}^n$ with the basic partition $\beta, \eta$ and a choice of nonbasic index $\eta_j$ via

$$\begin{aligned} \bar{z}_\eta &:= \boldsymbol{e_j} && \in \mathbb{R}^{n-m} \;; \\ \bar{z}_\beta &:= -A_\beta^{-1} A_{\eta_j} && \in \mathbb{R}^m \;. \end{aligned}$$

Note that every basic direction $\bar{z}$ is in the null space of $A$ :

$$A\bar{z} = A_\beta \bar{z}_\beta + A_\eta \bar{z}_\eta = A_\beta \left( -A_\beta^{-1} A_{\eta_j} \right) + A_\eta \boldsymbol{e_j} = -A_{\eta_j} + A_{\eta_j} = \mathbf{0} \,.$$

$$A\left(\hat{x} + \epsilon \bar{z}\right) = b \,,$$

Let $\bar{b} := \bar{x}_\beta = A_\beta^{-1} b$

**Theorem 3.5**
For a standard-form problem (P), suppose that $\bar{x}$ is a basic feasible solution relative to the basic partition $\beta, \eta$ . Consider choosing a nonbasic index $\eta_j$ . Then the associated basic direction $\bar{z}$ is a feasible direction relative to $\bar{x}$ if and only if

$$\bar{b}_i > 0 \text{ , for all } i \text{ such that } \bar{a}_{i,\eta_j} > 0 \text{ .}$$

For a nonempty convex set $S \subset \mathbb{R}^n$, a **ray** of $S$ is a $\hat{z} \neq \mathbf{0}$ in $\mathbb{R}^n$ such that $\hat{x} + \tau\hat{z} \in S$ for *all* $\hat{x} \in S$ and *all* positive $\tau \in \mathbb{R}$.

If the basic direction $\bar{z}$ is a ray, then we call it a **basic feasible ray**. We have already seen that $A\bar{z} = \mathbf{0}$. Furthermore, $\bar{z} \geq \mathbf{0}$ if and only if $\bar{A}_{\eta_j} := A_\beta^{-1} A_{\eta_j} \leq \mathbf{0}$.

**Theorem 3.6**
The basic direction $\bar{z}$ is a ray of the feasible region of (P) if and only if $\bar{A}_{\eta_j} \leq \mathbf{0}$.

A ray $\hat{z}$ of a convex set $S$ is an **extreme ray** if we *cannot* write

$$\hat{z} = z^1 + z^2 \text{ , with } z^1 \neq \mu z^2 \text{ being rays of } S \text{ and } \mu \neq 0 \text{ .}$$

**Theorem 3.7**
Every basic feasible ray of standard-form (P) is an extreme ray of its feasible region.

**Theorem 3.8**
Every extreme ray of the feasible region of standard-form (P) is a positive multiple of a basic feasible ray.