2.3 Newton's Method and Its Extension for Solving f(x)=0

Derivation of Newton's Method

- Taylor's Theorem Recap

Suppose $f \in C^2[a,b]$ and $p_0 \in [a,b]$ approximates solution p of $f(x) = 0$ with $f'(p_0) \neq 0$. Expand $f(x)$ about p_0:

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p-p_0)^2}{2} f''\big(\xi(p)\big)$$

$f(p) = 0$, and assume $(p - p_0)^2$ is negligible:

$$0 \approx f(p_0) + (p - p_0)f'(p_0)$$

Solving for p yields:

$$p \approx p_1 \equiv p_0 - \frac{f(p_0)}{f'(p_0)}$$

Making the above eq. an iterative eq., it gives the sequence $\{p_n\}_{n=0}^{\infty}$:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Remark: p_n is an improved approximation.

Newton's Method

1. Choose an initial approximation p_0 to solution of $f(x) = 0$.
2. Generate sequence $\{p_n\}_{n=0}^{\infty}$ by:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \quad \text{for } n \geq 1$$

p_n is an improved approximation.

Example 2.3.1 Consider the function $f(x) = \cos(x) - x = 0$, with $x \in \left[0, \frac{\pi}{2}\right]$. Approximate a root of f using Newton's method.

Algorithm: Newton's Method

INPUT initial approximation p0; tolerance TOL; maximum number of iterations N0.

OUTPUT approximate solution p or message of failure.

STEP1 Set i = 1.

STEP2 While i ≤ N0 do STEPs 3-6

STEP3 Set p = p0 – f(p0)/f'(p0).

STEP4 If |p-p0| < TOL then

OUTPUT (p);

STOP.

STEP5 Set i = i + 1.

STEP6 Set p0 = p.

STEP7 OUTPUT('The method failed');

STOP.

About Newton's Method

- Pros.
  1. Fast convergence: Newton's method converges fastest among methods we explore (quadratic convergence).
- Cons.
  1. $f'(x_{n-1})$ cause problems

  Remark: Newton's method works best if $|f'| \geq k > 0$

  2. Expensive: Computing derivative in every iteration
- We assume $|p - p_0|$ is small, then $|p - p_0|^2 \ll |p - p_0|$, and we can neglect the 2nd order term in Taylor expansion.

Remark: In order for Newton's method to converge we need a **good starting guess**.

Relation of Newton's method to fixed-point iteration

Newton's method can be viewed as fixed-point iteration with $g(x)$ defined to be:

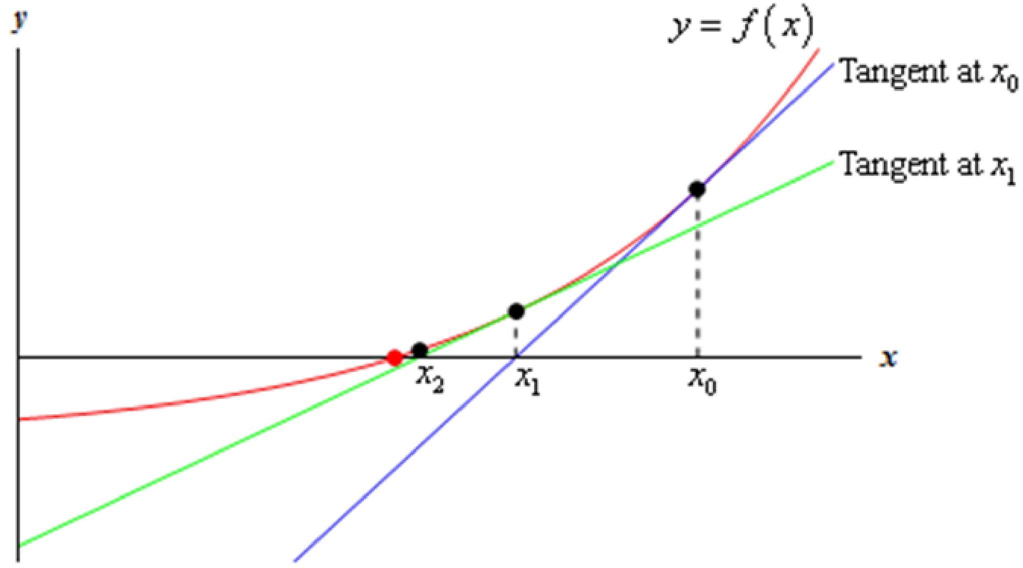
$$g(x) = x - \frac{f(x)}{f'(x)}$$

Convergence

Theorem 2.6

Let $f \in C^2[a, b]$ and $p \in [a, b]$ is that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$.

Geometric Interpretation of Newton's method



The tangent line of $f(x)$ which passes the point $(p_{n-1}, f(p_{n-1}))$ is: $y = f(p_{n-1}) + f'(p_{n-1})(x - p_{n-1})$. The x-intercept of this tangent line is: $x = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$.

So p_1, p_2 ... generated by Newton's method are x-intercepts of these tangent lines, respectively.

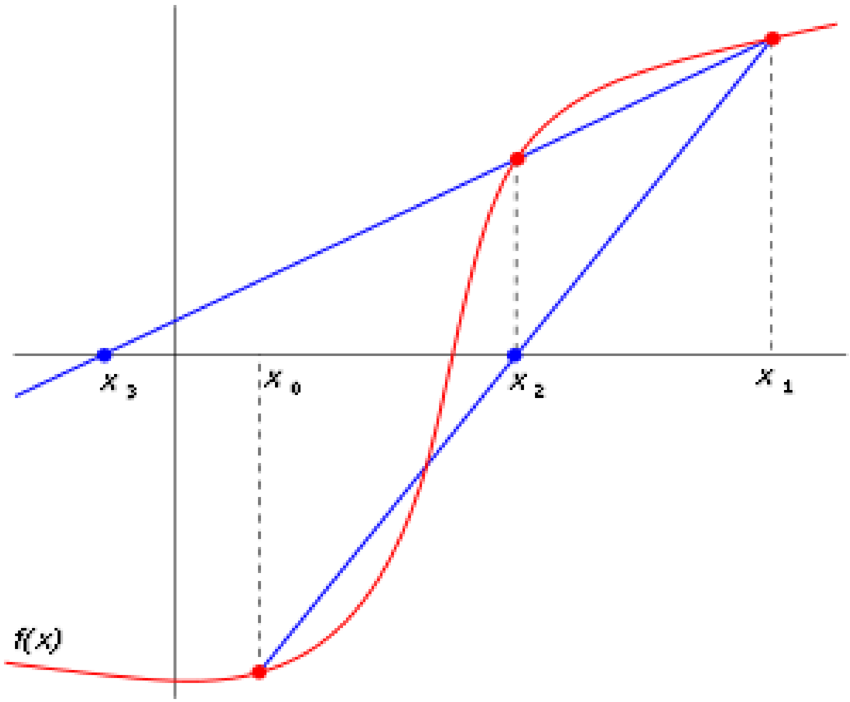
The Secant Method

- Approximate the derivative:

$$f'(p_{n-1}) \approx \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}$$

to get

$$p_n = p_{n-1} - \frac{f(p_{n-1})\,(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})} \tag{2.12}$$



Summary of Secant method

1. Make two initial guesses: p_0 and p_1
2. Use Eq. (2.12) to construct p_2, p_3, p_4 ... till accuracy is met.

Exercise 2.3.6 Consider the function $f(x) = e^x + 2^{-x} + 2cos(x) - 6$. Solve $f(x) = 0$ using the Secant method for $1 \leq x \leq 2$.

Algorithm: The Secant Method

INPUT initial approximation p0, p1; tolerance TOL; maximum number of iterations N0.

OUTPUT approximate solution p or message of failure.

STEP1 Set i = 2;

q0 = f(p0);

q1 = f(p1);

STEP2 While i ≤ N0 do STEPs 3-6

STEP3 Set p = p1 –q1(p1-p0)/(q1-q0).

STEP4 If |p-p1| < TOL then

OUTPUT (p);

STOP.

STEP5 Set i = i + 1.

STEP6 Set p0 = p1;

q0 = q1;

p1 = p;

q1 = f(p).

STEP7 OUTPUT(‘The method failed’);

STOP.