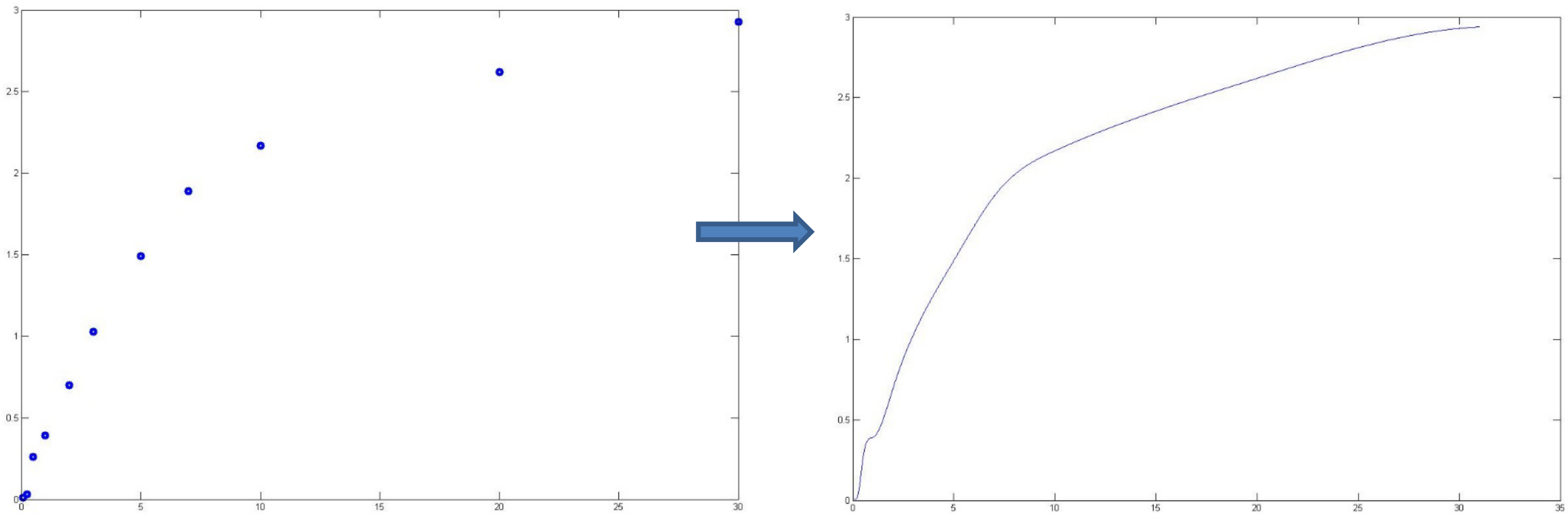
3.1 Interpolation and Lagrange Polynomial

Example. Daily Treasury Yield Curve Rates

Date	1 Mo	3 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
09/01/15	0.01	0.03	0.26	0.39	0.70	1.03	1.49	1.89	2.17	2.62	2.93

Suppose we want yield rate for a four-years maturity bond, what shall we do?

Solution: Draw a smooth curve passing through these data points (interpolation).



Ref: http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield

- **Interpolation problem**: Find a smooth function $P(x)$ which interpolates (passes) the data $\{(x_i, y_i)\}_{i=0}^{N}$.

- **Remark**: In this class, we always assume that the data $\{y_i\}_{i=0}^{N}$ represent measured or computed values of a underlying function $f(x)$, i.e., $y_i = f(x_i)$ Thus $P(x)$ can be considered as an approximation to f.

Polynomial Interpolation

Polynomials $P_n(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$ are commonly used for interpolation.

- Advantages for using polynomial: efficient, simple mathematical operation such as differentiation and integration.

Theorem 3.1 Weierstrass Approximation theorem

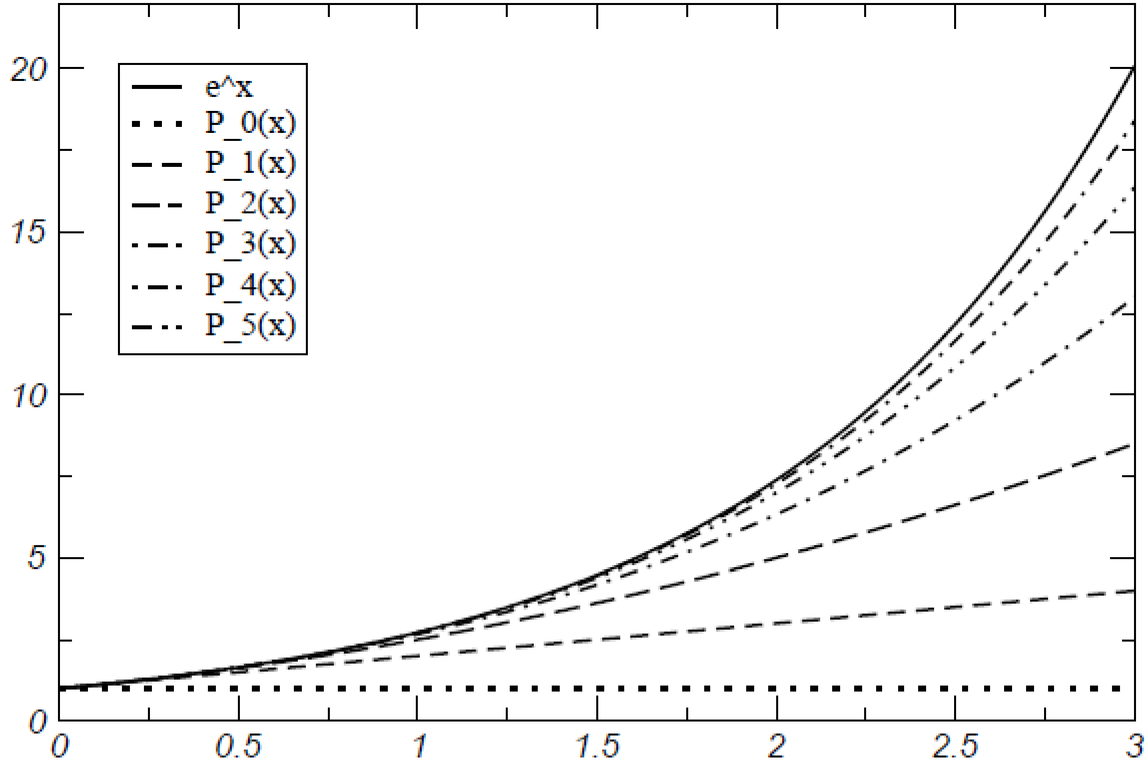
Suppose $f \in C[a, b]$. Then $\forall \epsilon > 0$, $\exists$ a polynomial $P(x)$: $|f(x) - P(x)| < \epsilon, \forall x \in [a, b]$.

Remark:

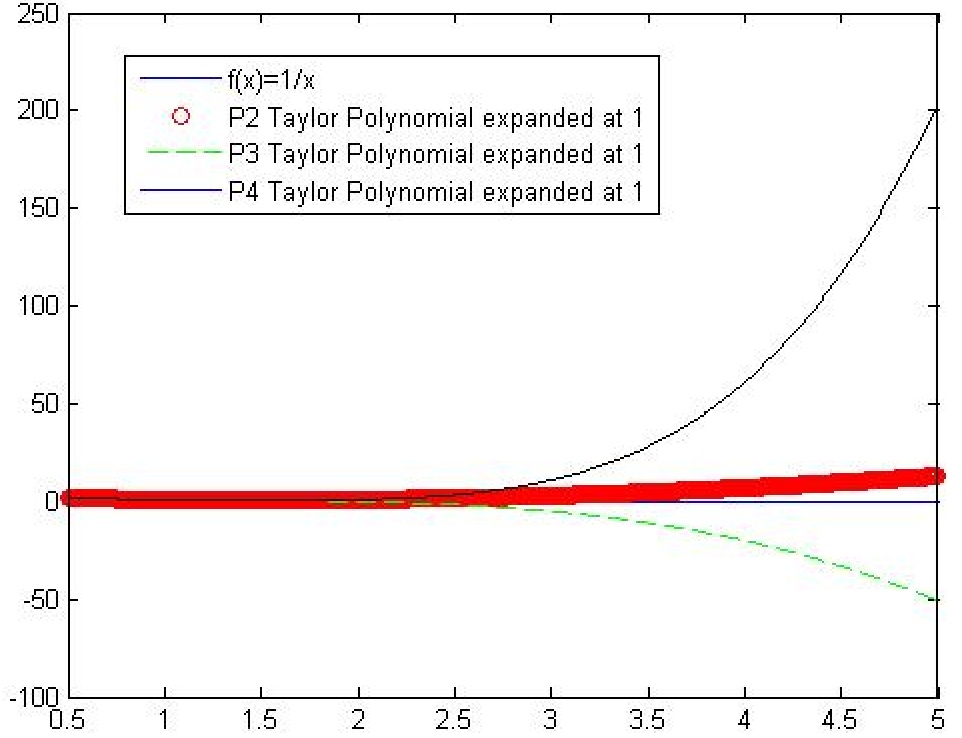
1. The bound is uniform, i.e. valid for all x in $[a, b]$. This means polynomials are good at approximating general functions.
2. The way to find $P(x)$ is unknown.

- **Question:** Can Taylor polynomial be used here?
- Taylor expansion is accurate in the neighborhood of **one** point. We need to the (interpolating) polynomial to pass many points.

- **Example**. Taylor polynomial approximation of e^x for $x \in [0,3]$



- **Example**. Taylor polynomial approximation of $\frac{1}{x}$ for $x \in [0.5,5]$. Taylor polynomials of different degrees are expanded at $x_0 = 1$



$2nd$-degree Lagrange Interpolating Polynomial

Goal: construct a polynomial of degree 2 passing 3 data points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$.

Step 1: construct a set of *basis polynomials* $L_{2,k}(x),\ k = 0,1,2$ satisfying

$$L_{2,k}(x_j) = \begin{cases} 1, & \text{when } j = k \\ 0, & \text{when } j \neq k \end{cases}$$

These polynomials are:

$$L_{2,0}(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)},$$

$$L_{2,1}(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)},$$

$$L_{2,2}(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

Step 2: form the 2nd-degree Lagrange interpolating polynomial $P(x)$:

$$P(x) = y_0 L_{2,0}(x) + y_1 L_{2,1}(x) + y_2 L_{2,2}(x)$$

Exercise 3.1.2(a) Use nodes $x_0 = 1, x_1 = 1.25, x_2 = 1.6$ to find 2nd Lagrange interpolating polynomial $P(x)$ for $f(x) = \sin(\pi x)$. And use $P(x)$ to approximate $f(1.4)$.

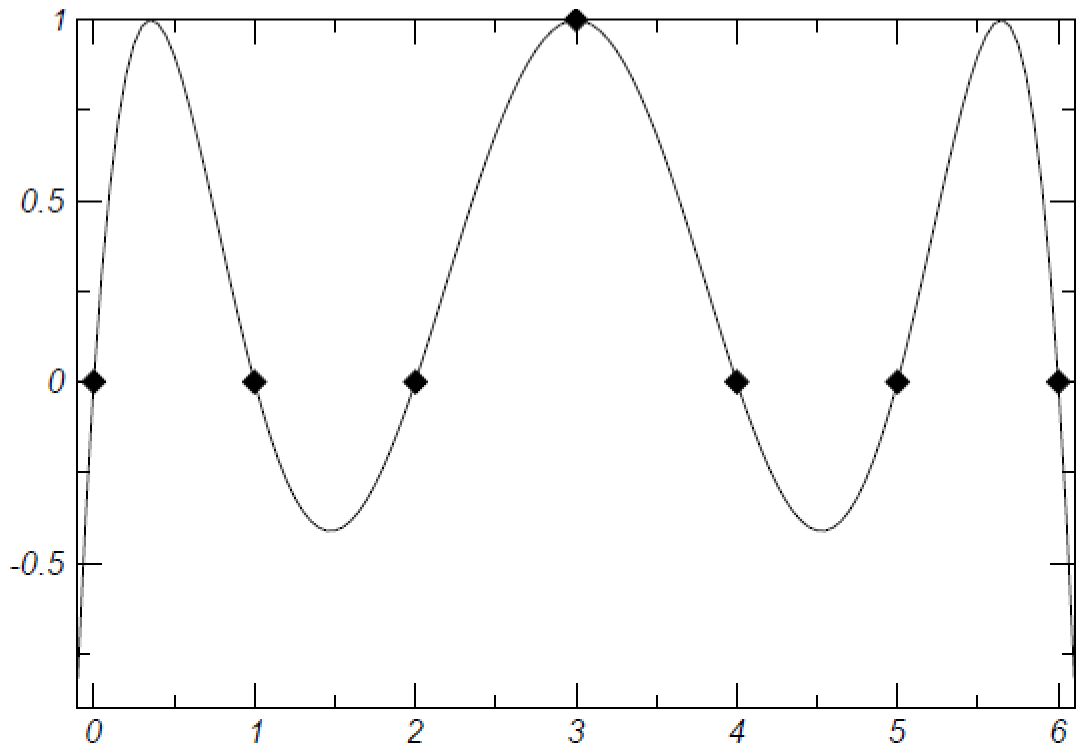
n-degree Interpolating Polynomial through $n+1$ Points

Constructing a Lagrange interpolating polynomial $P(x)$ passing through the points $(x_0, f(x_0))$, $\left(x_1, f(x_1)\right)$, $\left(x_2, f(x_2)\right), \dots, (x_n, f(x_n))$.

1. Define Lagrange basis functions $L_{n,k}(x) = \prod_{i=0, i\neq k}^{n} \frac{x-x_i}{x_k-x_i} = \frac{x-x_0}{x_k-x_0} \dots \frac{x-x_{k-1}}{x_k-x_{k-1}} \cdot \frac{x-x_{k+1}}{x_k-x_{k+1}} \dots \frac{x-x_n}{x_k-x_n}$ for $k = 0,1 \dots n$.

   Remark: $L_{n,k}(x_k) = 1$; $L_{n,k}(x_i) = 0, \ \forall i \neq k$

2. $P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x)$.



- $L_{6,3}(x)$ for points $x_i = i,\ i = 0, \ldots, 6.$

- **Theorem 3.2** If $x_0, \ldots, x_n$ are $n + 1$distinct numbers (called nodes) and f is a function whose values are given at these numbers, then a **unique polynomial** $P(x)$ of **degree at most** $\boldsymbol{n}$ exists with $P(x_k) = f(x_k), \text{for each } k = 0, 1, \ldots n.$

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x).$$

Where $L_{n,k}(x) = \prod_{i=0, i\neq k}^{n} \frac{x-x_i}{x_k-x_i}.$

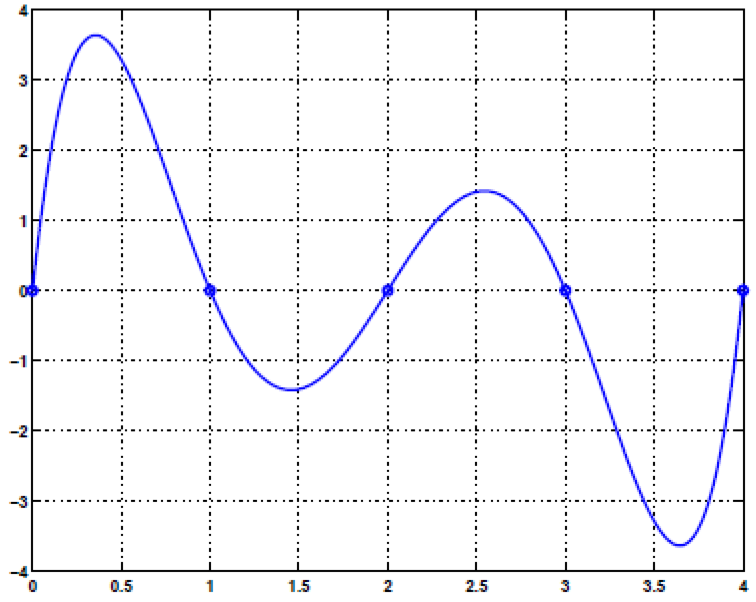
Error Bound for the Lagrange Interpolating Polynomial

Theorem 3.3 Suppose $x_0, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, \ldots, x_n$, and hence in (a, b), exists with $f(x) = P(x) +$

$$\frac{f^{(n+1)}\left(\xi(x)\right)}{(n+1)!}(x - x_0)(x - x_1) \ldots (x - x_n).$$

Where $P(x)$ is the Lagrange interpolating polynomial.

- Remark:
  1. Applying the error term may be difficult.
     $\xi(x)$ is generally unknown.
  2. $(x - x_0)(x - x_1) \ldots (x - x_n)$ is oscillatory.



Graph of $(x-0)(x-1)(x-2)(x-3)(x-4)$

Remark: In general, $|f(x) - p(x)|$ is small when x is close to the center of $[x_0,\ x_n]$.

  3. The error formula is important as they are used for numerical differentiation and integration.

Example 3. 2nd Lagrange polynomial for $f(x) = \frac{1}{x}$ on [2, 4] using nodes $x_0 = 2, x_1 = 2.75, x_2 = 4$ is $P(x) = \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}$. Determine the error form for $P(x)$, and maximum error when polynomial is used to approximate $f(x)$ for $x \in [2,4]$.

Exercise 3.1.6(a). Use appropriate Lagrange polynomials of degree 2 to approximate $f(0.43)$ if $f(0) = 1, f(0.25) = 1.64872,\ f(0.5) = 2.71828, f(0.75) = 4.48169$.

Example 4 Suppose a table is to be prepared for $f(x) = e^x, x \in [0,1]$. Assume the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent x-values, the step size is h. What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0,1]$.