

2.2 Fixed-Point Iteration

Definition 2.2. The number p is a **fixed point** for a given function $g(x)$ if $g(p) = p$.

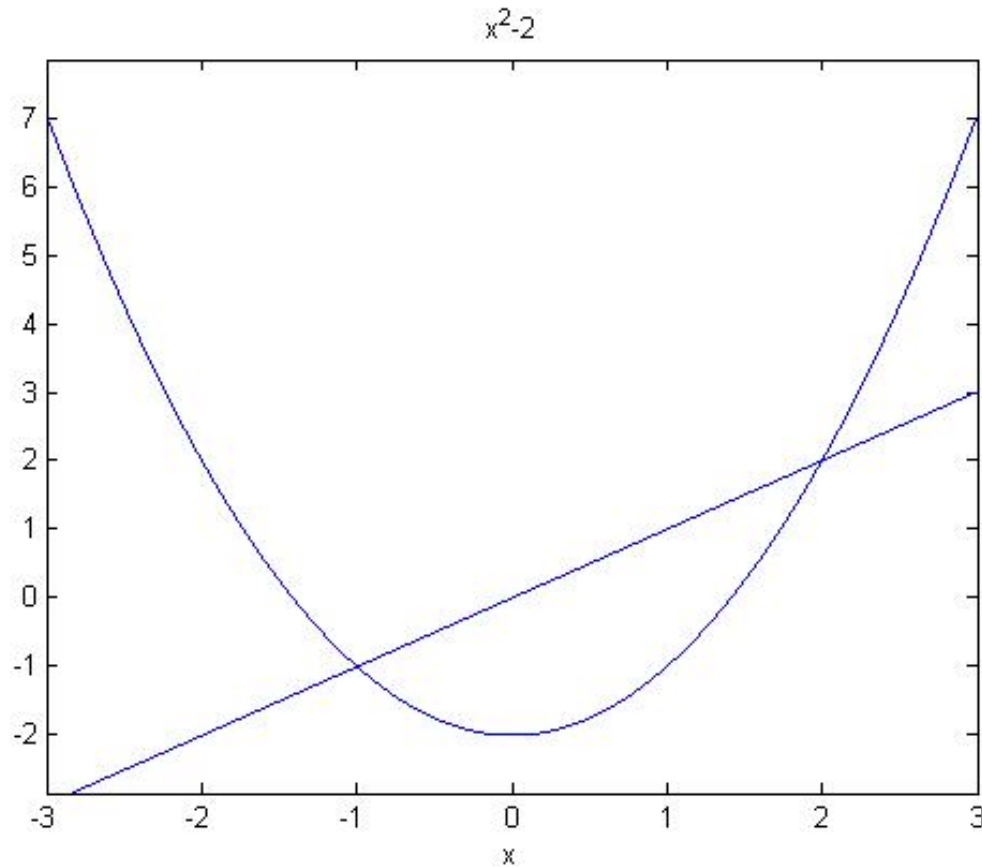
Geometric interpretation of fixed point.

- Consider the graph of function $g(x)$, and the graph of equation $y = x$.
- If they intersect, what are the coordinates of the intersection point?

Example 2.2.1.

Determine the fixed points of the function

$$g(x) = x^2 - 2.$$



Connection between fixed-point problem and root-finding problem

1. Given a root-finding problem, i.e., to solve $f(x) = 0$. Suppose a root is p , so that $f(p) = 0$.

There are **many ways to define** $g(x)$ with fixed-point at p .

For example, $g(x) = x - f(x)$,

$$g(x) = x + 3f(x),$$

...

2. If $g(x)$ has a fixed-point at p , then $f(x)$ defined by $f(x) = x - g(x)$ has a zero at p .

Sufficient conditions for existence and uniqueness of a fix point

Theorem 2.3. Existence and Uniqueness Theorem

- a. If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has **at least one fixed-point** in $[a, b]$
- b. If, in addition, $g'(x)$ exists on (a, b) and **a positive constant $k < 1$ exists** with
$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b),$$
then there is **exactly one fixed-point** in $[a, b]$.

Note:

1. $g \in C[a, b] \rightarrow g$ is continuous in $[a, b]$
2. $g(x) \in [a, b] \rightarrow$ range of g is in $[a, b]$

Example 2. Show $g(x) = \frac{x^2-1}{3}$ has a unique fixed point on $[-1, 1]$.

Example 3. Show that **Theorem 2.3** does not ensure a unique fixed point of $g(x) = 3^{-x}$ on the interval $[0, 1]$, even though a unique fixed point on this interval does exist.

Solution: $g'(x) = -3^{-x} \ln(3)$.

$g'(x) < 0$ on $[0, 1]$. So g is strictly decreasing on $[0, 1]$.

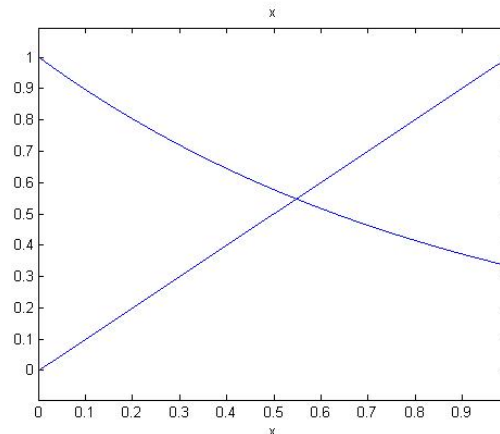
$g(1) = \frac{1}{3} \leq g(x) \leq g(0) = 1$, for $0 \leq x \leq 1$.

Part a of **Theorem 2.3** ensures there is at least one fixed point.

Since $|g'(0.01)| = |-3^{-0.01} \ln(3)| \approx 1.0866$,
 $|g'(x)| \not\leq 1$ on $(0, 1)$.

Since Part b of **Theorem 2.3** is **NOT satisfied**, **Theorem 2.3** can not determine uniqueness.

Graphs of 3^{-x} and $y = x$:



Fixed-Point Iteration Algorithm

- Choose an initial approximation p_0 , generate sequence $\{p_n\}_{n=0}^{\infty}$ by $p_n = g(p_{n-1})$.

- If the sequence converges to p , then

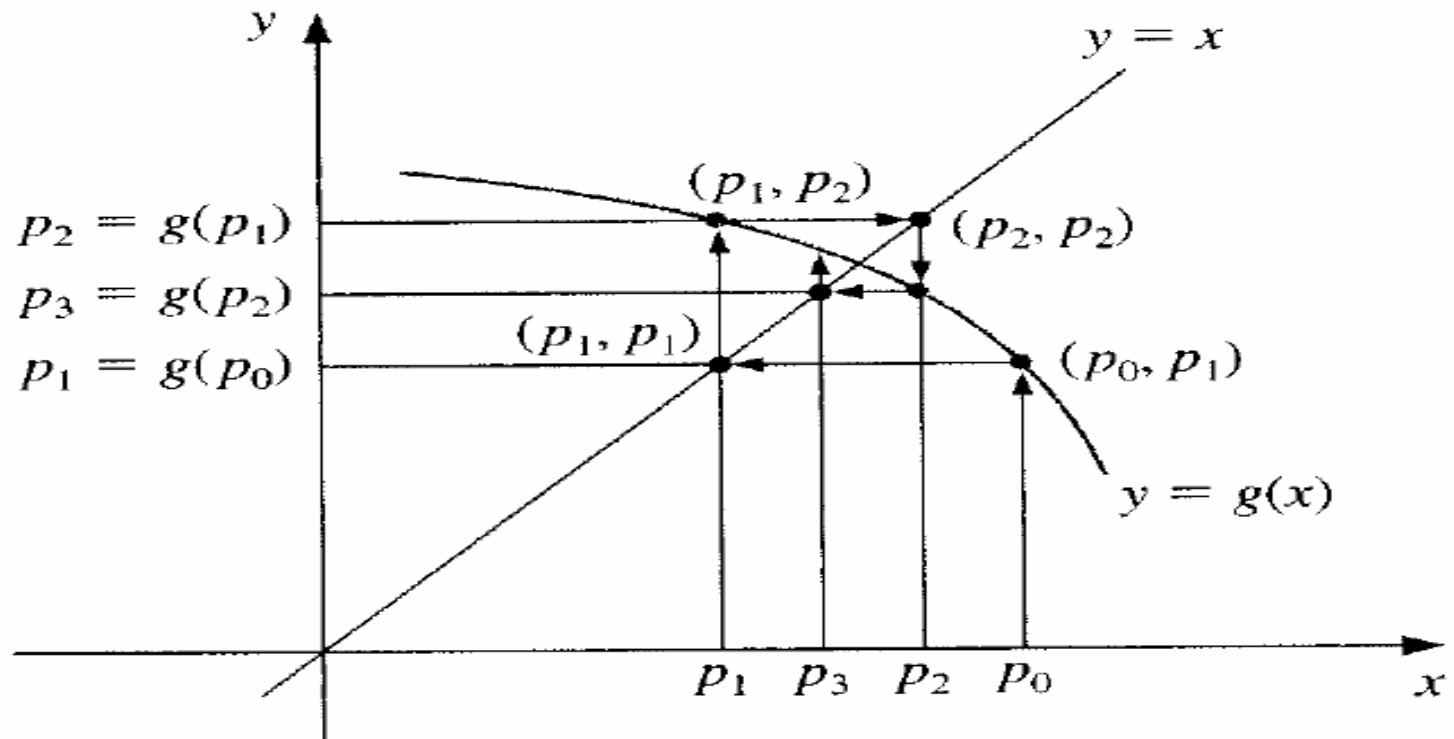
$$p = \lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} g(p_{n-1}) = g\left(\lim_{n \rightarrow \infty} p_{n-1}\right) = g(p)$$

A Fixed-Point Problem

Determine the fixed point of the function $g(x) = \cos(x)$ for $x \in [-0.1, 1.8]$.

Remark: See also the Matlab code.

The Algorithm



(a)

INPUT $\mathbf{p0}$; tolerance \mathbf{TOL} ; maximum number of iteration $\mathbf{N0}$.

OUTPUT solution \mathbf{p} or message of failure

STEP1 Set $i = 1$. // init. counter

STEP2 While $i \leq N0$ do Steps 3-6

STEP3 Set $\mathbf{p} = g(\mathbf{p0})$.

STEP4 If $|\mathbf{p} - \mathbf{p0}| < \mathbf{TOL}$ then

OUTPUT(\mathbf{p}); // successfully found the solution

STOP.

STEP5 Set $i = i + 1$.

STEP6 Set $\mathbf{p0} = \mathbf{p}$. // update $\mathbf{p0}$

STEP7 OUTPUT("The method failed after $\mathbf{N0}$ iterations");

STOP.

Convergence

Fixed-Point Theorem 2.4

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all $x \in [a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b)$$

Then, for any number p_0 in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1})$$

converges to the unique fixed point p in $[a, b]$.

Corollary 2.5

If g satisfies the above hypotheses, then bounds for the error involved using p_n to approximating p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

$$|p_n - p| \leq \frac{k^n}{1 - k} |p_1 - p_0|$$

Illustration Equation $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1,2]$. Use algebraic manipulation to obtain fixed-point iteration function g to solve this root-finding problem.