Implementing Black-Litterman using an Equivalent Formula and Equity Analyst Target Prices

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Abstract

We examine an alternative and equivalent Black and Litterman [1992] formula using classical multivariate analysis which is easier to interpret and which allows more general view formulations than the original formula. Specifically, the equivalent formula provides more intuitive explanation under the limiting case of deterministic views, and it is also easier to show that the resulting optimal portfolio as a combination of the market portfolio and a long-short view portfolio. The equivalent formula also allows for more convenient empirical implementations when views and expected return priors are correlated. We then use a numerical example to illustrate the equivalent formula, and also implement the formula in an optimal asset allocation setting using equity analysts' 12-month ahead target price forecasts for the period of 1999-2010. We show that the optimal portfolio outperforms the market (S&P 500) and this result is robust across different time periods and model parameter choices.

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THE mathematical formulation of modern portfolio theory using classical mean-variance optimization is elegant but has justifiably drawn criticism from practitioners due to the difficulty in obtaining reliable estimates of expected returns and covariances that are not readily observable and may vary significantly over time. These noisy estimates induce noise in the optimal portfolio allocation, characterized by unstable weights, corner solutions, and poor out-of-sample performance.

In a seminal paper, Black and Litterman [1992], BL hereafter, propose a novel way to incorporate investors' views into asset allocation decisions within the standard mean-variance optimization framework of Markowitz [1952]. The BL framework does not require expected return estimates for all securities since it incorporates investors' views into the market equilibrium implied returns. The views can involve multiple securities and can be relative performance views such as "stock A will outperform stock B by 1% over the next month", and the uncertainty of each view can be specified. The BL framework is a robust quantitative model that has proven to reduce the impact of estimation errors in the mean-variance framework and has become popular among practitioners. It also provides a convenient mathematical framework to test the effectiveness of alternative trading strategies.

In this paper, we present an equivalent BL formula derived from classical multivariate analysis, and use a numerical example to show that the equivalent formula provides more intuitive explanation than the original formula in the limiting case of deterministic views. Perhaps more importantly, through the equivalent formula it is also easier to show that the optimal BL portfolio can be seen as a combination of the market portfolio and a long-short "view portfolio". Finally, the more general assumptions held under the equivalent formula allow for faster and easier empirical implementations when the investor's views are not independent of the market conditions.

Da and Schaumburg [2011] provide evidence that within industry relative valuations implicit in analyst target prices provide valuable information to investors. Importantly, this finding is not limited to small stocks that are expensive to trade but also applies to the sample of S&P 500 stocks and does not require trading at the time of announcement. That provides a potentially important trading strategy for portfolio managers to incorporate into their existing portfolios. In this paper, we formally test the strategy within the BL framework using the entire sample of S&P 500 stocks and analysts' target price forecasts for the period of 1999-2010. We show that the optimized BL portfolio outperforms the market and that the result is robust across different time periods and parameter choices.

The paper is organized in the following way. We start by reviewing the BL model using the Bayesian approach, and then present a simpler formulation of the BL model using the classical multivariate analysis that is easier to interpret and allows for correlation between the uncertainty of the investor's views and the asset returns. We then show a numerical example followed by the empirical implementation that uses relative views implicit in analyst target prices to blend with market equilibrium under the BL framework, and show the performance of the optimal monthly-rebalanced BL portfolio.

An Equivalent Black-Litterman Formula

Consider a standard mean-variance portfolio optimization involving n assets. The necessary inputs are the expected returns, $\mu \in \mathbb{R}^n$, and the covariances, $V \in \mathbb{R}^{n\times n}$. Following the BL setup, we assume that the investor accurately measures V (e.g. using high frequency data) but only imperfectly observes μ so that she only can form a prior distribution of μ:

$$\mu \sim N(\mu_0, \Sigma) \tag{1}$$

In practice, however, the investor cannot be expected to provide the $\frac{1}{2}n(n+1)$ estimates required to specify the prior variance, Σ, and Black and Litterman [1992] therefore make the simplifying assumption that the prior variance is simply proportional to the return variance:

Assumption 1: $\Sigma = \tau V$.

Choosing τ much less than one captures the fact the variance (or covariance) of the expected return prior is typically smaller than that of the return itself.

It is well known that the portfolio allocation problem is particularly sensitive to the expected return estimates used. One of the main advantages of the BL framework is that the investor is not required to provide each of the n difficult to estimate expected returns, μ_0. Instead, Black and Litterman [1992] point out that, in equilibrium, there must exist a relation between current observed market capitalization weights and market "neutral" expected returns is given by

Assumption 2: $w_{mkt} = \frac{1}{\gamma}V^{-1}(\mu_0 - R_f\iota)$, where γ is the relative risk aversion of the average investor.[1]

Importantly, Assumption 2 avoids using historical average returns as measures of μ_0, which results in extreme positions in the optimal portfolio. Following BL, we shall maintain Assumptions 1-2 throughout, but assume that the investor receives new information about the performance of k portfolios with which she wants to update the expected return priors. The investor's new information can be represented by a matrix, $P \in \mathbb{R}^{k\times n}$, of weights of the k views on each of the n assets and a vector, $Q \in \mathbb{R}^k$, of returns to each of the k views.

$$P\mu = Q - \varepsilon \tag{2}$$

The fact that the views themselves are uncertain is captured by $\varepsilon \sim N(0, \Omega)$ where $\Omega \in \mathbb{R}^{k\times k}$. In the original BL model Ω is diagonal and the uncertainty of the view, ε, uncorrelated with μ but this is of course neither a necessary nor generally realistic feature.

Appendix A.1 shows that in a Bayesian decision framework with quadratic loss function, when the investor is provided with the views, her optimally updated prior is the conditional expectation – $E(\mu \mid views)$ or $E(\mu \mid y = Q)$. Appendix A.2 then shows:

$$E(\mu \mid views) = \left[\Sigma^{-1} + P'\Omega^{-1}P\right]^{-1}\left[\Sigma^{-1}\mu_0 + P'\Omega^{-1}Q\right] \tag{3}$$

[1] In Merton's [1973] ICAPM Assumption 2 also holds in various special cases.

which is the formula given in the original paper by Black and Litterman [1992].

Classical multivariate analysis

The conditional expectation can also be computed in the framework of classical multivariate normal analysis. Consider the $(n+k)\times 1$ vector of expected return priors (μ) and views ($y \equiv P\mu + \varepsilon$), $(\mu', y')'$, then under the assumption that μ and ε are jointly normally distributed with $\mathrm{Cov}(\mu, \varepsilon) = \Gamma$, the joint (prior) distribution of expected returns and views is given by:

$$\begin{pmatrix} \mu \\ y \end{pmatrix} \sim N\left(\begin{bmatrix} \mu_0 \\ Q \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right), \text{ where } \quad \begin{aligned} \Sigma_{11} &= \Sigma, \\ \Sigma_{12} &= \Sigma'_{21} = \Sigma P' + \Gamma, \\ \Sigma_{22} &= P\Sigma P' + \Gamma' P' + P\Gamma + \Omega \end{aligned} \tag{4}$$

The conditional expectation of μ given the views (2) can be computed as the projection (regression) of the n expected returns μ onto the k views $y \equiv P\mu + \varepsilon = Q$:[2]

$$E(\mu \mid y = Q) = \mu_0 + \Sigma_{12}\Sigma_{22}^{-1}(Q - Ey) \tag{5}$$

Substituting expression (4) into (5) with $\Gamma = 0$, we get:

$$E(\mu \mid y = Q) = \mu_0 + \Sigma P' \left[P\Sigma P' + \Omega\right]^{-1} [Q - P\mu_0] \tag{6}$$

Although not immediately obvious, the BL formula (3) and (6) are indeed the same expression and we show this equivalence in Appendix A.3.[3] More generally, when $\Gamma \neq 0$:

$$E(\mu \mid y = Q) = \mu_0 + (\Sigma P' + \Gamma)\left[P\Sigma P' + \Gamma' P' + P\Gamma + \Omega\right]^{-1} [Q - P\mu_0] \tag{7}$$

Comparison of the Equivalent and the Original Formula

The classical multivariate normal analysis has several advantages over the Bayesian inference approach. First, (6), derived from classical multivariate normal analysis, allows us to see more clearly what happens in the limiting case as $\Omega \to 0$ (i.e., when the views become certain or deterministic). The answer can be seen easily by setting $\Omega = 0$ in (6). The updated expected returns become:

$$E(\mu \mid \mathit{views}) \to \mu_0 + \Sigma P' \left[P\Sigma P'\right]^{-1} [Q - P\mu_0] \tag{8}$$

In contrast, this limiting case cannot be easily seen with (3) where Ω^{-1} is involved.

Second, (6) provides much better intuition than (3). Using (6), and Assumption 1, it is easy to show (see Appendix A.4) that the optimal BL portfolio weights (w_{BL}) are a combination of the

[2] Rao[1973] page 522 or Greene[2000] page 87.

[3] The equivalence follows from the celebrated Matrix Inversion Theorem from linear algebra - Golub and Van Loan [1996, page 51], as was also pointed out by Fabozzi et al [2006] and Da Silva et al [2009].

market portfolio (w_{mkt}) and a set of long and short portfolios (w_L and w_S respectively) representing an investor's views:

$$w_{BL} = \alpha_{mkt} w_{mkt} + \alpha_L w_L - \alpha_S w_S \ , \quad \alpha_{mkt} + \alpha_L - \alpha_S = 1 \ , \quad \alpha_L, \alpha_S, w_L, w_S \geq 0 \tag{9}$$

where the expressions for the weights $\{\alpha_{mkt}, \alpha_L, \alpha_S\}$ and portfolio weights $\{w_{BL}, w_{mkt}, w_L, w_S\}$ are provided in Appendix A.4. In the special case where all views are relative, i.e. $P\iota = 0$, the formula (9) simplifies since in that case $\alpha_{mkt} = 1$ and $\alpha_L = \alpha_S$ and the optimal BL portfolio is the sum of the market portfolio and a zero investment long-short portfolio: $w_{BL} = w_{mkt} + w_{LS}$, where

$$w_{LS} = \frac{1}{\gamma} P' \left[PVP' + \Omega/\tau \right]^{-1} [Q - P\mu_0] \tag{10}$$

He and Litterman [1999] discuss three simple intuitive properties of the BL model, which can be easily seen from (6) and (9):

– "The unconstrained optimal portfolio is the market equilibrium portfolio plus a weighted sum of portfolios representing an investor's views." This is a direct implication of (9).

– "The weight on a portfolio representing a view is positive when the view is more bullish than the one implied by the equilibrium." From the expression (6), it is clear that a positive $Q - P\mu_0$ for asset i is likely to result in a higher than equilibrium expected return and hence a positive portfolio weight in w_L for asset i.

– "The weight increases as the investor becomes more bullish on the view as well as when the investor becomes more confident about the view." It is easily seen from the expression (10) that a more bullish view (larger Q) will tend to lead to larger portfolio weights weights w_{LS} and similarly for a more confident view (smaller Ω).

Third, the classical multivariate normal analysis is more general than the standard Bayesian approach since there is no requirement that Ω be diagonal nor that ε is orthogonal to μ.[4] Therefore, even if views are correlated (i.e., Ω is not diagonal), (6) is still valid. More generally, we can easily extend the solution to the case where expected return and view are correlated (μ is correlated with ε) since equation (5) still holds, although the exact expressions of Σ_{12} and Σ_{22} have to change in order to incorporate the covariance matrix between μ and ε.[5] To achieve the same generalization in the Bayesian inference framework is more cumbersome since we have to rotate both expected returns and views before computing the posterior.

In summary, not only is the equivalent formula more intuitive, allowing us to easily see the limiting case of deterministic views and the optimal BL portfolio being the combination of the market portfolio and a long-short view portfolio, but the equivalent formula also allows for more convenient computations when views and priors on expected returns are correlated. In the next

[4] In general the Bayesian analysis can incorporate these features, but this is rarely if ever done.

[5] For example, views can be expressed as conditional on future economic outlook, thus will be correlated with future expected returns. A recent working paper by Kacperczyk, Van Nieuwerburgh, and Veldkamp [2011] provide consistent evidence that fund managers' skill can be time-varying and depend on business cycle.

section we show a numerical example of BL asset allocation using the equivalent formula that allows for this type of correlation between views and market conditions.

An Example of BL Asset Allocation that Allows View Correlation Using the Equivalent Formula

Consider a setting with a risk free rate $R_f = 0$ and four assets with a distribution of the expected returns given by $R \sim N(\mu, V)$ and a return prior $\mu \sim N(\mu_0, \Sigma)$ where $\Sigma = \frac{1}{10}V$ and

$$\mu_0 = \begin{bmatrix} 15.0 \\ 18.0 \\ 7.5 \\ 6.0 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 4 & 2 & \frac{1}{2} & \frac{1}{2} \\ 2 & 4 & 1 & 1 \\ \frac{1}{2} & 1 & 1 & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{4} & 1 \end{bmatrix} \tag{11}$$

The investor has two bullish views reflecting that she expects the first stock to outperform the second and third stock by 5 percent relative to what is implied by the prior. That is to say, the investor expects the first stock to outperform the second by 2 percent, and to outperform the third by 12.5 percent. These views can be expressed using (2) where:

$$P = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 2.0 \\ 12.5 \end{bmatrix} \tag{12}$$

In the BL world, the equilibrium implied market portfolio is $w_{mkt} = [0.2, 0.2, 0.4, 0.2]'$ and the view portfolios are given by $w_L = [1, 0, 0, 0]'$ and $w_S = [0, \frac{1}{2}, \frac{1}{2}, 0]'$ which is intuitive since the bullish view portfolio is positively correlated with stock 1 and negatively correlated with stock 2 and 3 while stock 4 is not involved in any view. The analyst does not need to be certain of her prediction and the uncertainty is parametrized by $\varepsilon \sim N(0, \Omega)$, where the view variance Ω determines the strength of her views. When views and priors are uncorrelated, i.e. $Cov(\mu, \varepsilon) = \Gamma = 0$, one can obtain the weights of the optimal BL portfolio expressed in (9) for various levels of view uncertainty using either (3) or (6):

View uncertainty	Expected return	Weights			BL portfolio weights
Ω	$E[\mu \mid view]$	α_{mkt}	α_L	α_S	$w_{BL}(= \alpha_{mkt} w_{mkt} + \alpha_L w_L - \alpha_S w_S)$
0	$[19.2, 17.2, 6.7, 5.8]'$	1.00	0.15	0.15	$[0.35, 0.125, 0.325, 0.2]'$
1	$[18.7, 17.3, 6.8, 5.8]'$	1.00	0.13	0.13	$[0.33, 0.135, 0.335, 0.2]'$
10	$[16.0, 17.8, 7.3, 6.0]'$	1.00	0.04	0.04	$[0.24, 0.18, 0.38, 0.2]'$
100	$[15.1, 18.0, 7.5, 6.0]'$	1.00	0.01	0.01	$[0.21, 0.195, 0.395, 0.2]'$

When the view is very uncertain (e.g. $\Omega = 100$), the expected returns $E[\mu\,|view]$ do not deviate much from the original expected returns μ_0 in (11) and the weights on the view portfolios, α_L, α_S, are close to zero. Also, the BL portfolio weights w_{BL} are close to the implied market weights w_{mkt}. However, as the confidence in the views grows ($\Omega \downarrow 0$), the expected returns $E[\mu\,|view]$ deviate substantially from the prior mean μ_0 and the weights on the view portfolios increase. BL portfolio weights w_{BL} thus deviate more from w_{mkt}.[6] In particular, the expected return is now much greater than the prior mean for stock 1 and much less than the prior mean for stock 2 and 3. Thus, more allocation is made to stock 1 and less to stock 2 and 3, as reflected in the BL portfolio weights w_{BL}. Note that, even though the view does not involve the fourth stock, its expected return is affected as confidence increases ($\Omega \downarrow 0$) since it is correlated with the portfolio views about which the investor is bullish.

Next, assume that the investor expects the performance of her view portfolios (which she thinks will earn an expected return of 5%) to be correlated with the expected return prior μ_0. However, if n is large, it would be an exceedingly difficult task to specify the individual correlations between each of the n stocks and k views required to specify the $n \times k$ covariance matrix Γ . Instead, the investor may be more comfortable specifying how her view portfolios are correlated with the market portfolio or some other set of benchmark portfolios. The task of specifying Γ can then be completed by the following assumptions:

Assumption 3:

i. The investor specifies the covariance, Λ, between the k view portfolios and a (small) set of m benchmark portfolios B (e.g. the market portfolio)

ii. Any portfolio whose prior is uncorrelated with the view portfolio prior, $P\mu$, is also uncorrelated with ε.

iii. Any portfolio whose prior is correlated with the view portfolio prior but uncorrelated with the benchmark portfolios, B, is uncorrelated with ε.

Put together, Assumption 3.i-iii pin down Γ and only require the investor to specify a much more limited number of parameters in the $m \times k$ covariance matrix, Λ (e.g. can take $m = 1$ and B equal to the market portfolio as in the example below). Assumption 3.ii combined with Assumption 1 ensures that we retain the intuitive feature of the original BL model, namely that the resulting optimal view portfolios, w_L, w_S only involve stocks about which the investor has a view while Assumption 3.iii states that the only source of common variation between views and priors are the portfolio(s) B specified in Assumption 3.i. The precise implementation details and formulas are given in Appendix A.5.

[6] Although not shown here to preserve space, a similar effect is evident as the view becomes more bullish, i.e. Q increases, α_L, α_S increase and expected returns deviate more from the prior mean.

Returning to our example, assume for simplicity that the investor specifies the same correlation between the market portfolio and each of her two views, i.e. $m = 1$ and $B =$ market portfolio. To focus on the effect of view correlation, we fix the view confidence, $\Omega = 1$, and let the correlation vary between the market and the views, ρ, from -1 to 1.

View correlation	Expected return	Weights			View Portfolio Weights (%)	
ρ	$E[\mu\,\vert view]$	α_{mkt}	α_L	α_S	w_L	w_S
-1.0	$[24.2, 9.5, 5.3, 3.9]'$	1.00	0.45	0.45	[100 0 0 0]	[0 97 3 0]'
-0.5	$[19.0, 16.1, 6.7, 5.5]'$	1.00	0.17	0.17	[100 0 0 0]	[0 71 29 0]'
-0.2	$[18.7, 17.0, 6.8, 5.7]'$	1.00	0.14	0.14	[100 0 0 0]	[0 58 42 0]'
0.0	$[18.7, 17.3, 6.8, 5.8]'$	1.00	0.13	0.13	[100 0 0 0]	[0 50 50 0]'
0.2	$[18.8, 17.6, 6.8, 5.9]'$	1.00	0.13	0.13	[100 0 0 0]	[0 43 57 0]'
0.5	$[19.1, 18.0, 6.8, 6.0]'$	1.00	0.14	0.14	[100 0 0 0]	[0 33 67 0]'
1.0	$[20.7, 18.8, 6.6, 6.2]'$	1.00	0.18	0.18	[100 0 0 0]	[0 18 82 0]'

Comparing the uncorrelated ($\rho = 0.0$) to the correlated cases, we see that while w_L is unchanged, the composition of w_S changes substantially depending on the correlation: as the correlation increases (decreases), more (less) weight is put on stock 3 relative to stock 2. Similarly, the weights put on the long/short portfolios, α_L, α_S, increase for non-zero correlations.

Empirical Implementation Using Equity Analyst Target Prices

Equity analysts provide 12-month ahead target price forecasts for the stocks they analyze. Given the target price, we can compute a *target price implied expected return* (TPER), defined as the ratio of consensus 12-month ahead target price to current market price: $TPER_t = TP_t/P_t - 1$. The resulting TPER summarizes the view held by the equity analysts on the stock and provides a natural instrument for implementing the BL model.

Rather than being generalists, most analysts specialize in a sector and typically cover about half a dozen stocks within the same industry. By analyzing the specifics of a handful of similar firms, the analyst is therefore well situated to rank the relative strength of each stock going forward, although she may have significantly less insight into the forecasting of macro factors which affect the performance of the sector as whole. Consistent with such intuition, Boni and Womack [2006] show that an investment strategy based on stock recommendation revisions within the same industry improves the return significantly compared to a similar strategy without industry control. Da and Schaumburg [2011] also find that relative ranking of TPER across individual stocks within sector is very informative although the absolute level of TPER is not. For the above reason, we only focus on relative views such as: stock (or portfolio) A will outperform stock (or portfolio) B in the same sector by $x\%$. For relative views, $\iota' P = 0$.

Data description

The target price data are provided by the Institutional Brokers' Estimate System (IBES). The IBES data covers the period from April 1999 to December 2010. At the end of each month during our sample period, we include only S&P500 stocks for which there are at least two (12 month ahead) target price announcements during that month. We do not "fill in the blanks" using older target prices in order to avoid introducing an upward bias in the target prices. The bias arises because analysts are more likely to issue a target price when they are in favor of a stock, as documented in Brav and Lehavy [2003].

We break down our sample into sectors according to the first two digits of Standard and Poor's GICS (Global Industry Classification Standard). Using IBES data, Boni and Womack [2006] show that the GICS sector and industry definitions match well with the areas of expertise of most analysts as defined by the set of stocks covered by each analyst. The GICS is therefore a natural choice for sector definition. Since there are few stocks in the Telecommunications Services sector, we group them with the Information Technology sector to form a combined Technology sector. The resulting 9 sectors are: *Energy, Materials, Industrials, Consumer Discretionary, Consumer Staples, Health Care, Financials, Technology and Utilities.* This classification is also consistent with the way sector ETFs are formed.

Exhibit 1
Summary statistics: TPERs of S&P500 stocks covered by IBES, 1999-2010

Year	Mean	Std Dev	Median	Quartile 1	Quartile 3	Monthly obs.	# stocks
1999	0.294	0.256	0.249	0.135	0.406	2374	347
2000	0.400	0.436	0.309	0.134	0.534	4215	396
2001	0.322	0.454	0.202	0.094	0.378	4512	402
2002	0.271	0.328	0.185	0.085	0.347	4647	414
2003	0.116	0.208	0.087	0.011	0.190	4785	413
2004	0.108	0.148	0.090	0.020	0.171	4927	429
2005	0.115	0.129	0.103	0.036	0.177	5055	442
2006	0.118	0.141	0.101	0.034	0.182	5321	471
2007	0.127	0.139	0.111	0.044	0.189	5565	493
2008	0.282	0.289	0.221	0.118	0.360	5621	509
2009	0.154	0.256	0.112	0.022	0.234	5679	509
2010	0.163	0.156	0.142	0.061	0.243	5383	505
Total	0.197	0.278	0.139	0.052	0.263	58084	663

Exhibit 1 reports the summary statistics of our S&P500 stock sample. The S&P500 universe which is the main focus of this paper distinguishes itself in several respects: First, S&P500 stocks receive the most attention and coverage by analysts. On average, analysts issue target prices for around 450 of the S&P500 stocks each month and the average number of target prices per S&P500 stock each month is 11.6 – significantly higher than that of an average stock in the IBES database (5.4). Therefore, the consensus target price used to compute $TPER$ for S&P500 stocks is less prone to outliers and presumably more accurate. Second, S&P500 stocks are on average more liquid and cheaper to trade, which makes it easier to bound the potential impact of transaction costs. Finally, the GICS sector assignment of S&P500 stocks is done directly by Standard & Poor's and does not rely on sometimes arbitrary mapping from SIC codes. The relative TPER within GICS sectors therefore provides a more precise signal of the deviation from fundamentals. The GICS was officially launched by Standard & Poor's and Morgan Stanley Capital International (MSCI) in 1999, so we avoid the issues of backfilling since the IBES data starts roughly the same time.

Empirical implementation

At the end of each month t from June 1999 to December 2010, we implement the long-short portfolio of (10) in the following way. Assume there are N_t S&P500 stock with TPERs in that month. Within each sector of the 9 sectors, we rank the S&P500 stocks into 9 groups according to their current month $TPER$s.[7] Within each sector, we then form a long-short portfolio where we long on stocks (equally-weighted) in the highest-TPER portfolio and short on stocks (equally-weighted) in the lowest-TPER portfolio. These portfolio weights are then summarized in the $9 \times N_t$ view matrix P. For example, if for simplicity we consider the market consisting of only 2 sectors with 3 stocks in each sector, and TPERs of the 6 stocks can be represented by $TPER$ = [3 4 0 5 8 -1]' (in percentage), then we long stock 2 and short stock 3 in the first sector, and long stock 5 and short stock 6 in the second sector. Thus:

$$P = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{13}$$

Let $TPER$ be a $N_t \times 1$ vector of TPERs and Z be a $N_t \times N_t$ diagonal matrix where the diagonal elements are the the standard deviation of target prices received from different analysts divided by the consensus target price, similar to the dispersion measure used in Diether, Malloy, and Scherbina [2002]. The uncertainty of the views Ω is then computed as $PZP\prime$ and the bullishness of the views $Q - P\mu_0$ is computed as $P{\cdot}TPER$.[8] Finally, the variance-covariance matrix V is estimated using

[7] In line with common practice in the empirical asset pricing literature, we exclude stocks with share prices below five dollars in order to ensure that the results are not unduly influenced by the bid-ask bounce. This price filter has little impact on the S&P500 sample, eliminating less than 1% of the stocks.

[8] We are implicitly assuming that $P\mu 0 = 0$ since stocks in the same industry will tend to have similar CAPM betas and expected returns. Notably, this short cut avoids the need to compute μ_0 using Assumption 2 which would involve the numerically challenging task of inverting a 500×500 matrix.

monthly returns in a 5-year rolling window. The parameter choice of γ and τ for the benchmark case are 5 and 0.1, respectively. The long-short and the BL portfolio weights can be computed using (10) and (25).

Empirical results

We compute the annual Sharpe ratios of the BL portfolio for the full sample period from July 1999 to December 2010, as well as for the sub-periods using the 5-year rolling windows. We choose a γ to be 5, and a τ to be 0.1. We also allow for views to be correlated with returns, and the correlation ρ to vary among -0.5, 0 (no corelation), and 0.5. The results are then compared to the annual market Sharpe ratios for the same period, as shown in Exhibit 2.

Exhibit 2
Annual Sharpe ratios of the market and the Black-Litterman portfolio

Period	Market SR	BL portfolio SR		
		$\rho = 0$	$\rho = -0.5$	$\rho = 0.5$
07/1999 - 12/2010	0.088	0.223	0.229	0.216
07/1999 - 12/2003	-0.171	0.020	0.022	0.019
01/2000 - 12/2004	-0.094	0.232	0.231	0.231
01/2001 - 12/2005	0.059	0.347	0.363	0.331
01/2002 - 12/2006	0.547	0.814	0.825	0.802
01/2003 - 12/2007	1.325	1.324	1.351	1.298
01/2004 - 12/2008	-0.275	-0.191	-0.182	-0.199
01/2005 - 12/2009	0.056	0.095	0.110	0.082
01/2006 - 12/2010	0.131	0.160	0.169	0.151

For the whole sample period, the annual Sharpe ratio (SR) of the BL portfolio is 0.223, much higher than 0.088, the annual Sharpe ratio of the market portfolio (S&P 500) for the same period. The sub-period comparisons in Exhibit 2 show the robustness of this result. The annual Sharpe ratios of the BL portfolio are higher than those of the market portfolio during all sub-periods except the period of 2003-2007 when the two ratios are virtually the same. When we allow for some significant view correlations with returns, the SR results do not deviate much as seen from the case when $\rho = -0.5$ and 0.5. SRs are slightly higher when views are negatively correlated with returns, and we speculate that is because negatively correlated views may provide slightly and relatively more valuable insights of stock performances.

We further plot the monthly and cumulative excess returns (over the risk free rates) of both portfolios during our sample period in Exhibit 3 and 4 respectively, assuming the full sample period and no view correlation.

Exhibit 3
Monthly excess returns

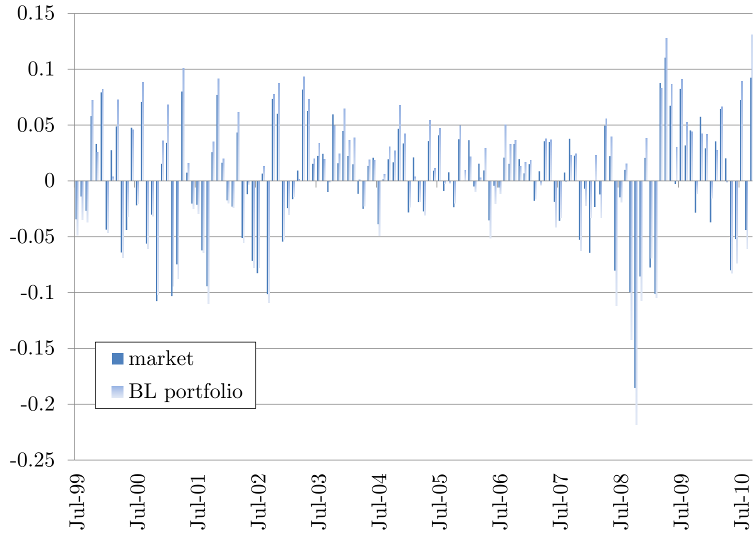


Exhibit 4
Cumulative monthly excess returns

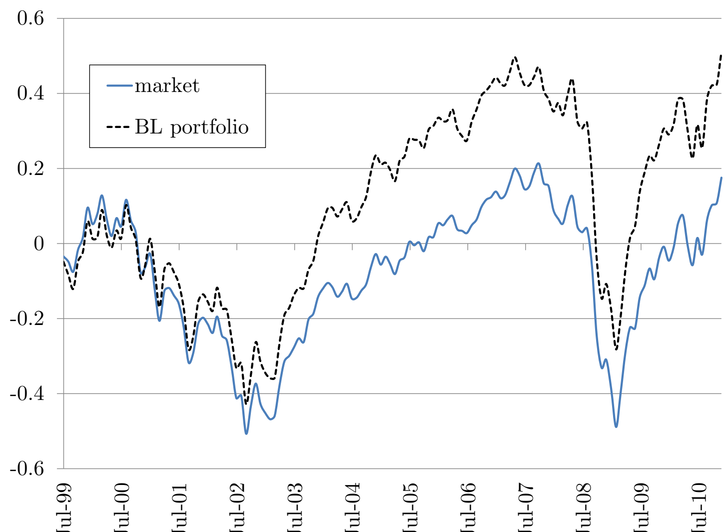
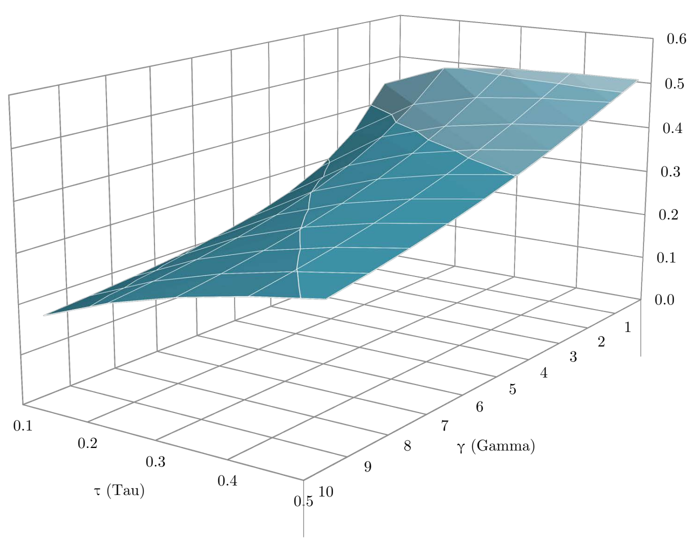


Exhibit 3 shows the monthly excess returns, and the BL portfolio appears to outperform the market portfolio during most of the months. Exhibit 4 shows the cumulative excess returns and again demonstrates that the BL portfolio outperforms the market portfolio. It also allows a visual inspection into the periods when the BL portfolio performs really well (or not so well). For example, during the financial crisis in late 2008 when the market fell sharply, the BL portfolio performed worse than the market but the performance picked up right after January 2009 and outperformed the market afterwards.

Exhibit 5
Robustness check of Sharpe ratios of the BL portfolio (full period)

		τ (tau)				
		0.1	0.2	0.3	0.4	0.5
γ	1	0.443	0.496	0.511	0.516	0.517
(gamma)	2	0.346	0.430	0.463	0.479	0.487
	3	0.287	0.376	0.419	0.443	0.457
	4	0.249	0.334	0.381	0.410	0.428
	5	0.223	0.302	0.350	0.381	0.402
	6	0.203	0.276	0.324	0.356	0.378
	7	0.189	0.256	0.302	0.334	0.357
	8	0.178	0.240	0.283	0.315	0.339
	9	0.169	0.226	0.268	0.299	0.322
	10	0.161	0.215	0.254	0.284	0.307

Exhibit 6
Black-Litterman portfolio Sharpe ratio sentitivity to BL parameters



To further examine the sensitivity of our results to parameter choice, we compute the annual Sharpe ratios for the full sample period using different combinations of γ and τ. We choose γ

to be integers from 1 through 10; a higher γ representing lower relative risk aversion and thus a lower weight on the long-short portfolio. We choose τ to be 0.1 to 0.5 with an increment of 0.1. A higher τ represents a higher proportion of variance-covariance of expected returns that is captured by that of the actual returns. The results are presented in Exhibit 5. Exhibit 5 shows that under all parameter combinations, the resulting Sharpe ratio of the BL portfolio is much higher than 0.088, which is the annual Sharpe ratio of the market portfolio for the whole sample period. We plot these computed Sharpe ratios in Exhibit 6. Intuitively, if the long-short portfolio strategy is effective, a lower γ and/or a higher τ will generate a higher Sharpe ratio, which is confirmed by results presented in Exhibit 5 and 6.

Conclusion

In this paper, we first derive an equivalent Black and Litterman [1992] formula using the classical multivariate analysis which is easier to interpret and to apply. We then illustrate the equivalent formula using a numerical example, and implement the formula in an optimal asset allocation setting using equity analysts' target price forecasts for the period of 1999-2010. We show that the optimal portfolio outperforms the market and this result is robust across different time periods and model parameter choices.

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A Appendix

A.1 Optimal decision in a Bayesian framework with quadratic loss function

The Bayes risk function $R(\cdot)$ associated with the estimator $\widehat{\theta}$ and data x is given by (see DeGroot[1970], Chapter 8):

$$R(\theta) = \int L(\theta, \widehat{\theta}) f(x|\theta) dx \tag{14}$$

where $L(\theta, \widehat{\theta})$ is a loss function, $f(y|\theta)$ is a proper pdf for the data x given θ. Given a prior density $f(\theta)$, The Bayesian approach amounts to find a $\widehat{\theta}$ that minimizes the expected risk across all possible θ:

$$\min_{\widehat{\theta}} E[R(\theta)] = \min_{\widehat{\theta}} \int \int L(\theta, \widehat{\theta}) f(x|\theta) f(\theta) dx d\theta \tag{15}$$

Interchanging the order of integration (assuming $E[R(\theta)]$ is finite) and using the Bayes rule: $f(x|\theta)f(\theta) = f(\theta|x)f(x)$, we have:

$$\begin{aligned} \min_{\widehat{\theta}} E[R(\theta)] &= \min_{\widehat{\theta}} \int \left[\int L(\theta, \widehat{\theta}) f(\theta|x) d\theta \right] f(x) dx \\ &= \min_{\widehat{\theta}} \int E\left[L(\theta, \widehat{\theta})|x \right] f(x) dx \end{aligned} \tag{16}$$

so that any $\widehat{\theta}$ which minimizes the expression in the square brackets will also minimize the expected risk. When the loss function is of quadratic form:

$$L(\theta, \widehat{\theta}) = (\theta - \widehat{\theta})' C (\theta - \widehat{\theta}) \tag{17}$$

where C is a positive definite symmetric matrix, the expected loss (16) is minimized at the conditional expectation $\widehat{\theta} = E[\theta|x]$.

A.2 Black-Litterman formula in a Bayesian framework

Since $y|\mu \sim N(P\mu, \Omega)$, we can write the likelihood function as:

$$f(y = Q|\mu) \propto \exp\left[-\frac{1}{2}(Q - P\mu)'\Omega^{-1}(Q - P\mu) \right] \tag{18}$$

Use Bayesian updating, the posterior pdf is:

$$\begin{aligned} f(\mu \mid views) &= f(\mu \mid y = Q) \propto f(y = Q \mid \mu) f(\mu) \\ &\propto \exp\left[-\frac{1}{2}(Q - P\mu)'\Omega^{-1}(Q - P\mu) \right] \times \exp\left[-\frac{1}{2}(\mu - \mu_0)'\Sigma^{-1}(\mu - \mu_0) \right] \\ &= \exp\left[-\frac{1}{2} \left\{ \begin{array}{c} \mu' P' \Omega^{-1} P \mu - Q' \Omega^{-1} P \mu - \mu' P' \Omega^{-1} Q + Q' \Omega^{-1} Q + \\ \mu' \Sigma^{-1} \mu - \mu_0' \Sigma^{-1} \mu - \mu' \Sigma^{-1} \mu_0 + \mu_0' \Sigma^{-1} \mu_0 \end{array} \right\} \right] \\ &= \exp\left[-\frac{1}{2} \left\{ \begin{array}{c} \mu' \left[\Sigma^{-1} + P' \Omega^{-1} P \right] \mu - \mu' \left[\Sigma^{-1} + P' \Omega^{-1} Q \right] - \\ \left[\mu_0' \Sigma^{-1} + Q' \Omega^{-1} P \right] \mu + \mu_0' \Sigma^{-1} \mu_0 + Q' \Omega^{-1} Q \end{array} \right\} \right] \\ &\propto \exp\left[-\frac{1}{2} \left\{ \begin{array}{c} \left(\mu - \left[\Sigma^{-1} + P' \Omega^{-1} P \right]^{-1} \left[\Sigma^{-1} \mu_0 + P' \Omega^{-1} Q \right] \right)' \\ \times \left[\Sigma^{-1} + P' \Omega^{-1} P \right] \\ \times \left(\mu - \left[\Sigma^{-1} + P' \Omega^{-1} P \right]^{-1} \left[\Sigma^{-1} \mu_0 + P' \Omega^{-1} Q \right] \right) \end{array} \right\} \right] \end{aligned} \tag{19}$$

Therefore,

$$E(\mu \mid views) = \left[\Sigma^{-1} + P'\Omega^{-1}P\right]^{-1}[\Sigma^{-1}\mu_0 + P'\Omega^{-1}Q] \tag{20}$$

A.3 Equivalence of the two Black-Litterman formulas

To show the equivalence between (3) and (6), we make use of the following matrix identity (see Rao [1973], page 33):

$$(A + BDB')^{-1} = A^{-1} - A^{-1}B(B'A^{-1}B + D^{-1})^{-1}B'A^{-1} \tag{21}$$

where $A \in \mathbb{R}^{m\times m}$, $D \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$, then,

$$\begin{aligned}
&(3) \Rightarrow \\
E(\mu \mid views) &= \left[\Sigma^{-1} + P'\Omega^{-1}P\right]^{-1}[\Sigma^{-1}\mu_0 + P'\Omega^{-1}Q] \\
&= [\Sigma - \Sigma P'(P\Sigma P' + \Omega)^{-1}P\Sigma][\Sigma^{-1}\mu_0 + P'\Omega^{-1}Q] \\
&= \mu_0 - \Sigma P'(P\Sigma P' + \Omega)^{-1}P\mu_0 + \Sigma P'\Omega^{-1}Q - \Sigma P'(P\Sigma P' + \Omega)^{-1}P\Sigma P'\Omega^{-1}Q \\
&= \mu_0 - \Sigma P'(P\Sigma P' + \Omega)^{-1}P\mu_0 + \Sigma P'\left[I - (P\Sigma P' + \Omega)^{-1}P\Sigma P'\right]\Omega^{-1}Q \\
&= \mu_0 - \Sigma P'(P\Sigma P' + \Omega)^{-1}P\mu_0 \qquad (22)\\
&+ \Sigma P'\left[(P\Sigma P' + \Omega)^{-1}(P\Sigma P' + \Omega) - (P\Sigma P' + \Omega)^{-1}P\Sigma P'\right]\Omega^{-1}Q \\
&= \mu_0 - \Sigma P'(P\Sigma P' + \Omega)^{-1}P\mu_0 + \Sigma P'(P\Sigma P' + \Omega)^{-1}\Omega\Omega^{-1}Q \\
&= \mu_0 + \Sigma P'(P\Sigma P' + \Omega)^{-1}(Q - P\mu_0) \\
&\Rightarrow (6).
\end{aligned}$$

A.4 Optimal weights on Black-Litterman portfolio

The optimal weights of the market portfolio (or the optimal risky portfolio that maximizes the Sharpe ratio) are given by (see Campbell, Lo and McKinlay [1997], pg 188, for example):

$$w_{mkt} = \frac{V^{-1}(E(R) - R_f\iota)}{\iota'V^{-1}(ER - R_f\iota)} \tag{23}$$

where ι denotes a $n \times 1$ vector of ones and R_f is the risk free rate and E(R) is the investor's expected return.

The BL model updates the expected return by incorporating personal views into equilibrium expected returns:

$$E(\mu \mid views) = \mu_0 + \Sigma_{12}\Sigma_{22}^{-1}[Q - P\mu_0] \tag{24}$$

Therefore the optimal weights in the BL model (4) is:

$$w_{BL} = \frac{V^{-1}(\mu_0 - R_f\iota + \Sigma_{12}\Sigma_{22}^{-1}[Q - P\mu_0])}{\iota'V^{-1}(\mu_0 - R_f\iota + \Sigma_{12}\Sigma_{22}^{-1}[Q - P\mu_0])} \tag{25}$$

The BL model portfolios can then be viewed as a combination of the market portfolio and a set of portfolios representing an investor's views by rewriting as:

$$w_{BL} = \alpha_{mkt} w_{mkt} + \alpha_L w_L - \alpha_S w_S \tag{26}$$

where

$$\alpha_{mkt} = \frac{\iota' V^{-1}(\mu_0 - R_f \iota)}{\iota' V^{-1}(\mu_0 - R_f \iota + \Sigma_{12}\Sigma_{22}^{-1}[Q - P\mu_0])} = \frac{\gamma}{\gamma + \iota' V^{-1}(\Sigma_{12}\Sigma_{22}^{-1}[Q - P\mu_0])} \tag{27}$$

$$\alpha_L = \frac{\iota' \left[V^{-1}\left\{(\Sigma_{12}\Sigma_{22}^{-1}(Q - P\mu_0)\right\}\right]^+}{\iota' V^{-1}(\mu_0 - R_f \iota + \Sigma_{12}\Sigma_{22}^{-1}[Q - P\mu_0])} = \frac{\iota' \left[V^{-1}\left\{(\Sigma_{12}\Sigma_{22}^{-1}(Q - P\mu_0)\right\}\right]^+}{\gamma + \iota' V^{-1}(\Sigma_{12}\Sigma_{22}^{-1}[Q - P\mu_0])} \tag{28}$$

$$\alpha_S = \frac{\iota' \left[V^{-1}\left\{(\Sigma_{12}\Sigma_{22}^{-1}(Q - P\mu_0)\right\}\right]^-}{\iota' V^{-1}(\mu_0 - R_f \iota + \Sigma_{12}\Sigma_{22}^{-1}[Q - P\mu_0])} = \frac{\iota' \left[V^{-1}\left\{(\Sigma_{12}\Sigma_{22}^{-1}(Q - P\mu_0)\right\}\right]^-}{\gamma + \iota' V^{-1}(\Sigma_{12}\Sigma_{22}^{-1}[Q - P\mu_0])} \tag{29}$$

and

$$w_{mkt} = \frac{V^{-1}(\mu_0 - R_f \iota)}{\iota' V^{-1}(\mu_0 - R_f \iota)} = \frac{1}{\gamma} V^{-1}(\mu_0 - R_f \iota) \tag{30}$$

$$w_L = \frac{\left[V^{-1}\left\{(\Sigma_{12}\Sigma_{22}^{-1}(Q - P\mu_0)\right\}\right]^+}{\iota' \left[V^{-1}\left\{(\Sigma_{12}\Sigma_{22}^{-1}(Q - P\mu_0)\right\}\right]^+} \tag{31}$$

$$w_S = \frac{\left[V^{-1}\left\{(\Sigma_{12}\Sigma_{22}^{-1}(Q - P\mu_0)\right\}\right]^-}{\iota' \left[V^{-1}\left\{(\Sigma_{12}\Sigma_{22}^{-1}(Q - P\mu)\right\}\right]^-} \tag{32}$$

and $[\cdot]^+$ denotes the positive part (element by element) of a vector and $[\cdot]^-$ denotes the negative part so that for any vector $X = [X]^+ - [X]^-$. Here γ denotes the representative investors risk aversion as defined in Assumption 2.

A.5 Specifying the Covariance between Views and Priors

To be specific, assume that the investor specifies the covariance between m (where $m \leq k$) benchmark portfolios (given by the $m \times n$ matrix of portfolio weights B and the k views as in Assumption 3.i:

$$Cov(B\mu, \varepsilon) = B\Gamma = \Lambda \in \mathbb{R}^{m \times k} \tag{33}$$

The set of portfolios whose priors are uncorrelated with the k views, P, are spanned by the $n \times n - k$ matrix $P^{\perp\prime}$ which can be computed as a basis for the orthogonal complement of $\Sigma P'$.[9] The restriction imposed by Assumption 3.ii is that

$$Cov(P^{\perp}\mu, \varepsilon) = P^{\perp}\Gamma = 0 \tag{34}$$

Finally, Assumption 3.iii states that, if S' is a $n \times (k - m)$ matrix that is a basis for the orthogonal complement of $\Sigma[B', P^{\perp\prime}]$, then $Cov(S\mu, \varepsilon) = S\Gamma = 0$.[10]

Together these assumptions completely characterize Γ in terms of the lower dimensional (and easier to specify) parameter Λ by solving the linear system of equations:

$$\begin{bmatrix} B \\ P^{\perp} \\ S \end{bmatrix} \Gamma = \begin{bmatrix} \Lambda \\ 0_{(n-m)\times k} \end{bmatrix} \tag{35}$$

[9] In `MATLAB` this is `null(`$P\Sigma$`)`.

[10] If $m = k$ then S is the empty matrix and Assumption 3.iii can be ignored.

Finally, note that when the number of assets n is large, one can avoid directly computing Σ^{-1} or V^{-1} in (27)-(32) by exploiting that $V^{-1}\Sigma_{12} = \tau\left(P' + \Sigma^{-1}\Gamma\right)$ and directly solve for $X = \Sigma^{-1}\Gamma$ by solving the linear system

$$\begin{bmatrix} B\Sigma \\ P^{\perp}\Sigma \\ S\Sigma \end{bmatrix} X = \begin{bmatrix} \Lambda \\ 0_{(n-m)\times k} \end{bmatrix} \tag{36}$$

since solving this equation tends to be more numerically stable than brute force matrix inversion.