Due: April 2

Problems for Chapter 1

Problem 1.1: $0 \leq x \leq 2\pi$

Run scheme 1 for $u_t(x,t) = u_x(x,t)$ assuming periodic boundary conditions:

$$V_j^{n+1} = V_j^n + \frac{\lambda}{2}(V_{j+1}^n - V_{j-1}^n) \tag{1.3}$$

$$V_{-1}^n = V_{N-1}^n$$
$$V_N^n = V_0^n$$

for all integer n, using:

$$\lambda = \frac{\Delta t}{\Delta x} = 0.9, \qquad \Delta x = \frac{2\pi}{N}.$$

First for $N = 20$ and then for $N = 40$, run the scheme until $n\Delta t \geq T > (n-1)\Delta t$. Plot the numerical solution V_j^n and the true solution $u(x_j, n\Delta t)$ for $j = 0, \ldots, N-1$ and *terminal times* $T = 5, 10,$ and 20.

Perform the above calculations for initial conditions:

$$f(x) = \sin(kx), \qquad \text{for } k = 1, 5, \text{ and } 10.$$

a). For each initial condition, construct a table with the values of the L^2-error:

$$\sqrt{\frac{1}{N}\sum_{j=0}^{N-1} |u(x_j, n\Delta t) - V_j^n|^2},$$

printing two columns, one for each of the values of N and one row for each of the terminal times T.

b). Construct a table as in (a), showing the values of the discrete energy:

$$E(n) = \frac{1}{N}\sum_{j=0}^{N-1}(V_j^n)^2$$

for the different experiments performed. On the basis of your results, discuss the growth of the numerical solution as a function of the terminal time T, the total number of time iterations n, and the initial function $f(x)$.

$a=1$, PDE: $u_t = u_x$, $0 \le x \le 2\pi$

em 1.2:

Run the stable scheme 2, Friedrichs' scheme assuming periodic boundary conditions:

$$V_j^{n+1} = \frac{V_{j+1}^n + V_{j-1}^n}{2} + a\frac{\Delta t}{2\Delta x}(V_{j+1}^n - V_{j-1}^n) \tag{1.6.1}$$

$$V_j^0 = f(x_j); \quad j = 0, \dots, N-1 \tag{1.6.2}$$

$$V_{-1}^n = V_{N-1}^n$$
$$V_N^n = V_0^n$$

for all integer n, using:

$$\lambda = \frac{\Delta t}{\Delta x} = 0.9, \qquad \Delta x = \frac{2\pi}{N}.$$

Run the scheme until $n\Delta t \geq T > (n-1)\Delta t$ and make a plot of the numerical solution V_j^n and the true solution $u(x_j, n\Delta t)$ for $j = 0, \dots, N-1$, for $N = 40$, 80 and 160, evaluated at the terminal times $T = 5$, 10, 20, 40 and 50, and do the same as in Problem 1.1 a) with the initial conditions:

$$f(x) = \sin(kx), \qquad \text{for } k = 1, 2, \text{ and } 5.$$

Problem 1.3:

Show that Friedrichs' scheme satisfies the relation:

$$\max_{0 \leq j \leq N-1} |V_j^{n+1}| \leq \max_{0 \leq j \leq N-1} |V_j^n|,$$

for all integer n.

Problem 1.4:

a). Do the same as in problem 1.2 for the upwind scheme:

$$V_j^{n+1} = V_j^n + a\frac{\Delta t}{\Delta x}(V_{j+1}^n - V_j^n) \tag{1.10.1}$$

$$V_j^0 = f(x_j); \quad j = 0, \dots, N-1. \tag{1.10.2}$$

for $a = 1$.

b). Run now the same scheme (1.10) when $a = -1$, that is, for $u_t = -u_x$, with:

$$\lambda = \frac{\Delta t}{\Delta x} = 0.9, \qquad \Delta x = \frac{2\pi}{N},$$

for $N = 10$, 20, 40; $T = 1$, 2, and 3, and initial conditions:

$$f(x) = \sin(kx), \qquad \text{for } k = 1, 2, \text{ and } 5.$$

Explain why this scheme is wrong for this problem.