v_*-TORSION SPACES AND THICK CLASSES

W. CHACHÓLSKI, W. G. DWYER, AND M. INTERMONT

1. Introduction

Suppose that p is some chosen prime number. A topological space is said to be *v_*-torsion* if its homotopy groups are concentrated at p and its v_n-periodic homotopy groups vanish for $n \geq 1$ (1.1). A similar definition can be made for spectra, and in this stable case it is easy to see that a spectrum is v_*-torsion if and only if it can be built (in the sense of [9]) from *any* finite spectrum with nontrivial mod p homology. The unstable analog of this statement is not true, but in this paper we provide a close substitute. For any space W, let $\mathcal{B}_W$ denote the class of spaces which can be built from W, and $\mathcal{T}(\mathcal{B}_W)$ the smallest class of spaces which contains $\mathcal{B}_W$ and is thick with respect to fibrations (1.6). We prove that a space X is v_*-torsion if and only if it belongs to $\mathcal{T}(\mathcal{B}_W)$ for *any* finite complex W with nontrivial reduced mod p homology. We go on to show that, although the category of v_*-torsion spaces is not closed under arbitrary homotopy colimits, it is closed in many cases with respect to realizations of simplicial objects. Finally, we consider the Stover-Blanc simplicial resolution $\mathrm{R}_W(X)$ of a space X, which comes up in a natural attempt to construct the W-cellular approximation $\mathrm{Cell}_W(X)$. We show that for appropriate W the fibres of the map from the geometric realization of $\mathrm{R}_W(X)$ to $\mathrm{Cell}_W(X)$ are v_*-torsion. Since v_*-torsion spaces are in some respects negligible objects, this can be interpreted as saying that the realization of the Stover-Blanc construction is closer to $\mathrm{Cell}_W(X)$ than one might suspect.

1.1. **v_*-torsion spaces.** Let p be a prime number, which is fixed for the rest of the paper. For each integer $n \geq 1$, let V_n be a pointed finite p-torsion complex *of type* n in the sense of chromatic homotopy theory [13], so that the reduced Morava K-theory module $\tilde{K}(i)_*V_n$ is trivial for $i < n$ and nontrivial for $i = n$. By a theorem of Hopkins-Smith [11], after replacing V_n by a large enough suspension we can find an integer $d > 0$ and a map $\omega : \Sigma^d V_n \to V_n$ which is a *v_n-self-map*, in the sense that it induces an isomorphism on $\tilde{K}(n)_*$ and induces the

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zero map on $\tilde{K}(i)_*$ for $i \neq n$. For a pointed space X, the *v_n-periodic homotopy groups* $v_n^{-1}\pi_*(X; V_n)$ are obtained by inverting the action of ω on $\pi_*(X; V_n)$ [2, 11.3].

1.2**. Definition.** Let $n \geq 1$ be an integer. An unpointed space X is *v_n-torsion* if for each choice of basepoint in X, the v_n-periodic homotopy groups $v_n^{-1}\pi_*(X; V_n)$ vanish.

1.3. *Remark.* By [11], the answer to the question of whether $v_n^{-1}\pi_*(X; V_n)$ is trivial does not depend upon the choice of the complex V_n or on the choice of the map ω.

Say that a group G is *concentrated at p* if it has a finite subnormal series

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_m = G$$

such that each quotient G_i/G_{i-1} is a p-primary torsion abelian group. We will say that an unpointed space X is *concentrated at p* if for each basepoint in X and each $k \geq 1$, the homotopy group $\pi_k(X)$ is concentrated at p.

1.4**. Definition.** An unpointed space X is said to be *v_*-torsion* if it is concentrated at p, and it is v_n-torsion for each $n \geq 1$.

1.5. *Remark.* The condition of being concentrated at p can be interpreted as being "v_0-torsion".

1.6**. Thick classes.** Let $\mathcal{C}$ be some class of unpointed spaces; all the classes we consider will be closed under retracts, and also *closed under equivalences*, in the sense that whenever $X \to Y$ is a weak equivalence, then $X \in \mathcal{C}$ if and only if $Y \in \mathcal{C}$. If $q : E \to B$ is a map of spaces, then F is said to be *a fibre of* q if for some choice of basepoint $b_0 \in B$, F is equivalent to the homotopy fibre of q over b_0. Consider the following three conditions on $q : E \to B$, where F runs through the fibres of q:

(1) $B \in \mathcal{C}$ and $E \in \mathcal{C}$ implies all $F \in \mathcal{C}$;
(2) $B \in \mathcal{C}$ and all $F \in \mathcal{C}$ implies $E \in \mathcal{C}$;
(3) $E \in \mathcal{C}$, all F connected, and all $F \in \mathcal{C}$ implies $B \in \mathcal{C}$.

1.7**. Definition.** Let $\mathcal{C}$ be a class of unpointed spaces, closed under retracts and equivalences. Then $\mathcal{C}$ is said to be *thick* (with respect to fibrations) if for any map $q : E \to B$ all three of the above conditions hold.

1.8. *Remark.* If the first condition holds, then $\mathcal{C}$ is said to be *closed under fibres*; if the second condition holds, then $\mathcal{C}$ is said to be *closed under extension by fibrations.*

1.9. *Examples.* The class of spaces concentrated at p is thick, as is the class of v_n-torsion spaces for fixed n, or the class of v_*-torsion spaces.

If $\mathcal{C}$ is a class of spaces, we write $\mathcal{T}(\mathcal{C})$ for the smallest thick class of unpointed spaces containing $\mathcal{C}$. If W is a pointed CW-complex, let $\mathcal{B}_W$ be the class of spaces *built from* W in the sense of [9]; $\mathcal{B}_W$ is the smallest class of spaces containing W and closed under equivalences and pointed homotopy colimits. Our first main theorem is the following one.

1.10. **Theorem.** *The class of v_*-torsion spaces is the intersection of the classes $\mathcal{T}(\mathcal{B}_W)$, where W ranges over all finite complexes such that $\tilde{H}_*(W;\mathbb{Z}/p)$ is nontrivial.*

1.11. *Remark.* The class $\mathcal{B}_W$ is naturally a class of pointed spaces, but in forming $\mathcal{T}(\mathcal{B}_W)$ we forget the basepoints and work in the unpointed category. The class $\mathcal{B}_W$ does not depend on the choice of basepoint in W.

1.12. *Remark.* The proof of 1.10 shows that $\mathcal{T}(\mathcal{B}_W)$ can also be described as the smallest class of spaces containing $\mathcal{B}_W$ and closed under fibres and extension by fibrations.

1.13. **Closure and Resolutions.** The class of v_*-torsion spaces is not closed under homotopy colimits; for instance, it clearly contains all discrete spaces, and any space is weakly equivalent to a homotopy colimit of discrete spaces. However, it is closed under one special kind of homotopy colimit.

1.14. **Theorem.** *Suppose that X is a simplicial space with the property that each space X_i, $i \geq 0$ is connected, is v_*-torsion, and has an abelian fundamental group. Then the geometric realization $|X|$ of X is also v_*-torsion.*

Note that by the geometric realization $|X|$ of a simplicial space X we mean its homotopy colimit; this is equivalent to the ordinary geometric realization if X satisfies suitable cofibrancy conditions.

Now for the resolution construction. Suppose that W is a pointed CW-complex. For any pointed space X, let $\mathrm{Cell}_W(X)$ denote the *W-cellular approximation to X* [9, 2.A]; this is the closest approximation to X by a space which is built from W. Stover [14] and Blanc [1] have constructed a natural simplicial resolution $\mathrm{R}_W(X)$ of X together with a map $|\mathrm{R}_W(X)| \to \mathrm{Cell}_W(X)$; this map is an equivalence if W is a sphere [14] and the question is whether or not it is an equivalence in general. Similar questions are considered in [6]. What we prove is the following.

1.15. **Theorem.** *Suppose that X is a pointed connected space, and that W is a pointed finite CW-complex which is 1-connected, concentrated at p, and homotopy equivalent to a suspension. Then the homotopy fibres of the map $|\,\mathrm{R}_W(X)| \to \mathrm{Cell}_W(X)$ are v_*-torsion.*

This paper is heavily dependent on ideas of Bousfield [2] and Dror-Farjoun [9].

1.16. **Organization of the paper.** Section 2 recalls some properties of nullification functors, which are a key tool in what we do; §3 uses these functors to give a proof of 1.10. Section 4 gives a recognition principle for v_*-torsion spaces. Section 5 has a brief discussion of the simplest kind of v_*-torsion spaces, the *polygems*, and §6 points out that certain simplicial constructions give polygems. The final section uses the results of the preceding three to prove 1.14 and 1.15.

1.17. **Notation and terminology.** If W is a space, $c(W)$ denotes the smallest integer k such that $\tilde{H}_*(W;\mathbb{Z}/p)$ is nonzero. If there is no such k, $c(k) = \infty$. Similarly, $\kappa(W)$ denotes the smallest integer n such that $\tilde{K}(n)_*(W)$ is nonzero. We say that a space is *connected* if it is pathwise connected, and that a map of spaces is an *equivalence* if it is a weak homotopy equivalence. The notation $\mathrm{Map}(X, Y)$ denotes the space of maps from X to Y, taken in the derived sense, i.e., before forming the mapping space, X is to be replaced by an equivalent cofibrant object and Y by an equivalent fibrant one. In the category of topological spaces, this involves replacing X by a CW-approximation; in the category of simplicial sets, replacing Y by a Kan complex. If X and Y are pointed, similar considerations apply to the space $\mathrm{Map}_*(X, Y)$ of pointed maps, or to its null component $\mathrm{Map}^0_*(X, Y)$.

2. Nullification

One of the main tools we use in this paper is the *nullification functor* P_W associated to a CW-complex W [2] [9]. Recall that a space X is said to be *W-null* if the map $W \to *$ induces an equivalence $X \to \mathrm{Map}(W, X)$. The nullification functor P_W assigns to any X a map $\epsilon_X : X \to P_W(X)$ from X to a W-null space $P_W(X)$; this is a *universal* map, in the sense that if Y is another W-null space, $\mathrm{Map}(\epsilon_X, Y)$ is an equivalence. The functor P_W is idempotent and $P_W(\epsilon_X)$ is an equivalence.

The space $P_W(X)$ is called the *nullification of X (with respect to W).* A map f is a said to be a *P_W-equivalence* if $P_W(f)$ is an equivalence. It is easy to see that X is W-null if and only if $P_W(X) \sim X$. If, on the other hand, $P_W(X) \sim *$, then X is said to be *killed by* W; we denote by $\mathcal{K}_W$ the class of all such spaces.

Our interest in P_W and its "kernel" $\mathcal{K}_W$ is explained by the following theorem

2.1. **Theorem.** [7] *For any pointed CW-complex* W*, the class* $\mathcal{K}_W$ *is the smallest class of spaces containing* $\mathcal{B}_W$ *and closed under equivalences and extension by fibrations. In particular* $\mathcal{T}(\mathcal{K}_W) = \mathcal{T}(\mathcal{B}_W)$

In this section we recall some properties of the functor P_W; we will use these properties in §3 to characterize $\mathcal{K}_W$ for suitable W and to prove 1.10.

2.2. *Remark.* If S^{k+1} is the $k+1$-sphere, then $P_{S^{k+1}}(X)$ is the *k'th Postnikov stage of* X. At what we hope is only a minor risk of confusion we will denote the nullification functor $P_{S^{k+1}}$ by the usual Postnikov notation P_k. For each choice of basepoint in X, the natural map $\pi_i X \to \pi_i P_k X$ is an isomorphism for $i \le k$, while $\pi_i P_k X$ is trivial for $i > k$.

The following proposition is easy to check directly from the above definitions. The space ΣW is the reduced suspension of W with respect to some chosen basepoint.

2.3. **Proposition.** *For any* W*,* $\mathcal{K}_{\Sigma W} \subset \mathcal{K}_W$.

2.4. *Remark.* This proposition implies that $\mathcal{T}(\mathcal{K}_{\Sigma W}) \subset \mathcal{T}(\mathcal{K}_W)$, and so in order to study the intersection which appears in 1.10, it is enough to consider $\mathcal{K}_W$ for W a suspension. The proposition also implies that $P_W P_{\Sigma W}(X) \sim P_W(X)$, so that at least up to canonical homotopy there is a natural map $P_{\Sigma W}(X) \to P_W(X)$. Similarly, $P_{\Sigma W} P_W(X) \sim P_W(X)$.

2.5. **Proposition.** [2, 4.4] *If* W *is connected, the functor* P_W *respects coproducts of spaces. If* W *is* k*-connected, then for any choice of basepoint in* X *the map* $X \to P_W(X)$ *induces isomorphisms* $\pi_i X \cong \pi_i P_W(X)$ *for* $i \le k$.

2.6. **Proposition.** [2, 4.3, 4.8] *If* $f : E \to B$ *is a map all of whose fibres are killed by* W*, then* $P_W(f)$ *is an equivalence. All of the fibres of the map* $X \to P_W(X)$ *are killed by* W.

2.7. **Proposition.** *(Partial nullification)* [2, 4.1] *Suppose that* B *is connected and that* $F \to E \to B$ *is a fibre sequence. Then there is a natural map*

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ P_W(F) & \longrightarrow & E' & \longrightarrow & P_{\Sigma W}(B) \end{array}$$

in the homotopy category of homotopy fibre sequences. All three vertical arrows are P_W-equivalences, and E' is ΣW-null.

2.8**. Proposition.** [2, 7.4] *If G is an abelian group which is concentrated at p and $n \geq c(W)$, then $P_W(K(G,n)) \sim *$.*

2.9**. The standard assumptions.** We will say that W *satisfies the standard assumptions* if W is a finite CW-complex which is is equivalent to a suspension, is not contractible, and is concentrated at p. Under these assumptions, W is simply connected and $c(W)$ is obtained by adding one to the connectivity of W.

The next proposition is clear from the way in which $P_W(X)$ is constructed by attaching copies of W and its suspensions to X.

2.10**. Proposition.** [2, 2.8] *If W satisfies the standard assumptions, then X is concentrated at p if and only if $P_W(X)$ is concentrated at p.*

2.11**. Definition.** A space X is said to be of *finite Postnikov type* if there is some integer k such that $X \sim P_k X$, i.e., if regardless of the choice of basepoint, $\pi_i X = 0$ for $i > k$.

2.12**. Proposition.** *(cf.* [2, 10.5]*) If W satisfies the standard assumptions and X is a space which is concentrated at p and of finite Postnikov type, then $P_W(X) \sim P_\nu(X)$ for $\nu = c(W) - 1$.*

Proof. We can assume that X is connected (2.5). It is clear that $P_\nu(X)$ is W-null, so it is only necessary to show that $X \to P_\nu(X)$ is a P_W-equivalence. Since X is of finite Postnikov type, it is enough to show that for $j > \nu$, the map $P_j(X) \to P_{j-1}(X)$ is a P_W-equivalence. But the fibre of this map is $K(\pi_j X, j)$, which is W-null by 2.8. The desired result follows from 2.6. □

2.13**. Proposition.** [3, 1.8] *Suppose that W satisfies the standard assumptions, that X is a space, and that F is a homotopy fibre of $P_{\Sigma W} X \to P_W(X)$. Then $F \sim K(G, c(W))$ for some abelian group G which is concentrated at p.*

2.14**. Definition.** A map $X \to Y$ of pointed, connected spaces is *a v_i^{-1}-equivalence* if it induces an isomorphism on $v_i^{-1}\pi_*(-; V_i)$.

2.15**. Proposition.** [2, 11.5, 11.8] *Suppose that W satisfies the standard assumptions. Then for any X the map $X \to P_W(X)$ is a v_i^{-1}-equivalence for $1 \leq i < \kappa(W)$, whereas $P_W(X)$ is v_i-torsion for $i \geq \kappa(W)$.*

2.16. *Remark.* Note in connection with 2.15 that Bousfield works with a functor P_{v_n} which is nullification with respect to the suspension of a particular complex of chromatic type n with minimal connectivity [2, 10.1]. However the arguments apply as stated to P_W.

3. Thick Classes

We are now ready to give a simple description of $\mathcal{T}(\mathcal{K}_W)$, at least when W satisfies the standard assumptions (2.9), and to prove 1.10.

3.1. **Definition.** A space X is said to be *almost killed by* W if $P_W(X)$ is concentrated at p and of finite Postnikov type. The class of all such spaces is denoted $\mathcal{K}_W^+$.

3.2. **Proposition.** *If W satisfies the standard assumptions, then $\mathcal{T}(\mathcal{K}_W) = \mathcal{K}_W^+$.*

3.3. **Proposition.** *If W satisfies the standard assumptions, then $\mathcal{K}_W^+$ is the class of spaces which are concentrated at p, and v_i-torsion for $1 \leq i < \kappa(W)$.*

Proof of 1.10. By 2.1 and 2.3, it is enough to show that a space X is v_*-torsion if and only if X belongs to $\mathcal{T}(\mathcal{K}_W)$ for any finite complex W which is nontrivial mod p and is, say, a double suspension. Pick such a W, and let C be the homotopy cofibre of the degree p map $W \to W$. Then C satisfies the standard assumptions and $C \in \mathcal{B}_W$, so that $\mathcal{B}_C \subset \mathcal{B}_W$, $\mathcal{K}_C \subset \mathcal{K}_W$ and $\mathcal{T}(\mathcal{K}_C) \subset \mathcal{T}(\mathcal{K}_W)$. It is thus enough to show that X is v_*-torsion if and only if $X \in \mathcal{T}(\mathcal{K}_W)$ for any W which satisfies the standard assumptions. But for such a W, $\mathcal{T}(\mathcal{K}_W) = \mathcal{K}_W^+$ (3.2), and so the result follows immediately from 3.3 and the fact that there are finite complexes W with $\kappa(W)$ arbitrarily large [12] [13, 3.4]. □

The rest of this section contains the proofs of 3.2 and 3.3.

3.4. **Lemma.** *If W satisfies the standard assumptions and $X \in \mathcal{K}_W^+$, then $P_W(X) \sim P_\nu(X)$, for $\nu = c(W) - 1$.*

Proof. By 2.12, $P_W(X) \sim P_W P_W(X)$ is equivalent to $P_\nu P_W(X)$. But by 2.5, $P_\nu P_W(X)$ is equivalent to $P_\nu(X)$. □

3.5. *Remark.* It follows from 2.5 and 3.4 that if W satisfies the standard assumptions, a space belongs to $\mathcal{K}_W^+$ if and only if all of its components do.

3.6. **Lemma.** *If W satisfies the standard assumptions, then $\mathcal{T}(\mathcal{K}_W)$ contains all spaces which are concentrated at p and of finite Postnikov type.*

Proof. Since $\mathcal{K}_W$ contains $*$ and $\mathcal{T}(\mathcal{K}_W)$ is closed under fibres, $\mathcal{T}(\mathcal{K}_W)$ is closed under taking loop spaces. Looping down Eilenberg-Mac Lane spaces (2.8) now shows that $\mathcal{T}(\mathcal{K}_W)$ contains discrete spaces of arbitrarily large cardinality, and taking fibres of maps between these discrete spaces shows that $\mathcal{T}(\mathcal{K}_W)$ contains all discrete spaces. For any X the homotopy fibres of $X \to \pi_0 X$ are the connected components of X; since $\mathcal{T}(\mathcal{K}_W)$ is closed under extension by fibres, it is now clear then that X belongs to $\mathcal{T}(\mathcal{K}_W)$ if and only if all of its components do. It is thus enough to show that any connected space X of the indicated type belongs to $\mathcal{T}(\mathcal{K}_W)$. Since every Eilenberg-Mac Lane space $K(G,n)$ in which G is concentrated at p and $n \geq 1$ belongs to $\mathcal{T}(\mathcal{K}_W)$ (2.8), the desired result is easy to prove by an induction on the Postnikov tower of X (slightly refined in the case of $\pi_1 X$). □

Proof of 3.2. Note by 2.13 that $\mathcal{K}_W^+ = \mathcal{K}_{\Sigma W}^+$. To obtain the inclusion $\mathcal{T}(\mathcal{K}_W) \subset \mathcal{K}_W^+$, it is enough to show that $\mathcal{K}_W^+$ is thick with respect to fibrations. By 3.5, it is sufficient to take a fibration sequence $F \to E \to B$ with B connected, and show that it satisfies the three conditions of 1.6. Consider the diagram

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ P_W(F) & \longrightarrow & E' & \longrightarrow & P_{\Sigma W}(B) \end{array}$$

from 2.7, in which $E \to E'$ is a P_W-equivalence and E' is ΣW-null. Say that a space Y is *good* if it is concentrated at p and has finite Postnikov type. Then $P_W(F)$ is good if and only if $F \in \mathcal{K}_W^+$; E' is good if and only if $E' \in \mathcal{K}_{\Sigma W}^+ = \mathcal{K}_W^+$, or, since $E \to E'$ is a P_W-equivalence, if and only if $E \in \mathcal{K}_W^+$; and $P_{\Sigma W}(B)$ is good if and only if $B \in \mathcal{K}_{\Sigma W}^+ = \mathcal{K}_W^+$. The desired result follows immediately from the long exact homotopy sequence of a fibration. Note that $P_W(F)$ is connected if and only if F is connected.

It remains to show that $\mathcal{K}_W^+ \subset \mathcal{T}(\mathcal{K}_W)$. Pick $X \in \mathcal{K}_W^+$, and consider the map $X \to P_W(X)$. By 2.6, the homotopy fibres of this map are in $\mathcal{K}_W$; by 3.6, the target of the map is in $\mathcal{T}(\mathcal{K}_W)$. Since $\mathcal{T}(\mathcal{K}_W)$ is closed under extension by fibrations, $X \in \mathcal{T}(\mathcal{K}_W)$.

Proof of 3.3. By 3.5, we can assume that X is connected and show that X belongs to $\mathcal{K}_W^+$ if and only if it is concentrated at p and v_i-torsion for $1 \leq i \leq n$. This follows from results of Bousfield. Let $\nu = c(W) - 1$, and F the fibre of the Postnikov map $X \to P_\nu(X)$. By 2.10, 2.6, and 3.4, $X \in \mathcal{K}_W^+$ if and only if X is concentrated at p and $P_W(F) \sim *$.

Since the map $F \to X$ induces an isomorphism on $v_i^{-1}\pi_*(-; V_i)$ for all i, the result follows from 2.15. □

4. Recognizing v_*-torsion spaces

The aim of this section is to prove the following proposition, which gives a way of recognizing v_*-torsion spaces.

4.1. **Proposition.** *Suppose that W is a pointed space that satisfies the standard assumptions, and that X is a pointed connected space. Then the following three conditions are equivalent:*

(1) *X is v_*-torsion;*
(2) *$\mathrm{Map}^0_*(W, X)$ and $P_W(X)$ are v_*-torsion;*
(3) *$\mathrm{Cell}_W(X)$ and $P_W(X)$ are v_*-torsion.*

This is a consequence of two more detailed statements, which depend on work of Bousfield and complement 2.15.

4.2. **Proposition.** *Suppose that W satisfies the standard assumptions, and that $f : X \to Y$ is a map of pointed connected spaces. If $i \geq \kappa(W)$, then $\mathrm{Map}^0_*(W, f)$ is a v_i^{-1}-equivalence (2.14) if and only if f is a v_i^{-1}-equivalence. Moreover, $\mathrm{Map}^0_*(W, X)$ is v_i-torsion for $i < \kappa(W)$.*

4.3. **Proposition.** *Suppose that W satisfies the standard assumptions and that X is a connected pointed space. Then $\mathrm{Cell}_W(X) \to X$ is a v_i^{-1}-equivalence for $i \geq \kappa(W)$, whereas $\mathrm{Cell}_W(X)$ is v_i-torsion for $i < \kappa(W)$.*

4.4. *Remark.* Taken together, 2.15 and 4.3 suggest that $\mathrm{Cell}_W(X) \to X \to P_W(X)$ is a fibre sequence. This is not quite the case [7], but the sequence certainly acts like a fibre sequence as far as v_*-periodic homotopy is concerned.

Proof of 4.2. Suppose that $i \geq \kappa(W)$. Then $\kappa(V_i \wedge W) = i$; this comes from applying the Kunneth formula for Morava K-theory, and using the fact that $\widetilde{K}(i)_* W \neq 0$ if $i \geq \kappa(W)$ [13, 1.5.2]. It is also clear that any v_i-self-map of V_i induces a v_i-self-map of $V_i \wedge W$. By adjointness, then, there is an isomorphism

$$v_i^{-1}\pi_*(X; V_i \wedge W) \cong v_i^{-1}\pi_*(\mathrm{Map}^0_*(W, X); V_i)\,,$$

as well as a similar isomorphism for Y. Since the question of whether a map is a v_i^{-1}-equivalence does not depend upon the complex of type i used to compute the v_i-periodic homotopy groups, the result follows.

To see that the v_i-periodic homotopy of $\mathrm{Map}^0_*(W, X)$ vanishes for $i < \kappa(W)$, consider the tower

$$V_i \wedge W \to \Sigma^d V_i \wedge W \to \Sigma^{2d} V_i \wedge W \to \cdots$$

where $\Sigma^d V_i \to V_i$ is the v_i-self-map used to construct the v_i-periodic homotopy groups. By the Kunneth theorem, the maps in this tower induce the zero homomorphism on $\tilde{K}(n)_*$ for all $n \geq 1$. It follows from the nilpotence theorem [8] that the tower is pro-trivial (i.e. composites of sufficiently long chains of maps are null homotopic). Mapping the tower into X, however, and taking a colimit, gives the v_i-periodic homotopy of $\mathrm{Map}^0_*(W, X)$. See the proof of [2, 11.4]) for another instance of this argument. □

Proof of 4.3. The first statement is a consequence of 4.2, since the map $\mathrm{Cell}_W(X) \to X$ induces an equivalence on $\mathrm{Map}_*(W, -)$. The fact that $\mathrm{Cell}_W(X)$ is v_i-torsion for $i < \kappa(W)$ follows from 3.3 together with the chain $\mathrm{Cell}_W(X) \in \mathcal{B}_W \subset \mathcal{K}_W \subset \mathcal{K}^+_W$. □

Proof of 4.1. Note that X is concentrated at p if and only if $P_W(X)$ is concentrated at p (2.10). The space $\mathrm{Cell}_W(X)$ is always concentrated at p, since $P_W(\mathrm{Cell}_W(X)) \sim *$; $\mathrm{Map}^0_*(W, X)$ is concentrated at p by inspection. The first two conditions are now equivalent by 2.15 and 4.3; the first and third by combining 2.15 with an application of 4.2 to the map $X \to *$.

5. Polygems

The purpose of this short section is to recall some properties of *polygems*, which are in some sense the simplest v_*-torsion spaces.

5.1. **Definition.** [9, 4.B] A *generalized Eilenberg-Mac Lane space*, or *GEM*, is a space which is equivalent to a (possibly infinite) product of Eilenberg-Mac Lane spaces whose homotopy groups are abelian.

5.2. *Remark.* A space X is a GEM if and only if X is equivalent to a topological abelian group.

5.3. **Definition.** The class of *polygems* is the smallest class of spaces which contains GEMs and is closed under fibres and under extension by fibrations.

5.4. *Example.* A connected space X of finite Postnikov type is a polygem if and only if $\pi_1(X)$ is solvable.

5.5. *Remark.* All of the components of a polygem are polygems. If X is a polygem and W is a pointed finite CW-complex, then for any choice of basepoint in X, the spaces $\mathrm{Map}(W, X)$, $\mathrm{Map}_*(W, X)$ and $\mathrm{Map}^0_*(W, X)$ are also polygems.

It is easy to see that any GEM is v_i-torsion for $i \geq 1$. Since the class of spaces which are v_i-torsion for $i \geq 1$ is closed under fibres and extension by fibrations (in fact it is thick with respect to fibrations), we immediately obtain the following proposition.

5.6. **Proposition.** *Any polygem X is v_i-torsion for $i \geq 1$. If in addition X is concentrated at p, then X is v_*-torsion.*

6. Simplicial spaces

In this section we use some results from Bousfield and Friedlander [4] in order to prove the following two theorems, which identify certain spaces as polygems (§5).

6.1. **Theorem.** *Let X be a simplicial pointed topological space with the property that each space X_i is connected and has an abelian fundamental group. Suppose there is an integer m such that for any $i \geq 0$ and $j > m$, $\pi_j X_i$ is trivial. Then $|X|$ is a polygem.*

6.2. **Theorem.** *Let X be a simplicial pointed topological space with the property that each space X_i is connected and has an abelian fundamental group. Let A be a pointed finite-dimensional CW-complex which is equivalent to a suspension. Then the homotopy fibre of the natural map*

$$|\operatorname{Map}^0_*(A, X)| \to \operatorname{Map}^0_*(A, |X|)$$

is a polygem.

Since we will be using simplicial machinery from [4], we will prove these theorems in the case in which X is a simplicial pointed simplicial set and A is a finite-dimensional pointed simplicial set which is weakly equivalent to a suspension. The results as stated can be obtained by using the singular complex functor and the geometric realization functor to compare spaces to simplicial sets.

6.3. *Warning. For the remainder of this section, the word "space" will mean "simplicial set".* In particular, a *simplicial space* is a *bisimplicial set* [4, §B] [10, IV].

Theorem 6.1 is a special case of a more general result. If X is a simplicial pointed space, then $\pi_j(X)$ $(j \geq 1)$ is a simplicial group, and we let $\pi_i\pi_j(X)$ denote the i'th homotopy group of this object.

6.4. *Remark.* There is a first quadrant spectral sequence with $E^2_{i,j} = \pi_i\pi_j(X)$, which converges to $\pi_*|X|$ if each space X_i is connected [4, B.5].

6.5. **Theorem.** *Let X be a simplicial pointed space with the property that each space X_i is connected and has an abelian fundamental group.*

Suppose there is an integer m such that for any $i \geq 0$ and $j > m$, $\pi_i \pi_j X$ is trivial. Then $|X|$ is a polygem.

Proof. This is a corollary of the proof of [4, B.5]. According to the inductive fibration argument used in that proof, it is enough to show that if there is a single integer t such that for all $i \geq 0$, $\pi_j(X_i) = 0$ for $j \neq t$, then $|X|$ is a GEM. But, as indicated in the proof, in this case there is a natural bisimplicial map $X \to Y$, where Y is the simplicial space with Y_i given by the minimal Eilenberg-Mac Lane complex $K(\pi_t X_i, t)$; see also the discussion following [10, IV.4.12]. Since Y is a simplicial object in the category of simplicial abelian groups, its homotopy colimit, which is equivalent to its diagonal [4, B.1] [5, XII.3.4], is a simplicial abelian group and hence a GEM [10, III.2.20]. □

The proof of 6.2 depends on the following result of Bousfield and Friedlander

6.6. **Theorem.** [4, B.4] [10, IV.4.9] *Suppose that $F \to E \to B$ is a sequence of simplicial spaces, with B a simplicial pointed space, and each B_i and E_i connected. Assume that for any $i \geq 0$ the sequence $F_i \to E_i \to B_i$ is a homotopy fibre sequence. Then $|F| \to |E| \to |B|$ is also a homotopy fibre sequence.*

Proof of 6.2. Let $M(\text{–})$ stand for $\mathrm{Map}_*^0(A, \text{–})$, so that we have to study the homotopy fibre of the natural map $|MX| \to M|X|$. Suppose that $d = \dim(A)$. Let B_i be the Postnikov stage $P_d X_i$, and F_i the fibre of the map $X_i \to B_i$. Since each space F_i is d-connected, an induction over the skeleta of A using 6.6 shows that $|MF| \to M|F|$ is an equivalence. There is a commutative diagram

$$\begin{array}{ccccc} |MF| & \longrightarrow & |MX| & \longrightarrow & |MB| \\ \sim\downarrow & & \downarrow & & \downarrow \\ M|F| & \longrightarrow & M|X| & \longrightarrow & M|B| \end{array} \tag{6.7}$$

in which both rows are fibration sequence by 6.6. Note in this connection that because of the connectivity of F_i, $MF_i = \mathrm{Map}_*(A, F_i)$ and $MF_i \to MX_i \to MB_i$ is a fibration sequence. Similar remarks apply to the bottom row, since $|F|$ is d-connected [10, IV.4.13]. It follows that the right hand square in 6.7 is a homotopy fibre square, and so the homotopy fibre of $|MX| \to M|X|$ is the same as the homotopy fibre of $|MB| \to M|B|$. Since A is a suspension, the fundamental groups of the spaces B_i are abelian, and so MB satisfies the assumptions of 6.1 and $|MB|$ is a polygem. Similarly, B satisfies the assumptions of 6.1, and so $|B|$ and hence $M|B|$ are polygems (5.5). It follows that the homotopy fibre of $|MB| \to M|B|$ is also a polygem.

7. Closure and resolution

In this section we prove 1.14 and 1.15. The proof of 1.14 depends on the following slightly more elaborate result.

7.1. **Proposition.** *Suppose that W satisfies the standard assumptions (2.9), and that X is a simplicial space with the property that for any $i \geq 0$, the space X_i is connected, has an abelian fundamental group, and belongs to $\mathcal{K}_W^+$. Then the realization $|X|$ also belongs to $\mathcal{K}_W^+$.*

Proof. Choosing a basepoint in X_0 and propagating it to the other spaces X_i with degeneracy maps converts X into a simplicial pointed space. Now apply the functor P_W in each simplicial degree, and let F_i be the fibre of $X_i \to P_W(X_i)$, so that $F \to X \to P_W(X)$ is a sequence of simplicial spaces which in each degree is a fibration sequence. By 2.5 and 6.6, $|F| \to |X| \to |P_W(X)|$ is a homotopy fibre sequence of spaces. By 3.4 and 5.6, $P_W(X_i)$ has trivial homotopy in dimensions $\geq c(W)$, and so (6.1) $|P_W(X)|$ is a polygem. It is clear from 6.4 that $|P_W(X)|$ is concentrated at p, and so this space belongs to $\mathcal{K}_W^+$. For any i the space F_i is killed by W (2.6), and hence $|F|$ is killed by W [2, 2.5] and belongs to $\mathcal{K}_W^+$. Since $\mathcal{K}_W^+ = \mathcal{T}(\mathcal{K}_W)$ (3.2) and so is closed under extension by fibrations, $|X| \in \mathcal{K}_W^+$. □

Proof of 1.14. Note that as in the proof of 1.10, a space X is v_*-torsion if and only if it belongs to $\mathcal{K}_W^+$ for each space W which satisfies the standard conditions. Now apply 7.1. □

Proof of 1.15. Let $M(-) = \mathrm{Map}_*^0(W, -)$, $R = \mathrm{R}_W(X)$, and $C = \mathrm{Cell}_W(X)$. Then R is a simplicial pointed space which is augmented by a map to X and in each degree is equivalent to a wedge of suspensions of W. Essentially by construction [14] [1], the natural map $|M(R)| \to M(X)$ is a weak equivalence; in fact R is designed so that the homotopy spectral sequence (6.4) for $\pi_*|M(R)|$ collapses to $\pi_* M(X)$. Consider the natural sequence

$$|M(R)| \to M(|R|) \to M(X)\,.$$

in which the composite is an equivalence. Let F_1 be the fibre of the left hand map and F_2 the fibre of the right hand map. Then $F_1 \sim \Omega F_2$, and F_1 is a polygem (6.2). In particular F_1 is v_i-torsion for $i \geq 1$ (5.6), so the same is true of F_2. It follows that $M(|R|) \to M(X)$ is a v_i^{-1}-equivalence for $i \geq 1$ (2.14). The space $|R|$ is built from W, and so the map $|R| \to X$ lifts canonically (up to homotopy) to a map $|R| \to C$. Since $C \to X$ induces an equivalence $M(C) \to M(X)$, it's clear that the map $M(|R|) \to M(C)$ is a v_i^{-1}-equivalence for $i \geq 1$; hence (4.2) $|R| \to C$ is a v_i^{-1}-equivalence for $i \geq \kappa(W)$. However

$|R| \to C$ is trivially a v_i^{-1}-equivalence for $i < \kappa(W)$, since both of these spaces are built from, hence killed by, W, and consequently are v_i-torsion for $1 \leq i < \kappa(W)$ (3.3). It follows that the fibre F of $|R| \to C$ is concentrated at p and is v_i-torsion for $i \geq 1$; in other words, F is v_*-torsion. □

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Yale University, New Haven, CT 06520 USA
E-mail address: chacholskiwojciech@math.yale.edu

University of Notre Dame, Notre Dame IN 46556 USA
E-mail address: dwyer.1@nd.edu

Kalamazoo College, Kalamazoo MI, 49006 USA
E-mail address: intermon@kzoo.edu