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## SELF HOMOTOPY EQUIVALENCES OF POSTNIKOV CONJUGATES<sup>1</sup>

E. DROR, W. G. DWYER AND D. M. KAN

**ABSTRACT.** The purpose of this note is to generalize a result of Wilkerson [4] and show that the self homotopy equivalences of the Postnikov approximations of a space  $X$  determine, in a rather simple manner, simultaneously the self homotopy equivalences of  $X$  and the self homotopy equivalences of the Postnikov conjugates of  $X$  (i.e. the spaces with the same Postnikov approximations as  $X$ ).

**1. Statement of results.** For a connected CW-complex  $X$ , let  $EX$  denote its space of self homotopy equivalences,  $B_{EX}$  the classifying space of  $EX$  and  $X^{(n)}$  the  $n$ th Postnikov approximation of  $X$ . Consider the homotopy inverse limit space [1, p. 301]

$$V = \text{ho proj lim}_{\leftarrow} B_{EX^{(n)}}$$

i.e. the space obtained by turning the tower  $\{B_{EX^{(n)}}\}$  into a tower of fibrations and taking the ordinary limit of the latter. Then it is reasonable to expect that this space  $V$  is in some way related to  $B_{EX}$  and it turns out that, in fact, *one component of  $V$* , the one “containing” the identity maps of the  $X^{(n)}$ , *has exactly the homotopy type of  $B_{EX}$* . To describe the homotopy types of the (possible) other components of  $V$  one needs the *Postnikov conjugates* of  $X$ , i.e. the spaces  $Y$  for which each  $Y^{(n)}$  has the same homotopy type as  $X^{(n)}$  (such spaces need *not* have the same homotopy type as  $X$ , as the homotopy equivalences  $Y^{(n)} \rightarrow X^{(n)}$  are *not* required to be compatible). Our main result then is:

**1.1 THEOREM.** (i) *There is a canonical 1-1 correspondence between the components of  $V$  and the homotopy types of the Postnikov conjugates of  $X$ .*

(ii) *For every Postnikov conjugate  $Y$  of  $X$ , the corresponding (see (i)) component  $V_Y \subset V$  has the homotopy type of  $B_{EY}$ .*

As the homotopy groups of the homotopy inverse limit of a tower fit into short exact sequences involving a  $\text{proj lim}$  and a  $\text{proj lim}^1$  term [1, p. 310], Theorem 1.1 implies the following “known” result:

**1.2 COROLLARY.** *For every Postnikov conjugate  $Y$  of  $X$  and every integer  $i \geq 0$ , there is a short exact sequence*

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$$* \rightarrow \text{proj lim}^1 \pi_{i+1} EX^{(n)} \rightarrow \pi_i EY \rightarrow \text{proj lim} \pi_i EX^{(n)} \rightarrow *$$

in which each of the maps  $\pi_j EX^{(k)} \rightarrow \pi_j EX^{(k-1)}$  involved in the  $\text{proj lim}$  and  $\text{proj lim}^1$  terms differs from the obvious one by the action of an element of  $\pi_0 EX^{(k-1)}$  (which depends on  $Y$ ).

1.3 COROLLARY. *There is a canonical 1-1 correspondence between the elements of the set  $\text{proj lim}^1 \pi_0 E_X^{(n)}$  and the homotopy types of Postnikov conjugates of  $X$ .*

As  $\pi_0 EX^{(n)}$  is the group of homotopy classes of self homotopy equivalences of  $X^{(n)}$ , this corollary is just Theorem I of Wilkerson [4]. In fact, it was our attempt to understand this result of Wilkerson which led to our Theorem 1.1.

1.4 Generalizations. If  $X$  has a base point, then one can consider the space  $E_* X$  of base point preserving self homotopy equivalences and it is not hard to verify that Theorem 1.1 and its proof remain valid if one replaces everywhere  $E$  by  $E_*$ . One could go even further and consider, for instance, the space  $E_\pi X$  of base point preserving self homotopy equivalences which induce the identity on all the homotopy groups. Again Theorem 1.1 and its proof remain valid if one replaces everywhere  $E$  by  $E_\pi$ , except that one has to suitably restrict the notion of Postnikov conjugate.

1.5 Organization of the proof. In order to keep the rather technical proof of our theorem as reasonable as possible, we reduce (in §2) Theorem 1.1 to a minimal simplicial version 2.1 and then proceed (in §3) to prove the latter. This has the considerable advantage that one then can use isomorphisms instead of homotopy equivalences as, for minimal simplicial sets, these two notions coincide. We will freely use the simplicial terminology and notation of [1] and [2].

**2. Reduction to the minimal simplicial case.** Let  $M$  be a connected simplicial set which is minimal [2, p. 35] and denote by  $M^{(n)}$  its  $n$ th Postnikov approximation [2, p. 34]. By a Postnikov conjugate of  $M$  we then mean any minimal simplicial set  $N$  for which each  $N^{(n)}$  is isomorphic to  $M^{(n)}$  (again this does not imply that  $N$  is isomorphic to  $M$ , as the isomorphisms  $N^{(n)} \approx M^{(n)}$  are not required to be compatible). If  $AM$  denotes the simplicial group of automorphisms of  $M$  [2, p. 74] and  $\overline{W}$  is the simplicial classifying functor [2, p. 87], then one can form the homotopy inverse limit [1, p. 295]

$$L = \text{ho proj lim } \overline{W}AM^{(n)}$$

and formulate the following minimal simplicial version of Theorem 1.1.

2.1 THEOREM. (i) *There is a canonical 1-1 correspondence between the components of  $L$  and the (isomorphism classes of) Postnikov conjugates of  $M$ .*

(ii) *For every Postnikov conjugate  $N$  of  $M$ , the corresponding (see (i)) component  $L_N \subset L$  has the homotopy type of  $\overline{W}AN$ .*

THEOREM 1.1 now readily follows from this minimal simplicial version by applying Theorem 2.1 to a *minimal subcomplex of the singular complex* of  $X$  and observing that, for a *minimal simplicial set*  $M$ , every self homotopy equivalence is an automorphism and, in fact,  $EM = AM$ .

It thus remains to prove Theorem 2.1, and this we will do in §3.

**3. Proof of Theorem 2.1.** The main part of the proof consists of, given a Postnikov conjugate  $N$  of  $M$  and a choice of isomorphisms  $q: N^{(n)} \approx M^{(n)}$ , constructing an explicit map  $\overline{WAN} \rightarrow L$ . Once this is done, it will not be hard to show that

3.1(i) *The correspondence  $N \rightarrow L_N$ , which assigns to  $N$  the component of  $L$  containing the image of  $\overline{WAN}$  under this map, does not depend on the choice of the  $q$ 's and, in fact, induces a 1-1 correspondence between the (isomorphism classes of) Postnikov conjugates of  $M$  and the components of  $L$ , and*

3.1(ii) *The map  $\overline{WAN} \rightarrow L_N$  is a homotopy equivalence.*

We start with constructing a commutative diagram

$$\begin{array}{ccccccc}
 \dots & \rightarrow & \overline{WAN}^{(n+1)} \times I/n+1 & \rightarrow & \overline{WAN}^{(n)} \times I/n & \rightarrow & \dots \\
 & & \downarrow & & \downarrow & & \\
 \dots & \rightarrow & \overline{WAM}^{(n+1)} & \rightarrow & \overline{WAN}^{(n)} & \rightarrow & \dots
 \end{array} \tag{3.2}$$

in which  $I/n$  denotes the (contractible) simplicial set that has [1, p. 292] a  $k$ -simplex for every nondecreasing sequence of integers  $(n, n_0, \dots, n_k)$ . (The existence of this commutative diagram is equivalent to the statement that the diagram

$$\begin{array}{ccccccc}
 \dots & \rightarrow & \overline{WAN}^{(n+1)} & \rightarrow & \overline{WAN}^{(n)} & \rightarrow & \dots \\
 & & \downarrow & & \downarrow & & \\
 \dots & \rightarrow & \overline{WAM}^{(n+1)} & \rightarrow & \overline{WAM}^{(n)} & \rightarrow & \dots
 \end{array}$$

in which the vertical maps are induced by the  $q$ 's, commutes up to compatible homotopies [3]). The horizontal maps in (3.2) are the obvious ones and the vertical map  $\overline{WAN}^{(n)} \times I/n \rightarrow \overline{WAM}^{(n)}$  is given by the formula

$$((b_{k-1}, \dots, b_0), (n, n_0, \dots, n_k)) \rightarrow (a_{k-1}qb_{k-1}q^{-1}a_k^{-1}, \dots, a_0qb_0q^{-1}a_1^{-1})$$

in which  $a_{k-i}: M^{(n)} \approx M^{(n)}$  denotes the *unique* isomorphism such that the diagram

$$\begin{array}{ccccc}
 N^{(n)} & \xrightarrow{q} & M^{(n)} & & \\
 \text{proj} \swarrow & & & & \searrow \text{proj} \\
 N^{(n)} & \xrightarrow{q} & M^{(n)} & \xrightarrow{a_{k-i}} & M^{(n)}
 \end{array}$$

commutes and in which  $q, q^{-1}, a_i$  and  $a_i^{-1}$  are abbreviations for  $q \times \Delta[j], q^{-1} \times \Delta[j], a_i \times \Delta[j]$  and  $a_i^{-1} \times \Delta[j]$  with appropriate  $j$ . A straightforward but long calculation then shows that the function  $\overline{WAN}^{(n)} \times I/n \rightarrow \overline{WAM}^{(n)}$

so defined is actually a simplicial map and that diagram (3.2) indeed commutes.

Next we observe that another long but straightforward calculation yields

3.3. *The diagram (3.2) induces an isomorphism*

$$\text{ho proj lim } \overline{WAN}^{(n)} \overset{r}{\approx} \text{ho proj lim } \overline{WAM}^{(n)} = L$$

by assigning to every  $k$ -simplex of  $\text{ho proj lim } \overline{WAN}^{(n)}$ , i.e. compatible collection of maps  $\Delta[k] \times I/n \rightarrow \overline{WAN}^{(n)}$ , the composition

$$\Delta[k] \times I/n \rightarrow \Delta[k] \times I/n \times I/n \rightarrow \overline{WAN}^{(n)} \times I/n \rightarrow \overline{WAM}^{(n)}.$$

Finally we obtain the desired map  $\overline{WAN} \rightarrow L$  as the obvious [1, p. 297] composition

$$\begin{aligned} \overline{WAN} &= \text{proj lim } \overline{WAN}^{(n)} \rightarrow \text{ho proj lim } \overline{WAN}^{(n)} \\ &\overset{r}{\approx} \text{ho proj lim } \overline{WAM}^{(n)} = L. \end{aligned}$$

One now proves 3.1(i) by verifying what happens to the vertex of  $\overline{WAN}$  under the map  $\overline{WAN} \rightarrow L$ . The argument is straightforward and is essentially a  $\overline{W}$ -translation of the proof of Theorem I of Wilkerson [4].

To prove 3.1(ii) one notes that the projections  $N^{(n+1)} \rightarrow N^{(n)}$  are fibrations and that therefore the induced maps  $AN^{(n+1)} \rightarrow AN^{(n)}$  are also fibrations (even though they need not be onto). It readily follows [1, p. 254 and p. 310] that the induced [1, p. 297] map

$$\pi_i \overline{WAN} = \pi_i \text{proj lim } \overline{WAN}^{(n)} \rightarrow \pi_i \text{ho proj lim } \overline{WAN}^{(n)}$$

is an isomorphism for all  $i \geq 1$  and the desired result thus follows from 3.3.

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