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PSEUDO-ISOTOPY AND INVARIANT THEORY—I

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§1. INTRODUCTION AND STATEMENT OF RESULTS

Let M be a compact C^∞-manifold (possibly with boundary ∂M), and let $I = [0, 1]$ be the closed unit interval. A pseudo-isotopy of M is a C^∞-diffeomorphism

$$f: M \times I \to M \times I$$

such that $f|M \times 0 = id$; the space of all such pseudo-isotopies (with the C^∞ Whitney topology) is denoted $P(M)$. This paper looks at the problem of computing the rational homotopy groups $\pi_i P(M) \otimes \mathbf{Q}$ for $i \ll \dim M$, under the assumption that M is simply-connected. These homotopy groups are very closely related to corresponding rational homotopy groups of the space of diffeomorphisms of M [1–4].

According to [1, 3] there is a natural suspension map

$$\Sigma: P(M) \to P(M \times I)$$

(essentially given by $\Sigma(f) = f \times id_I$). Since Σ induces an isomorphism $\pi_i P(M) \to \pi_i P(M \times I)$ for $i \ll \dim M$‖ [1], we can construct the direct limit $\mathcal{P}(M) = \varinjlim_k P(M \times I^k)$ of the suspension maps and conclude that $\pi_i P(M) \approx \pi_i \mathcal{P}(M)$ for $i \ll \dim M$. However, up to rational homotopy type there is a direct product decomposition [4]

$$A(M) \approx B^2\mathcal{P}(M) \times h(M; s)$$

where $B^2\mathcal{P}(M)$ is a canonical 2-fold delooping of $\mathcal{P}(M)$, $h(M; s)$ is the infinite loop space associated to the generalized homology of M with coefficients in the sphere spectrum s, and the space $A(M)$ is Waldhausen's algebraic K-theory of M. For this reason computing $\pi_i P(M) \otimes \mathbf{Q}$ for $i \ll \dim M$ is the same (mudolo $\pi_{i+2} h(M; s) \otimes \mathbf{Q} = H_{i+2}(M; \mathbf{Q})$) as computing $\pi_{i+2} A(M) \otimes \mathbf{Q}$.

These applications of the algebraic K-theory of topological spaces are our principal motivation for undertaking this study. However, until the recognized technical problems in the paper [3] linking the algebraic K-theory $A(M)$ via the Whitehead space Wh(M) to pseudo-isotopy theory are cleared up, it is perhaps more appropriate to view this paper as bearing the alternate title, "Algebraic K-theory of Spaces and Invariant Theory—I".

Now let X be any simply-connected space, for instance, $X = M$. In this paper we start with a differential graded algebra (DGA) K which is a model for the chains on the loop space ΩX of X, i.e. K is an augmented DGA over $\mathbf{Q}$ such that the

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‖It is sufficient to assume $i < (n/6) - 7$. However, it is generally believed that this stability result will eventually be improved.

augmentation induces an isomorphism $K_0 \xrightarrow{\cong} \mathbb{Q}$ of the degree zero part of K with $\mathbb{Q}$ and the homology of K, $H_*(K)$, is isomorphic to the graded ring $H_*(\Omega X; \mathbb{Q})$. For example, if X is a simply-connected space and ΩX is the Moore loop space, then the subcomplex of the rational singular chain complex of ΩX generated by all singular simplices with their vertices at the identity loop is a model K. Other more economical models for ΩX are constructed in [5, 6].

Now recall the construction of a differential graded coalgebra (DGC) from a differential graded Lie algebra (DGL) given in [7, pp. 290–292]. Let L be a reduced ($L_0 = 0$) DGL, and let ΣL denote the suspension of the graded vector space underlying L. That is, $(\Sigma L)_{l+1}$ consists of elements Σx where x is an element of L homogeneous of degree l. Let $S(\Sigma L)$ be the graded symmetric coalgebra of ΣL,

$$S(\Sigma L) = \bigotimes_{i \text{ even}} \Lambda^*(\Sigma L_i) \otimes \bigotimes_{j \text{ odd}} S^*(\Sigma L_j),$$

where $\Lambda^*(\ )$ and $S^*(\ )$ denote the ordinary exterior coalgebra and symmetric coalgebra respectively. Denote by $[x_1 | \ldots | x_N]$ the image of the element $\Sigma x_1 \otimes \ldots \otimes \Sigma x_N$ of the tensor product $\Sigma L^{\otimes N}$ in $S(\Sigma L)$, so that the elements of $S(\Sigma L)$ are linear combinations of these symbols subject to relations of graded commutativity. Thus, for a fixed n, the homogeneous elements of degree n are combinations of elements of this form from

$$\Lambda^{p_1}(\Sigma L_{i_1}) \otimes \ldots \otimes \Lambda^{p_k}(\Sigma L_{i_k}) \otimes S^{q_1}(\Sigma L_{j_1}) \otimes \ldots \otimes S^{q_l}(\Sigma L_{j_l}).$$

$\mathscr{C}(L)$ may then be defined to be $S(\Sigma L)$ with the differential $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2$ where the differentials d_1 and d_2 arise from the differential of L and the Lie algebra structure of L_1, respectively.

For the computation of the rational homotopy groups of $A(X)$, we will have to consider the DGC $\mathscr{C}(L)$ for $L = M_n(\bar{K})$ where $\bar{K}$ is the augmentation ideal of a model K for ΩX. The $(n \times n)$ matrices $M_n(\bar{K})$ are a DGL with the usual graded bracket $[M, N] = MN - (-1)^{|M||N|}NM$ on homogeneous elements and with differential inherited from the differential on K. $GL_n(\mathbb{Z})$ acts on $M_n(K)$ by conjugation (see [8, pp. 187–189], and choosing a basis for $\bar{K}_l$ breaks up the representation on $(M_n(\bar{K}))_l$ into a direct sum of copies of the adjoint representation ad_n of $GL_n(\mathbb{Z})$. By naturality there is an action of $GL_n(\mathbb{Z})$ on $\mathscr{C}(M_n(\bar{K}))$. These actions are compatible with the "upper" inclusions $GL_n(\mathbb{Z}) \to GL_{n+1}(\mathbb{Z})$, $M_n(\bar{K}) \to M_{n+1}(\bar{K})$ and the induced inclusion $\mathscr{C}(M_n(\bar{K})) \to \mathscr{C}(M_{n+1}(\bar{K}))$.

Now let us consider the chain complex of representations of $GL_n(\mathbb{Z})$

$$\ldots \xrightarrow{\mathrm{d}} \mathscr{C}_{i+1}(M_n(\bar{K})) \xrightarrow{\mathrm{d}} \mathscr{C}_i(M_n(\bar{K})) \xrightarrow{\mathrm{d}} \ldots \quad . \tag{1}$$

For the proof of the main theorem (Theorem 1.2), we observe that the representation of $GL_n(Z)$ in each degree is completely reducible to a direct sum

$$\mathscr{C}_i(M_n(\bar{K}) = \mathscr{C}_i(M_n(\bar{K}))^{GL_n(\mathbb{Z})} \oplus \sum_\alpha (\oplus A_\alpha) \tag{2}$$

where $\mathscr{C}_i(M_n(\bar{K}))^{GL_n(\mathbb{Z})}$ is the subspace on which $GL_n(\mathbb{Z})$ acts trivially and each A_α is a finite-dimensional, nontrivial irreducible representation which is the restriction of a finite-dimensional irreducible complex representation of $GL_n(\mathbf{R})$. Furthermore, the part of Schur's lemma stating that for any group G there are no nonzero G-maps between nonisomorphic irreducible complex G-modules implies first that the homology group $H_i(\mathscr{C}(M_n(\bar{K})))$ decomposes in the same manner, and second, that the invariant chains

$$\dots \xrightarrow{d} \mathscr{C}_{i+1}(M_n(\bar{K}))^{GL_n(\mathbf{Z})} \xrightarrow{d} \mathscr{C}_i(M_n(\bar{K}))^{GL_n(\mathbf{Z})} \xrightarrow{d} \dots \tag{3}$$

form a subcomplex of (1) which may be used to compute the invariant homogogy $H_i(\mathscr{C}(M_n(\bar{K})))^{GL_n(\mathbf{Z})}$.

By naturality and compatibility the complexes (1) and (3) for $M_n(\bar{K})$ map to those for $M_{n+1}(\bar{K})$. It will be made clear in Lemma 2.2 that the map of invariant chains is an isomorphism in degrees $i \leq n-1$. If we put $GL(\mathbf{Z}) = \varinjlim_n GL_n(\mathbf{Z})$ and $M(\bar{K}) = \varinjlim_n M_n(\bar{K})$, we have a DGC $\mathscr{C}(M(\bar{K}))$ with an action of $GL(\mathbf{Z})$. Piecing together the complexes (3) for various n and using the isomorphisms $\mathscr{C}_i(M_n(\bar{K}))^{GL_n(\mathbf{Z})} \to \mathscr{C}_i(M_{n+1}(\bar{K}))^{GL_{n+1}(\mathbf{Z})}$ for $i \leq n-1$, we obtain a single chain complex $\varinjlim_n \mathscr{C}(M_n(\bar{K}))^{GL_n(\mathbf{Z})}$ which is just $\mathscr{C}(M(\bar{K}))^{GL(\mathbf{Z})}$. Let us summarize:

PROPOSITION 1.1. *The natural chain map*

$$\mathscr{C}(M_n(\bar{K}))^{GL_n(\mathbf{Z})} \to \mathscr{C}(M(\bar{K}))^{GL(\mathbf{Z})}$$

is an isomorphism in degrees less than n. Thus,

$$H_i(\mathscr{C}(M_n(\bar{K})))^{GL_n(\mathbf{Z})} \to H_i(\mathscr{C}(M(\bar{K})))^{GL(\mathbf{Z})}$$

is an isomorphism for $i < n-1$.

$H_i(\mathscr{C}(M(\bar{K})))^{GL(\mathbf{Z})} = \varinjlim_n H_i(\mathscr{C}(M_n(\bar{K})))^{GL_n(\mathbf{Z})}$ will be called the i-th invariant homology of the DGL $M(\bar{K})$. It turns out that if K is a DGA model for ΩX, $H_q(\mathscr{C}(M(\bar{K})))^{GL(\mathbf{Z})}$ only depends on X. Denote $H_q(\mathscr{C}(M(\bar{K})))^{GL(\mathbf{Z})}$ by $\mathrm{Inv}_q(X)$ if K is a DGA model for ΩX. We may also consider the cochain complex of invariant functions $\mathrm{inv}^*\mathscr{C}(M(\bar{K}))$ on $\mathscr{C}(M(\bar{K}))$, which may be identified with $\mathrm{Hom}(\mathscr{C}(M(\bar{K}))^{GL(\mathbf{Z})}, \mathbf{Q})$. Denote the invariant cohomology $H^q(\mathscr{C}(M(\bar{K})))^{GL(\mathbf{Z})}$ by $\mathrm{Inv}^q(X)$ if K is a DGA model for ΩX. It follows from the discussion above that $\mathrm{Inv}^q(X) = \varprojlim_n H^q(\mathrm{Inv}\mathscr{C}(M_n(\bar{K})))$.

Here is the main theorem of the paper.

THEOREM 1.2. *There are isomorphisms*

$$H_m(A(X); \mathbf{Q}) \cong \sum_{p+q=m} H_p(GL(\mathbf{Z}); \mathbf{Q}) \otimes \mathrm{Inv}_q(X)$$

and

$$H^m(A(X);\mathbf{Q}) \cong \sum_{p+q=m} H^p(GL(\mathbf{Z});\mathbf{Q})\otimes \mathrm{Inv}^q(X).$$

Let us outline the proof of this theorem.

Suppose that X is a simplicial set with one zero simplex and no nondegenerate one simplices. (Any one-connected topological space is weakly equivalent to the realization of such an X.) Let $\mathbf{Q}G(X)$ be the simplicial rational group ring of the Kan loop group $G(X)$, and let

$$\alpha: \mathbf{Q}G(X) \to \mathbf{Q} \tag{4}$$

be the augmentation homomorphism from $\mathbf{Q}G(X)$ to the constant simplicial ring $\mathbf{Q}$. There is an induced homomorphism of simplicial rings of $n\times n$ matrices

$$M_n(\alpha)\colon M_n(\mathbf{Q}G(X)) \to M_n(\mathbf{Q}), \tag{5}$$

and, in line with [4], we define $\widehat{GL}_n(\mathbf{Q}G(X))$ to be the pullback of $M_n(\alpha)$ over the natural inclusion $GL_n(\mathbf{Z}) \to M_n(\mathbf{Q})$.

$$\begin{tikzcd}
\widehat{GL}_n(\mathbf{Q}G(X)) \arrow[r] \arrow[d] & M_n(\mathbf{Q}G(X)) \arrow[d, "M_n(\alpha)"] \\
GL_n(\mathbf{Z}) \arrow[r] & M_n(\mathbf{Q}).
\end{tikzcd}$$

Matrix multiplication induces a simplicial monoid structure on the direct limit $\widehat{GL}(\mathbf{Q}G(X))$ of the upper inclusion maps $\widehat{GL}_n(\mathbf{Q}G(X)) \to \widehat{GL}_{n+1}(\mathbf{Q}G(X))$. The plus construction $B\widehat{GL}(\mathbf{Q}G(X))^+$ with respect to the perfect commutator subgroup of $\pi_1 B\widehat{GL}(\mathbf{Q}G(X)) = GL(\mathbf{Z})$ is a space which is denoted $A(X)$. This is rationally equivalent to the $A(X)$ of [4] (see [23]). (Note that in [4, p. 42] $A(X)$ is described as $B\widehat{GL}(Q[G(X)])^+$, where $Q[G(X) = \Omega^\infty S^\infty(G(X)\cup *)$, so that the homotopy groups of $Q[G(X)]$ are the unreduced stable homotopy groups of $G(X)$.) For us $A(X) = B\widehat{GL}(\mathbf{Q}[G(X)])^+$, where $\mathbf{Q}[\,\cdot\,]$ is essentially the rational group ring functor, so that the higher homotopy groups of $\mathbf{Q}[G(X)]$ are the unreduced rational homology groups of $G(X)$. The fact that these two definitions agree rationally follows ultimately from the fact that stable homotopy and rational homology are rationally the same.

Write $R = \mathbf{Q}G(X)$, and let $\widehat{GL}_n{}^0(R)$ be the kernel of the map $\widehat{GL}_n(R) \to GL_n(\mathbf{Z})$ provided in the definition of $\widehat{GL}_n(R)$ above. Let $\widehat{GL}^0(R) = \varinjlim_n \widehat{GL}_n{}^0(R)$. The exact sequence of simplicial monoids

$$1 \to \widehat{GL}^0(R) \to \widehat{GL}(R) \to GL(\mathbf{Z}) \to 1$$

induces a fibration sequence

$$B\widehat{GL}^0(R) \to B\widehat{GL}(R) \to BGL(\mathbf{Z}) \tag{7}$$

to which there corresponds a Serre spectral sequence

$$E^2_{p,q} = H_p(GL(\mathbf{Z}); H_q(B\widehat{GL}^0(R);\mathbf{Q})) \Rightarrow H_{p+q}(A(X);\mathbf{Q}). \tag{8}$$

Our main lemma (Lemma 3.1) identifies $H_i(B\widehat{GL}_n{}^0(R);\mathbf{Q})$ and $H_i(\mathscr{C}(M_n(\bar{K}))$ as isomorphic $GL_n(\mathbf{Z})$ modules when K is a model for ΩX via an application of Quillen's rational homotopy theory[4]. In order to prove this fact, we shall complete the simplicial ring R with respect to the augmentation ideal a of the obvious simplicial augmentation map $R\to\mathbf{Q}$. For details, see §3. The discussion above of the complete reducibility of the $GL_n(\mathbf{Z})$ on $H_*(\mathscr{C}(M_n(\bar{K})))$, together with the additional fact that homology of $GL_n(\mathbf{Z})$ with coefficients in the modules A_α vanishes in a range tending to infinity with n[2, 9], implies that the E^2 term of the spectral sequence (8) of the limit fibration ($n\to\infty$) is a tensor product

$$H_*(GL(\mathbf{Z});\mathbf{Q})\otimes\mathrm{Inv}_*(X).$$

By comparison of this spectral sequence with a related spectral sequence which collapses, we are able to conclude it collapses, completing the proof of the theorem.

Now, since $A(X)$ is an infinite loop space, $H_*(A(X);\mathbf{Q})$ is an abelian Hopf algebra, and $\pi_*A(X)\otimes\mathbf{Q}$ is isomorphic to the space of primitive elements of $H_*(A(X);\mathbf{Q})$[10, p. 263]. On the other hand, such a Hopf algebra is a tensor product of the Hopf algebras generated by a basis for the primitives, and each one of these is an exterior algebra or a polynomial algebra on one generator depending on the degree of the basic primitive elements([10, §7]). Therefore, from a factorization of the Poincaré series of $H_*(A(X);\mathbf{Q})$ into products of terms of the form $(1+t^{2k+1})$ or $(1-t^{2k})^{-1}$, one can read off the dimensions of the rational homotopy groups in degree n by counting the number of factors $(1\pm t^n)^{\pm1}$. We point out that it follows from our argument that $\mathrm{Inv}_*(X)$ becomes a Hopf algebra and $H_*(A(X);\mathbf{Q})$ is isomorphic to $H_*(GL(\mathbf{Z});\mathbf{Q})\otimes\mathrm{Inv}_*(X)$ as Hopf algebras. There are dual statements about $\mathrm{Inv}^*(X)$, $H^*(A(X);\mathbf{Q})$, and $H^*(GL(\mathbf{Z});\mathbf{Q})\otimes\mathrm{Inv}^*(X)$. Theoretically we can compute $\pi_iA(X)\otimes\mathbf{Q}$ if a model K for ΩX is explicitly given, and we can choose economical models for practical purposes. We shall come back to this point by taking advantage of some good models in a future paper.

Tools for computing $\mathrm{Inv}_*(X)$ are the spectral sequences arising from the following considerations. $\mathscr{C}(M(\bar{K}))$ has a natural bigrading, and the decomposition of the differential $\mathrm{d}=\mathrm{d}_1+\mathrm{d}_2$ permits the interpretation of $\mathscr{C}(M(\bar{K}))$ as bicomplex with vertical differential d_1 arising from the differential on $\bar{K}$ and horizontal differential d_2 derived from the Lie product. This bicomplex structure is inherited by $\mathscr{C}(M(\bar{K}))^{GL(\mathbf{Z})}$ and gives rise to a pair of spectral sequences.

PROPOSITION 1.3. *Let K be a DGA model for ΩX.*

(*a*) *There are two spectral sequences ${}_{(1)}E^r$ and ${}_{(2)}E^r$ convergent to graded vector spaces associated to* $\mathrm{Inv}_*(X)$. *The E^2 terms are respectively*

$${}_{(1)}E^2_{p,q}=H_p(H_q(\mathscr{C}(M(\bar{K}))^{GL(\mathbf{Z})},\mathrm{d}_1),\mathrm{d}_2)$$

and

$${}_{(2)}E^2_{p,q}=H_q(H_p(\mathscr{C}(M(\bar{K})),\mathrm{d}_2),\mathrm{d}_1).$$

(b) ${}_{(1)}E^2_{p,*}$ *may be identified with* $H_p(\mathscr{C}(M(H(\bar{K})))^{GL(\mathbf{Z})}$.

(c) *If X is a product of Eilenberg–MacLane spaces or a suspension, then* ${}_{(i)}E^2_{p,q}={}_{(i)}E^\infty_{p,q}$ *for i = 1, 2.*

There are dual results for $\mathrm{Inv}^*(X)$.

Proofs of Theorem 1.2 and Proposition 1.3 will be given in §3. See §2 for some examples and further discussion of both spectral sequences.

Burghelea has claimed results similar to ours[11]. He also published a calculation in a separate paper[8]. This is our Example 2.3. We would like to thank J. C. Moore for useful discussions.

§2. INVARIANT THEORY AND SIMPLE EXAMPLES

Let us first recall some results in classical invariant theory and develop the extension required for this paper. Let $\underline{k}$ be a field of characteristic 0, and let ad_n denote the adjoint representation of $GL_n(\underline{k})$ on $M_n(\underline{k})$, the $n \times n$ matrices with entries in $\underline{k}$. Two questions with which invariant theory is concerned are "What are the $GL_n(\underline{k})$-invariant, $\underline{k}$-valued polynomial functions of some number of $(n \times n)$-matrices?" and "Given a set of generators for this algebra, what are all the relations among them?" Procesi[12, p. 313] answers the first question by proving a first fundamental theorem: Any polynomial invariant of m matrices $M_1, \ldots, M_m$ is a polynomial in the invariants Tr $M_{m_1} \ldots M_{m_p}$, $M_{m_1} \ldots M_{m_p}$ running over all possible noncommutative monomials. By the processes of polarization and restitution[13, pp. 5, 6], the answer to the first question follows from the answer to the question, "What are the multilinear invariants of N matrices in $M_n(\underline{k})$?" Procesi shows that this question may be answered by interpreting the following classical result[13, p. 130]: Let V denote the standard n-dimensional representation of $GL_n(\underline{k})$, and let $GL_n(\underline{k})$ act diagonally on the tensor power $V^{\otimes N}$. Then the space of endomorphisms of $V^{\otimes N}$ commuting with the action of $GL_n(\underline{k})$ is precisely the span of the endomorphisms of $V^{\otimes N}$ given by permuting the factors. The invariant theory interpretation of this result is as follows. Let $\sigma \in S_N$, the permutation group on N letters, be written as a product of disjoint cycles, including cycles of length one:

$$\sigma = (i_1 \ldots i_r) \ldots (j_1 \ldots j_s).$$

Define μ_σ: $ad_n^{\otimes N} \to \underline{k}$ by

$$\mu_\sigma(M_1 \otimes \ldots \otimes M_N) = Tr(M_{i_1} \ldots M_{i_r}) \ldots Tr(M_{j_1} \ldots M_{j_s}).$$

Then $\{\mu_\sigma | \sigma \in S_N\}$ spans the space of invariant multilinear functions of N matrices. We shall use this fact to extend Procesi's result to our situation.

To answer the second question Procesi first determines all the multilinear trace identities in terms of Young diagrams[12, p. 318]. In particular, there is a fundamental multilinear trace identity $F(M_1, \ldots, M_{n+1}) = 0$ for $(n \times n)$-matrices, which is related to the fact that $\Lambda^{n+1} V = 0$. For example, if $n = 2$, we have

$$TrM_1M_2M_3 + TrM_1M_3M_2 - TrM_1M_2TrM_3$$

$$- TrM_2M_3TrM_1 - TrM_3M_1TrM_2 + TrM_1TrM_2TrM_3 = 0$$

for any three (2×2)-matrices. Thinking of the polynomial invariants of m $(n \times n)$-matrices as a quotient of the polynomial algebra on symbols $TrM_{m_1} \ldots M_{m_p}$ where cyclic permutations of the factors of $M_{m_1} \ldots M_{m_p}$ are not distinguished, the second fundamental theorem identifies the ideal of relations as the ideal generated by all $F(X_1, \ldots, X_{n+1})$ where $X_1, \ldots, X_{n+1}$ run over all possible monomials. From all of this we need to observe our second basic fact that because the fundamental multilinear trace identity involves $(n+1)$ matrices, $\{\mu_\sigma | \sigma \in S_N\}$ is linearly independent when

$N < n$. Therefore, the natural inclusions $M_n(\underline{k}) \to M_{n+1}(\underline{k})$ and $GL_n(\underline{k}) \to GL_{n+1}(\underline{k})$ induces an isomorphism of the invariant linear functions on $\sum_{0 \le N < n} (ad_{n+1})^{\otimes N}$ to those on $\sum_{0 \le N < n} (ad_n)^{\otimes N}$. We also remark that a glance at Weyl[13, p. 70] shows that in fact the $GL_n(\underline{k})$ invariant linear functions are the same as the $SL_n(\underline{k})$ invariant ones on these representation spaces.

For the extension let $(p) = (p_1, \ldots, p_k)$ and $(q) = (q_1, \ldots, q_l)$ be a k-tuple and an l-tuple of natural numbers with $N = \Sigma p_i + \Sigma q_j$, and let $\Lambda^{(p)} \otimes S^{(q)}(ad_n)$ denote the representation $\Lambda^{p_1}(ad_n) \otimes \ldots \otimes \Lambda^{p_k}(ad_n) \otimes S^{q_1}(ad_n) \otimes \ldots \otimes S^{q_l}(ad_n)$, a tensor product of exterior and symmetric powers. We will describe the linear invariant functions on $\Lambda^{(p)} \otimes S^{(q)}(ad_n)$ in terms of those on $(ad_n)^{\otimes N}$.

There is a canonical equivariant projection $(ad_n)^{\otimes N} \to \Lambda^{(p)} \otimes S^{(q)}(ad_n)$ and an equivariant section which we now describe. Consider the product of permutation groups $S_{(p),(p)} = S_{p_1} \times \ldots \times S_{p_k} \times S_{q_1} \times \ldots \times S_{q_l}$ and the obvious embedding into S_N. Define $\epsilon_{(p),(q)}: S_{(p)(q)} \to \{\pm 1\}$ by the formula

$$\epsilon_{(p),(q)}(\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_l) = \text{sign}\, \sigma_1 \ldots \text{sign}\, \sigma_k.$$

The section is given by

$$M_1 \wedge \ldots \wedge M_{p_1} \otimes \ldots \otimes M_{N-q_l+1} \cdot \ldots \cdot M_N \to$$

$$\frac{1}{|S_{(p),(q)}|} \sum_{\sigma \in S_{(p),(q)}} \epsilon_{(p),(q)}(\sigma) M_{\sigma(1)} \otimes \ldots \otimes M_{\sigma(N)}$$

where $|S_{(p),(q)}|$ is the order of $S_{(p),(q)}$. Following the projection by the section gives a projection $Q^*_{(p),(q)}$ of $(ad_n)^{\otimes N}$ to itself onto a subspace identifiable with $\Lambda^{(p)} \otimes S^{(q)}(ad_n)$. Taking duals, we can make a similar statement identifying the space $\text{Inv}(\Lambda^{(p)} \otimes S^{(q)}(ad_n))$ of invariant linear functions on $\Lambda^{(p)} \otimes S^{(q)}(ad_n)$ with a subspace of $\text{Inv}(ad_n^{\otimes N})$. This identification we will freely use below. The basic facts we will refer to are recorded in

LEMMA 2.1. (i) *There is a projection* $Q_{(p),(q)}: \text{Inv}(ad_n^{\otimes N}) \to \text{Inv}(ad_n^{\otimes N})$ *given by*

$$Q_{(p),(q)}(\mu_\tau) = \frac{1}{|S_{(p),(q)}|} \sum_{\sigma \in S_{(p),(q)}} \epsilon_{(p),(q)}(\sigma)\, \mu_{\sigma\tau\sigma^{-1}}$$

whose image is identified with $\text{Inv}(\Lambda^{(p)} \otimes S^{(q)}(ad_n))$. *Thus,* $\text{Inv}(\Lambda^{(p)} \otimes S^{(q)}(ad_n))$ *is generated by* $\{Q_{(p),(q)}(\mu_\tau)\}$ *subject to the relations*

$$Q_{(p);(q)}(\mu\tau) = \epsilon_{(p),(q)}(\sigma) Q_{(p),(q)} \mu_{\sigma\tau\sigma^{-1}}, \text{ for } \sigma \in S_{(p),(q)}.$$

(ii) *The natural map* $\text{Inv}(\Lambda^{(p)} \otimes S^{(q)}(ad_{n+1})) \to \text{Inv}(\Lambda^{(p)} \otimes S^{(q)}(ad_n))$ *is an isomorphism for* $n > N = \Sigma p_i + \Sigma q_j$.

(iii) *These spaces of invariant functions are the same for both* $SL_n(\underline{k})$ *and* $GL_n(\underline{k})$.

Now we will describe the invariant linear functions on $\mathscr{C}(M_n(\bar{K}))$ denoted by $\text{Inv}^*\mathscr{C}(M_n(\bar{K}))$. This will be mostly elaboration of notation. Choose k homogeneous elements of even degree $(m)^0 = (m_1, \ldots, m_k)^0$ and l homogeneous elements of odd degree $(m)^1 = (m_{k+1}, \ldots, m_{k+l})^1$ in $\bar{K}$. Then all elements of $\mathscr{C}(M_n(\bar{K}))$ of the form

$$[M_1 m_1 | \ldots | M_{p_1} m_1 | M_{p_1+1} m_2 | \ldots | M_{n-q_l+1} m_{k+l} | \ldots | M_N m_{k+l}]$$

where $M_i \in M_n(\mathbf{Q})$ and $M_i m = mM_i = M_i({}^{m}\cdot\cdot{}_{m})$, span a linear subspace $\mathscr{C}_{(p)(m)^0;(q)(m)^1}$ of $\mathscr{C}(M_n(\bar{K}))$. Clearly $\mathscr{C}(M_n(\bar{K}))$ is the sum of all $\mathscr{C}_{(p)(m)^0;(q)(m)^1}$ (direct sum if the homogeneous elements form a $\mathbf{Q}$ basis of $\bar{K}$), and each $\mathscr{C}_{(p)(m)^0;(q)(m)^1}$ is isomorphic to $\Lambda^{(p)}\otimes S^{(q)}(ad_n)$ as a representation of $GL_n(\mathbf{Q})$. Therefore $\mathrm{Inv}^*\mathscr{C}(M_n(\bar{K}))$ is a direct sum of $\mathrm{Inv}^*(\mathscr{C}_{(p)(m)^0;(q)(m)^1})$, and we have via Lemma 2.1 a description of this in terms of functions $Q_{(p)(m)^0;(q)(m)^1}(\mu_\sigma)$.

Example 2.1. Let $\{m_1, \ldots, m_p\}$ be a basis for $\bar{K}_r$. Then some of the invariant linear functions $\mathscr{C}(M_n(\bar{K}))_{r+1} \to \mathbf{Q}$ are $Q_{(1)(m_i)} \in (\mu_{(1)})\ ([M_1 m_1], \ldots, [M_p m_p]) = TrM_i$.

Example 2.2. Let x be a homogeneous element of odd degree r in $\bar{K}$. Then the permutation (1)(23) induces two elements of $\mathrm{Inv}^{5r+3}\mathscr{C}(M_n(\bar{K}))$.

$$Q_{(2)(x^2)^0;(1)(x)^1}\mu_{(1)(23)}([M_1x|M_2x|M_3x^3], [M_1'x|M_2'x^2|M_3'x^2]) = 0$$

$$Q_{(2,1)(x^3,x)^1}\mu_{(1)(23)}([M_1x|M_2x|M_3x^3], [M_1'x|M_2'x^2|M_3'x^2])$$

$$= \tfrac{1}{2}(TrM_1 TrM_2M_3 + TrM_2 TrM_1M_3).$$

Given this much of a description of $\mathrm{Inv}^*\mathscr{C}(M_n(\bar{K}))$, we can apply Lemma 2.1 to obtain a lemma used in the proofs of Proposition 1.1 and Theorem 1.2.

LEMMA 2.2. *The natural maps*

$$\mathscr{C}_i(M_n(\bar{K}))^{GL_n(Z)} \to \mathscr{C}_i(M_{n+1}(\bar{K}))^{GL_{n+1}(Z)}$$

and

$$\mathrm{Inv}^i\mathscr{C}(M_{n+1}\bar{K})) \to \mathrm{Inv}^i\mathscr{C}(M_n(\bar{K}))$$

are isomorphisms for $i < n$.

We conclude this section with some examples. It turns out that one can compute $\pi_*(A(X))\otimes\mathbf{Q}$ fairly effectively for simply-connected spaces X which are rationally equivalent either to suspensions or to certain products of Eilenberg–MacLane spaces. We also have a partial computation for $X = \mathbf{C}P^2$ and a conjecture that $\pi_*(A(\mathbf{C}P^n))\otimes\mathbf{Q}$ may be fairly simple.

It is easy to see that if $X \to X'$ is a map of simply-connected spaces which is k-connected mod torsion, then the induced map $A(X) \to A(X')$ is also k-connected mod torsion. This implies that for simply-connected X the rational type of the Postnikov stage $P_kA(X)$ depends only on the rational type of P_kX. It is often the case with a simply-connected manifold M that P_kM, for some small k, is rationally equivalent to a suitable product of Eilenberg–MacLane spaces or to P_kX for a suspension X; in this case the calculations below give information about the pseudoisotopy space of M.

Our calculations are made with the spectral sequences of Proposition 1.3. Effective use of these spectral sequences depends upon exploiting the flexibility of the algebraic situation to choose a particularly tractable DGA model K for ΩX. In an appendix we show that the models of [5, 6] may be used. In general these are DGA's with a highly nontrivial differential and with an underlying algebra which is free. Nevertheless, in favorable cases we may work with DGA's which have zero differential, or are not free as algebras, or both. In the examples below we mention the DGA's used and postpone further explanation of the computational process to a second appendix.

In each case below we actually give (part of) the Poincaré series of $\mathrm{Inv}^*(X)$. By Theorem 1.2 the reader may compute the corresponding Poincaré series of $A(X)$ by multiplying with the Poincaré series of $H_*(GL(\mathbf{Z});\mathbf{Q})$, which is $\prod_{k\geq 1}(1+t^{4k+1})$.

Example 2.3. If $X = K(\mathbf{Z}, 2l)$, we take $K = E(x)$, the exterior algebra on one generator x in degree $2l-1$ with differential zero. The computation is very easy and leads to

$$P.S.(\mathrm{Inv}^*(X)) = \prod_{k\geq 1}(1-t^{2kl})^{-1}.$$

Example 2.4. $X = K(\mathbf{Z}, 2)\times K(\mathbf{Z}, 4)$; $K = E(x_1, x_2)$, the exterior algebra on x_1 and x_2 when $|x_1| = 1$, $|x_2| = 3$ and $dx_1 = dx_2 = 0$. Even in this case the calculation is nontrivial. Through degree 10

$$P.S.(\mathrm{Inv}^*(X)) = (1-t^2)^{-1}(1-t^4)^{-2}(1+t^5)(1-t^6)^{-2}(1+t^7)(1-t^8)^{-3}(1+t^9)(1-t^{10})^3.$$

(Recall that

$$P.S.H.^*(h(X;s);\mathbf{Q}) = (1-t^2)^{-1}(1-t^4)^{-2}(1-t^6)^{-2}(1-t^8)^{-3}(1-t^{10})^{-3}$$

through degree 10.)

In case $X = \prod_{s=1}^{q} K(\mathbf{Z}, 2l_s)$, one may take $K = E(x_1, \ldots, x_q)$, the exterior algebra on generators x_s in degree $2l_s - 1$ with $dx_s = 0$.

Example 2.5. $X = K(\mathbf{Z}, 2l+1)$; $K = P[x]$, the polynomial algebra on x in degree $2l$ with $dx = 0$. It follows from Theorem 8.6 of [22] that

$$P.S.\mathrm{Inv}^*(X) = \prod_{k\geq 1}(1+t^{2kl+1}).$$

Example 2.6. $X = K(\mathbf{Z}, 2n+1)\times K(\mathbf{Z}, 2n+3)$; $K = P[x_1, x_2]$, the polynomial algebra on x_1 and x_2 in degrees $2n$ and $2n+2$, respectively. $dx_1 = dx_2 = 0$. We have computed part of the Poincaré series of $\mathrm{Inv}^*(X)$:

$$\begin{aligned}(1+t^{2n+1})(1+t^{2n+3})\cdot &\left\{\frac{(1+t^{4n+1})(1+t^{4n+3})(1+t^{4n+5})}{(1-t^{4n+4})}\right\}\\ &\cdot\left\{\frac{(1+t^{6n+1})(1+t^{6n+3})(1+t^{6n+5})(1+t^{6n+7})}{(1-t^{6n+4})(1-t^{6n+6})}\right\}\\ &\cdot\left\{\frac{(1+t^{8n+1})(1+t^{8n+3})\ldots(1+t^{8n+9})}{(1-t^{8n+4})(1-t^{8n+6})(1-t^{8n+8})}\right\}.\end{aligned}$$

When $n > 4$, this is the Poincaré series of $\mathrm{Inv}^*(X)$ in degrees less than $10n$.

In more generality, if $X = \prod_{s=1}^{q} K(\mathbf{Z}, 2l_s + 1)$, one may take $K = P[x_1, \ldots, x_q]$, the polynomial algebra on $\{x_1, \ldots, x_q\}$ with $|x_s| = 2l_s$ and $dx_s = 0$.

Example 2.7. If $X = S^{2n}$, $K = T[x]$, the free algebra on x. $|x| = 2n - 1$ and $dx = 0$. We have computed the Poincaré series of $\mathrm{Inv}^*(X)$ up to degree $10n - 4$:

$$(1 - t^{2n})^{-1}(1 - t^{6n-2})^{-1}(1 - t^{10n-4})^{-1}.$$

In general, if $X = \Sigma Y$, the suspension of a connected complex, a free associative algebra with vanishing differential models ΩX.

Example 2.8. Consider $X = \mathbf{C}P^2$. $\Omega\mathbf{C}P^2$ has the DGA model $T[x_1, x_2;\ dx_1 = 0, dx_2 = x_1{}^2]$, the free associative algebra on x_1 degree 1 and x_2 in degree 3. The Poincaré series of $\mathrm{Inv}^*(X)$ through degree 8 is

$$(1 - t^2)^{-1}(1 - t^4)^{-1}(1 - t^6)^{-1}(1 - t^8)^{-1},$$

a calculation based on the second spectral sequence of Proposition 1.3.

Conjecture:

$$P.S.\mathrm{Inv}^*(\mathbf{C}P^n) = \prod_{k=1}^{n}(1 - t^{2k})^{-1}\prod_{l\geq 1}(1 - t^{2ln+2})^{-1}(1 - t^{2ln+4})^{-1}.$$

Observe the difference between this conjectural formula and the formula of Example 2.3 when $l = 1$.

We also conjecture that the second spectral sequence will be of more interest than the first, in general. If the model K for ΩX is constructed as in [6] and used in the second spectral sequence for $\mathrm{Inv}^*(X)$, we conjecture that $E_2 = E_\infty$. If this is true, then $\mathrm{Inv}^*(X)$ can be determined efficiently from K.

§3. PROOF OF THEOREM 1.2.

Suppose that X is a simplicial set with one zero simplex and no nondegenerate one simplex. (Any one-connected topological space is weakly equivalent to the realization of such an X.) Recall that we have the augmentation homomorphism

$$\alpha: R = \mathbf{Q}G(X) \to \mathbf{Q}$$

from the simplicial rational group ring of the Kan loop group $G(X)$ to the constant simplicial ring $\mathbf{Q}$. To prove Theorem 1.2, it suffices to show that the spectral sequence (8) of §1 can be identified as

$$\begin{aligned} E^2_{p,q} &= H_p(GL(\mathbf{Z}); H_q(B\widehat{GL}^0)(R); \mathbf{Q})) \\ &= H_p(GL(\mathbf{Z}); \mathbf{Q})\otimes\mathrm{Inv}_q(X) \end{aligned} \tag{9}$$

and it collapses.

We first observe that $\mathrm{Inv}_q(X)$ is independent of the choice of the DGA model K for ΩX. For this purpose, let us consider the category $\mathcal{D}$ of differential graded algebras (DGA's) K over $\mathbf{Q}$ such that the augmentation induces an isomorphism $K_0 \xrightarrow{\cong} \mathbf{Q}$ of the degree 0 part of K with $\mathbf{Q}$. Call a morphism $K \to K'$ an *h-isomorphism* if it induces isomorphisms

$$H_i(K) \cong H_i(K'),\quad i \geq 0. \tag{10}$$

Two DGA's are said to be *h-equivalent* if they are related by the equivalence relation generated by h-isomorphisms. It is not difficult to see that if K and K' are h-equivalent DGA's, then $\mathrm{Inv}_*(K)$ and $\mathrm{Inv}_*(K')$ are isomorphic. A DGA *model* K for ΩX is an object in $\mathcal{D}$ h-equivalent to the normalized group ring $N\mathbf{Q}G(X) = NR$ [14, p. 68]. These remarks imply that $\mathrm{Inv}_q(X)$ is independent of the choice of the model K, and we may assume that $K = N\mathbf{Q}G(X)$ in order to prove the theorem.

The first step in working with the spectral sequence is to study the local coefficient system in the E^2-term. Let $\bar{K}$ be the augmentation ideal of K. Form the DGL $M_n(\bar{K})$ of $n \times n$ matrices, and note that $GL_n(\mathbf{Z})$ acts on $M_n(\bar{K})$ by conjugation. By functoriality this carries over to an action of $GL_n(\mathbf{Z})$ on $\mathscr{C}(M_n(\bar{K}))$ and on $H_*(\mathscr{C}(M_n(\bar{K})))$.

LEMMA 3.1. *$H_*(B\widehat{GL}_n{}^0)(R);\mathbf{Q})$ is isomorphic to $H_*(\mathscr{C}(M_n(\bar{K})))$ in a way that respects the natural $GL_n(\mathbf{Z})$ actions.*

The proof of this lemma appears at the end of this section.

Before stating Lemma 3.2, recall that in §1 and §2 we described $\mathscr{C}(M_n(\bar{K}))$ and its decomposition as a representation of $GL_n(\mathbf{Z})$. In particular, there is a subcomplex of trivial representations $\mathscr{C}(M_n(\bar{K}))^{GL_n(\mathbf{Z})}$, and there are stabilizations $\mathscr{C}(M_n(\bar{K}))^{GL_n(\mathbf{Z})} \to \mathscr{C}(M_{n+1}(\bar{K}))^{GL_{n+1}(\mathbf{Z})}$ induced by $\mathscr{C}(M_n(\bar{K})) \to \mathscr{C}(M_{n+1}(\bar{K}))$. These stabilizations of the subcomplexes are isomorphisms in degrees less than n (Lemma 2.2). Therefore, we may piece the complexes $\mathscr{C}(M_n(\bar{K}))^{GL_n(\mathbf{Z})}$ together into a complex identifiable with $\mathscr{C}(M(\bar{K}))^{GL(\mathbf{Z})}$, the subcomplex of $\mathscr{C}(M(\bar{K})) = \varinjlim_n \mathscr{C}(M_n(\bar{K}))$ consisting of all trivial representations of $GL(\mathbf{Z})$ in $\mathscr{C}(M(\bar{K}))$.

LEMMA 3.2. $H_i(\mathscr{C}(M(\bar{K}))^{GL(\mathbf{Z})}) = \varinjlim_n H_i(B\widehat{GL}_n{}^0(R);\mathbf{Q})^{GL_n(\mathbf{Z})} = H_i(B\widehat{GL}{}^0(R);\mathbf{Q})^{GL(\mathbf{Z})}$.

Proof. Write $\mathscr{C}$ for $\mathscr{C}(M(\bar{K}))$ and $\mathscr{C}(n)$ for $\mathscr{C}(M_n(\bar{K}))$. It follows from Lemma 3.1 that $\mathscr{C}$ is the chain complex for computing $H_*(B\widehat{GL}{}^0(R);\mathbf{Q})$ together with the $GL(\mathbf{Z})$ action. Since the isomorphism of Lemma 3.1 sends $H_i(\mathscr{C}(n))^{GL_n(\mathbf{Z})}$ to $H_i(B\widehat{GL}_n{}^0(R);\mathbf{Q})^{GL_n(\mathbf{Z})}$ and $H_i(\mathscr{C}(n))^{GL_n(\mathbf{Z})}$ may be computed from $\mathscr{C}(n)^{GL_n(\mathbf{Z})}$ for $i < n$, we have

$$\begin{aligned} H_i(BGL^0(R);\mathbf{Q})^{GL(\mathbf{Z})} &\cong H_i(\mathscr{C})^{GL(\mathbf{Z})} \\ &\cong H_i(\mathscr{C}^{GL(\mathbf{Z})} \\ &\cong H_i(\mathscr{C}(n)^{GL_n(\mathbf{Z})}) \\ &\cong H_i(\mathscr{C}(n))^{GL_n(\mathbf{Z})} \\ &\cong H_i(B\widehat{GL}_n{}^0(R);\mathbf{Q})^{GL_n(\mathbf{Z})} \end{aligned}$$

for $i < n$. This proves the lemma.

Now we return to the spectral sequence of the fibration (7).

LEMMA 3.3. *The homology Serre spectral sequence of the fibration* (1) *has the* E^2*-term given by*

$$E^2_{p,q} = H_p(GL(\mathbf{Z}); \mathbf{Q}) \otimes \mathrm{Inv}_q(X).$$

Proof. Since homology commutes with direct limits, we have

$$\begin{aligned} E^2_{p,q} &= \varinjlim_n H_p(GL_n(\mathbf{Z}); H_q(B\widehat{GL}_n{}^0(R); \mathbf{Q})) \\ &= \varinjlim_n H_p(GL_n(\mathbf{Z}); H_q(\mathscr{C}(n))) \end{aligned}$$

by Lemma 3.1. But by the vanishing theorem of [2] we have that for $p+q \ll n$

$$\begin{aligned} H_p(GL_n(\mathbf{Z}); H_q(\mathscr{C}(n))) &= H_p(GL_n(\mathbf{Z}); H_q(\mathscr{C}(n)^{GL_n(\mathbf{Z})})) \\ &\cong H_p(GL_n(\mathbf{Z}); \mathbf{Q}) \otimes H_q(\mathscr{C}(n)^{GL_n(\mathbf{Z})}) \\ &\cong H_p(GL(\mathbf{Z}); \mathbf{Q}) \otimes \mathrm{Inv}_q(X) \end{aligned}$$

by definition of $\mathrm{Inv}_q(X)$.

The proof of Theorem 1.2 is completed with the observation that the spectral sequence of Lemma 3.3 collapses: $E^2 = E^\infty$. This is seen as follows. Consider the commutative diagram of fibrations

$$\begin{tikzcd} B\widehat{GL}^0(R) \arrow[r] \arrow[d, "a"] & B\widehat{GL}(R) \arrow[r, shift right, "p_1"'] \arrow[d, "b"] & BGL(\mathbf{Z}) \arrow[l, shift right, "s_1"'] \arrow[d, "c"] \\ F \arrow[r] & BGL(R)^+ \arrow[r, shift right, "p_2"'] & BGL(\mathbf{Z})^+ \arrow[l, shift right, "s_2"'] \end{tikzcd} \tag{12}$$

where s_1 is the obvious cross-section induced by $1 \to G(X)$, the maps p_2 and s_2 are derived from p_1 and s_1, respectively, and F is the homotopy fibre of the map p_2. (For instance, the section s_2 can be gotten as follows. Consider first

$$\begin{tikzcd} BGL(R) \arrow[r, shift right, "p_1"'] & BGL(\mathbf{Z}). \arrow[l, shift right, "s_1"'] \end{tikzcd}$$

The attachement of cells to $BGL(\mathbf{Z})$ to form $BGL(\mathbf{Z})^+$ we carry up to $BGL(\mathbf{R})$ by s_1. This procedure gives us

$$\begin{tikzcd} BGL(R)^+ \arrow[r, shift right, "p_2"'] & BGL(\mathbf{Z})^+ \arrow[l, shift right, "s_2"'] \end{tikzcd}$$

where s_2 is a section of p_2.) The projection $p_2: A(X) \to BGL(\mathbf{Z})^+$ is a map of infinite loop spaces, and the fiber F is also an infinite space. Using the cross section $s_2: BGL(\mathbf{Z})^+ \to A(X)$, we can define a homotopy equivalence

$F \times BGL(\mathbf{Z})^+ \to A(X)$ sending (f, x) to $f \cdot s_2(x)$, where $\cdot$ denotes the multiplication on $A(X)$. Therefore, the commutative diagram (12) may be replaced by the commutative diagram

$$\begin{tikzcd}
B\widehat{GL}^0(R) \arrow[r] \arrow[d, "a"] & B\widehat{GL}(R) \arrow[r, shift right, "p_1"'] \arrow[d, "b'"] & BGL(\mathbf{Z}) \arrow[l, shift right, "s_1"'] \arrow[d, "c"] \\
F \arrow[r] & F \times BGL(\mathbf{Z})^+ \arrow[r, "pr_2"] & BGL(\mathbf{Z})^+
\end{tikzcd} \tag{12'}$$

where pr_2 denotes the projection onto the second factor of $F \times BGL(\mathbf{Z})^+$. The spectral sequence of the first row maps to the trivial spectral sequence of the second row, which has $E^2_{p,q} = E^\infty_{p,q} = H_p(GL(\mathbf{Z})) \otimes H_q(F; \mathbf{Q})$. Since b and hence b' are homology equivalences, it follows from the comparison theorem for spectral sequences [15] that the two spectral sequences are isomorphic and $\mathrm{Inv}_q(X) \xrightarrow{\cong} H_q(F; \mathbf{Q})$. Thus the spectral sequences of the first row collapses because the one of the second row does. So we obtain our theorem:

$$H_{p+q}(A(X); \mathbf{Q}) \cong \sum_{p+q} H_p(BGL(\mathbf{Z}); \mathbf{Q}) \otimes \mathrm{Inv}_q(X). \tag{13}$$

We now make a comment on the computation of $\mathrm{Inv}_*(X)$. The proof of the theorem has used the particular *DGA* $K = N\mathbf{Q}G(X)$. However, in computing $\mathrm{Inv}_*(X)$ we may replace K by any *DGA* h-equivalent to K. In an appendix we construct h-equivalences of K to certain other *DGA*'s, in particular, to $C_*(\Omega|X|: \mathbf{Q})$. Then by [5, 6]$C^*(\Omega|X|; \mathbf{Q})$ is h-equivalent to a free associative algebra with differential.

Before proceeding to the proof of 3.1 we remark on the proof of Proposition 1.3. As noted previously, the existence of the spectral sequences is established by standard considerations of bicomplexes. Then 1.3(b) is essentially obvious. For part (c), note that if X is a product of Eilenberg–MacLane spaces, then we may choose a DGA model K for ΩX which is a graded-commutative algebra. Since $H_*(K) = H_*(\Omega X; \mathbf{Q})$ is a free graded-commutative algebra, there is a map of DGA's

$$l: H_*(K) \to K$$

lifting homology classes to representative cycles. ($H_*(K)$ is viewed as a *DGA* with vanishing differential.) This map induces *DGL* maps

$$M_n(\bar{H}_*(K)) \to M_n(\bar{K})$$

and

$$M(\bar{H}_*(K)) \to M(\bar{K})$$

which give isomorphisms on passing to homology. Of course, the $GL(\mathbf{Z})$ actions are respected. Therefore, there is a map

$$\mathscr{C}(M(\bar{H}_*K))^{GL(\mathbf{Z})} \to \mathscr{C}(M(\bar{K}))^{GL(\mathbf{Z})}$$

which induces a map of the respective spectral sequences, and it is an isomorphism on the E^2-terms. The first spectral sequence collapses, however, because one of the differentials of the bicomplex is zero.

As to the second half of (c), recall that $H_*(\Omega\Sigma Y;\mathbb{Q})$ is a free associative algebra by the Bott–Samelson Theorem[16]. The argument proceeds similarly, for lifting generators of $H_*(\Omega\Sigma Y;\mathbb{Q})$ to representative cycles in any algebra K modeling $\Omega\Sigma Y$ gives an equivalence of *DGA*'s which we use exactly as before.

Proceeding now to the proof of 3.1, let $\mathfrak{A} = Ker(\alpha:\mathbb{Q}G(X)\to\mathbb{Q})$ be the augmentation ideal, and write $\hat{R}$ for the completion of $R=\mathbb{Q}G(X)$ with respect to powers of $\mathfrak{A}$.

PROPOSITION 3.4. (1) *The homomorphism* p: $R\to R/\mathfrak{A}^k$ *is* $k-1$ *connected.* (2) *The homomorphism* i: $R\to\hat{R}$ *is a weak homotopy equivalence.*

(2) follows immediately from (1) by letting k tend to infinity and using an argument as in the Proof of 3.4 of [7]. Before we prove (1) we recall a result due to Curtis. Let F be the dimension-wise free group functor from the category of pointed simplicial sets to the category of simplicial groups.

Lemma 3.5.[17] *Let T be a functor from groups to abelian groups which commutes with direct limits and takes the trivial group to the trivial group. Suppose that for each* 1*-reduced simplicial set K, $T(FK)$ is q-connected. Then for each* 2*-reduced simplicial set X, $T(G(X))$ is q-connected.*

(A simplicial set Y is r-reduced if Y_i consists of a single i-simplex for $0\le i\le r-1$.)

Proof of Proposition 3.4, part (1). Let $(\bar{\mathbb{Q}}FK)^k$ be the k-th power of the augmentation ideal of $\mathbb{Q}FK$. Recall that $R=\mathbb{Q}GX$ for X 2-reduced. In view of Lemma 3.5, it is enough to show that $(W\bar{\mathbb{Q}}FK)^k$ is $(k-1)$-connected whenever K is a 1-reduced simplicial set.

Let F^+ be the dimension-wise free monoid functor from the category of pointed simplicial sets to the category of simplicial monoids. Consider the map of fibration sequences induced by the natural inclusion i: $F^+K\to FK$:

$$\begin{tikzcd}
(\bar{\mathbb{Q}}F^+K)^k \arrow[r] \arrow[d, "a"] & \mathbb{Q}F^+K \arrow[r] \arrow[d, "b"] & \mathbb{Q}F^+K/(\mathbb{Q}F^+K)^k \arrow[d, "c"] \\
(\bar{\mathbb{Q}}FK)^k \arrow[r] & \mathbb{Q}FK \arrow[r] & \mathbb{Q}FK/\bar{\mathbb{Q}}FK)^k.
\end{tikzcd} \tag{14}$$

The formula

$$\begin{aligned} 1 &= x(1+(1-x)+\ \dots\ +(1-x)^{k-1} \\ &= (1-(1-x))(1+(1-x)+\ \dots\ +(1-x)^{k-1}) \end{aligned} \tag{15}$$

shows each $x\in K$ is a unit in $\mathbb{Q}F^+K/(\bar{\mathbb{Q}}F^+K)^k$, so that there is a map d inverse to c. Therefore, the map c is an isomorphism. The homotopy map b_*: $\pi_*\mathbb{Q}F^+K\to\pi_*\bar{\mathbb{Q}}FK$ can be identified with the homology map i_*: $H_*(F^+K;\mathbb{Q})\to H_*(FK;\mathbb{Q})$. Since i is a weak equivalence[20], b is also a weak equivalence. By a long exact homotopy sequence argument and the Five Lemma, a is a weak equivalence.

To complete the proof, it suffices to show that $(\bar{\mathbb{Q}}F^+K)^k$ is $(k-1)$-connected. However, an argument along the lines of [14] shows that there is a direct sum decomposition

$$\mathbb{Q}F^+K=\mathbb{Q}\oplus\bar{\mathbb{Q}}K\oplus(\bar{\mathbb{Q}}K\otimes\bar{\mathbb{Q}}K)\oplus\dots$$

in terms of which $(\bar{\mathbb{Q}}F^{+}K)^{k}$ consists of all the summands with k or more tensor factors. (Here the symbol "$\otimes$" is used for the dimension-wise tensor product of simplicial $\mathbb{Q}$ modules, for which Dold uses "$\times$" [14].) Since $\pi_0\bar{\mathbb{Q}}K = \bar{H}_0(K;\mathbb{Q}) = 0$, the necessary connectivity result follows directly from the Eilenberg–Zilber Theorem and the Künneth formula.

Proof of 3.1. For m a natural number, consider the homomorphism $R \to R/\mathfrak{A}^m$, where $\mathfrak{A}^m$ is the m-th power of the augmentation ideal $\mathfrak{A}$ of R. $R/\mathfrak{A}^m$ is also augmented, with augmentation ideal $\mathfrak{A}/\mathfrak{A}^m$. Clearly we may construct a simplicial monoid $\widehat{GL}_n{}^0(R/\mathfrak{A}^m)$ and a simplicial Lie algebra $M_n(\mathfrak{A}/\mathfrak{A}^m)$. To $M_n(\mathfrak{A}/\mathfrak{A}^m)$ we may associate the DGC $\mathscr{C}(NM_n(\mathfrak{A}/\mathfrak{A}^m))$. It follows from Proposition 3.4 that the quotient map $R \to R/\mathfrak{A}^m$ induces $GL_n(\mathbb{Z})$-module isomorphisms

$$H_i(B\widehat{GL}_n{}^0(R);\mathbb{Q}) \to H_i(B\widehat{GL}_n{}^0(R/\mathfrak{A}^m);\mathbb{Q})$$

and

$$H_i(\mathscr{C}(M_n(\bar{K})) \to H_i(\mathscr{C}(NM_n(\mathfrak{A}/\mathfrak{A}^m)))$$

if $i < m$. But $\widehat{GL}_n{}^0(R/\mathfrak{A}^m)$ is a nilpotent simplicial group, and $M_n(\mathfrak{A}/\mathfrak{A}^m)$ is the Lie algebra associated to it by the machinery of [7, pp. 257–279]. It follows from [7, p. 210] and naturality that $H_i(B\widehat{GL}_n{}^0(R/\mathfrak{A}^m);\mathbb{Q})$ and $H_i(\mathscr{C}(NM_n(\mathfrak{A}/\mathfrak{A}^m)))$ are isomorphic as $GL_n(\mathbb{Z})$-modules. The lemma follows as we let m tend to infinity.

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APPENDIX I

In [6] Chen constructs a *DGA* equivalence between the rational singular chain complex of the Moore loop space ΩM of M and a *DGA* K which is free associative as an algebra. These *DGA*'s seem to be particularly useful for computing $\mathrm{Inv}_*(M)$ from the spectral sequence of Proposition 1.3. In this appendix we justify their utilization by sketching the construction of a chain of *DGA* equivalences linking the rational singular chains on ΩM to $N\mathbf{Q}G(X)$, where X is the Eilenberg subcomplex of the singular complex $\mathrm{Sing}(M)$ consisting of simplices with their one-skeleton at the basepoint of M.

From now on we will work in the category of compactly generated spaces. Let $B(\cdot)$ denote the Milgram classifying space of a topological monoid[18] or any other reasonable classifying space functor. We assume it known that there is a natural weak homotopy equivalence $M\to B(\Omega M)$.

Let $\mathrm{Sing}(\Omega M)$ be the singular complex of ΩX; this is a simplicial monoid, so, by the argument of [19, ii, §4, Proposition 3], it is possible to find a free simplicial monoid F together with a weak equivalence $F\to \mathrm{Sing}(\Omega M)$. It follows from [20] that the natural inclusion of F into its dimension-wise group completion $F[F^{-1}]$ is a weak equivalence.

Let H denote $F[F^{-1}]$, let $|\cdot|$ denote the realization functor[21, p. 55], and let $\bar{W}$ be the simplicial classifying space functor[21, p. 87]. Since $B(|H|)$ and $|\bar{W}H|$ are both base spaces of principal $|H|$-fibrations with contractible total space, there is a weak homotopy equivalence $|\bar{W}H|\to B(|H|)$.

There is thus a chian of weak equivalences

$$|\bar{W}H|\to B|H|\leftarrow B|F|\to B|\mathrm{sing}\Omega M|\to B\Omega M \tag{17}$$

where the far right-hand one comes from the adjointness of $|\cdot|$ and $\mathrm{sing}(\cdot)$. Using this chain we can lift the weak equivalence $M\to B\Omega M$ to a weak equivalence $M\to|\bar{W}H|$. It follows that there are weak equivalences[21, p.33]

$$X = E_2\mathrm{sing}M\to E_1\mathrm{sing}|\bar{W}H|\leftarrow \bar{W}H \tag{18}$$

which induce weak equivalences of simplicial groups

$$GX\to GE_1\mathrm{sing}|\bar{W}H|\leftarrow G\bar{W}H. \tag{19}$$

Adjointness[21, p. 122] provides a weak equivalence of simplicial groups

$$G\bar{W}H\to H \tag{20}$$

while the argument above gives weak equivalences

$$H\leftarrow F\to \mathrm{sing}\Omega X. \tag{21}$$

The desired result is deduced by applying the normalized rational monoid ring functor $N\mathbf{Q}(\cdot)$ to diagrams (19)–(21).

APPENDIX II

We have already mentioned that the basic tools for computing $\mathrm{Inv}_*(X)$ are the spectral sequences of Proposition 1.3, and that a good choice of a *DGA* K modeling ΩX promotes the spectral sequences from theoretical instruments to calculational devices. In most of the examples our computations are limited by the present elementary level of our technique. We have a good description of the bigraded vector space $\mathrm{Inv}\mathscr{C}(M_n(\bar{K}))$ of invariant functions of $\mathscr{C}(M_n(\bar{K}))$, and we simply calculate the vertical and horizontal differentials δ_1 and δ_2 from the rule

$$\delta_i f([x_1|\ldots|x_n]) = f(\mathrm{d}_i[x_1|\ldots|x_n]) \tag{22}$$

where

$$\mathrm{d}_1[x_1|\ldots|x_N] = \sum_{1\le i\le N}(-1)^{\delta(i)}[x_1|\ldots|\mathrm{d}x_i|\ldots|x_N] \tag{23}$$

with

$$\delta(i) = (|x_1|+1) + \ldots + (|x_{i-1}|+1) + 1(|x_j| = \deg x_j),$$

and

$$\mathrm{d}_2[x_1|\ldots|x_N] = \sum_{N\ge i\ge 1}\sum_{k\ge 1}(-1)^{\epsilon(i,k)}[x_1|\ldots|\hat{x}_i|\ldots|x_{i+k-1}|[x_i, x_{i+k}]|\ldots|x_N] \tag{24}$$

with

$$\epsilon(i,k) = (|x_1|+1)\ldots(|x_{i+k-1}|+1) + (|x_i|+1)\{(|x_{i+1}|+1) + \ldots + (|x_{i+k-1}|+1)\}.$$

Then we try to work out the spectral sequences.

In spite of the primitivity of this technique, we can see hope for eventually obtaining a closed formula for the Poincaré series of $\mathrm{Inv}_*(X)$ in many more cases. This is why we have called attention to the models of [5, 6] for loop spaces. We conjecture that the second spectral sequence constructed from such input data collapses: $E_2 = E_\infty$. We suspect that from this the Poincaré series could be obtained with a moderate effort.

Before presenting two illustrative examples of computations of δ_2 we recall notation from §2. Write $(m)^0 = (m_1, \ldots, m_k)^0$ for a sequence of homogeneous elements of even degree, and $(m)^1 = (m_{k+1}, \ldots, m_{k+l})^1$ for a sequence of homogeneous elements of odd degree. We denoted $\mathscr{C}_{(p)(m)^0;(q)(m)^1}$ the subrepresentation of $\mathscr{C}(M_n(\bar{K}))$ isomorphic to

$$\Lambda^{p_1}(ad_n)\otimes\ldots\otimes\Lambda^{p_k}(ad_n)\otimes S^{q_1}(ad_n)\otimes\ldots\otimes S^{q_l}(ad_n)$$

and an element of it by

$$[M_1m_1|\ldots|M_{p_1}m_1|\ldots|M_{N-q_l+1}m_{k+l}|\ldots|M_Nm_{k+l}] \tag{25}$$

where $N = \Sigma p_i + \Sigma q_j$ and each M_m is a rational matrix.

The first example (see Example 2.4) is for $K = E(x_1, x_2)$, the exterior algebra on x_1 in degree 1 and x_2 in degree 3 with $\mathrm{d}x_1 = \mathrm{d}x_2 = 0$, and the second (see Example 2.5) is $K = P[x]$, the polynomial algebra on x in $2n$ with $\mathrm{d}x = 0$. The first example illustrates the resolution of the primary problem in the description of δ_2, and the second illustrates a resolution of the secondary problem: bookkeeping.

Example 1. $\bar{K}$ has a basis $\{x_1, x_2, x_1x_2\}$, and therefore $\mathscr{C}(M_n(\bar{K}))$ is a direct sum of representations

$$\mathscr{C}_{t;r,s} = \mathscr{C}_{(t),(x_1x_2)^0;(r,s)(x_1,x_2)^1}$$

where a typical element may be written

$$[M_1x_1| \dots |M_rx_1|M_{r+1}x_2| \dots |M_{r+s}x_2|M_{r+s+1}x_1x_2| \dots |M_{r+s+t}x_1x_2]. \tag{26}$$

Since $0 = \mathrm{d}\colon K \to K$, $\delta_1 = 0$. In terms of the decomposition

$$\mathrm{Inv}^{\mathrm{d}}\mathscr{C}(M(\bar{K})) = \bigoplus_{\mathrm{d}=2r+4s+5t} \mathrm{Inv}\,\mathscr{C}_{t;r,s}$$

d_2 induces a differential $\delta_2 = \bigoplus \delta_2^{t;r,s}$.

$$\delta_2^{t;r,s}\colon \mathrm{Inv}\,\mathscr{C}_{t;r,s} \to \mathrm{Inv}\,\mathscr{C}_{t-1;r+1,s+1} \tag{27}$$

is given by the formula

$$\delta_2^{t;r,s} Q_{t;r,s}\mu_\sigma = \frac{(s+1)(r+1)}{t}\sum_{k=1}^{t}(-1)^k Q_{t-1;r+1,s+1}(\mu_{i^+_{r,s}}(\tau_k\sigma\tau_k^{-1}) - \mu_{i^-_{r,s}}(\tau_k\sigma\tau_k^{-1})) \tag{28}$$

where $\tau_k = (r+s+1 \dots r+s+k) \in S_N$ $(N = r+s+t)$ and $i^+_{r,s}, i^-_{r,s}$ are certain set maps $S_N \to S_{N+1}$. $i^+_{r,s}$ is the composition

$$\begin{tikzcd} S_N \arrow[r, "\tau_{r,s}"] & S_N \arrow[r, "\hat{1}"] & S_{N+1} \arrow[r, "\lambda(1,r+2)"] & S_{N+1}, \end{tikzcd}$$

and $i^-_{r,s}$ is the composition

$$\begin{tikzcd} S_N \arrow[r, "\tau_{r,s}"] & S_N \arrow[r, "\hat{1}"] & S_{N+1} \arrow[r, "\rho(1,r+2)"] & S_{N+1}, \end{tikzcd}$$

where $\tau_{r,s}(\sigma) = (r+1 \dots r+s+1)(\sigma)(r+s+1 \dots r+1)$, $\hat{1}$ is the embedding onto permutations fixing 1, and $\lambda(1, r+2)$ and $\rho(1, r+2)$ are the left and right translations by $(1, r+2) \in S_{N+1}$.

The formula arises, of course, from interpretation of the definition $\delta_2(Q_{t;r,s}\mu_\sigma) = Q_{t;r,s}\mu_\sigma \circ \mathrm{d}_2$. It is helpful to note that the groups $S_r \times S_s \times S_t$ and $S_{r+1} \times S_{s+1} \times S_{t-1}$ have a subgroup $S_r \times S_s \times S_{t-1}$ "in common". Under the embedding of $S_r \times S_s \times S_t$ in S_N suggested by (26), $S_r \times S_s \times S_{t-1}$ is the subgroup fixing $r+s+1$ and the $\{\tau_k\}$ are right coset representatives. Arrange the embedding of $S_r \times S_s \times S_{t-1}$ in $S_{r+1} \times S_{s+1} \times S_{t-1}$ so that when the latter is embedded in S_{N+1} in the obvious way, the former fixes 1 and $r+2$. Then

$$\{(1 \dots p)(r+2 \dots r+1+q) | 1 \le p \le r+1, 1 \le q \le s+1\}$$

may be used as left coset representatives.

Example 2. In $K = P[x]$, $\bar{K}$ has the basis $\{x, x^2, \dots\}$. $\mathscr{C}(M_n(\bar{K})) = \bigoplus \mathscr{C}_{(r)}$ where

$$\mathscr{C}_{(r)} = \mathscr{C}_{(r_1, r_2, \dots);(x, x^2, \dots)^0}$$

is the space of elements which are linear combinations of elements of the form

$$[M_1 x | \dots | M_{r_1} x | M_{r_1+1} x^2 | \dots | M_{r_1+r_2} x^2 | \dots].$$

Again, $\delta_1 = 0$, and one sees that $\delta_2 = \bigoplus \delta_2^{(r)}$, where

$$\delta_2^{(r)}: \operatorname{Inv} \mathscr{C}_{(r)} \to \bigoplus_{r_k \geq 1} \bigoplus_{\substack{i+j=k \\ i \leq j}} \operatorname{Inv} \mathscr{C}_{(r)(i,j)}. \tag{29}$$

Here $r(i,j) = (\dots, r_i + 1, \dots, r_j + 1, \dots, r_k - 1, \dots)$ if $i < j$ and $r(i,i) = (\dots, r_i + 2, \dots, r_k - 1, \dots)$. Let $R_n = r_1 + \dots + r_{n-1}$, $R_1 = 0$, $R_\infty = \sum_{i=1}^{\infty} r_i$.
Then the component

$$\operatorname{Inv} \mathscr{C}_{(r)} \to \operatorname{Inv} \mathscr{C}_{r(i,j)}$$

is given by

$$Q_{(r)} \mu_\sigma \to \sum_{q=1}^{r_k} \frac{(r_i + 1)(r_j + 1)}{r_k} (-1)^{R_i + R_j + R_k + q} Q_{(r)(i,j)} \times \left(\mu_{i^+_{r(i,j)}(\tau_q \sigma \tau_q^{-1})} - \mu_{i^-_{r(i,j)}(\tau_q \sigma \tau_q^{-1})}\right) \tag{30}$$

where $\tau_q = (R_k + 1 \dots R_k + q) \in S_{R_\infty}$ and $i^+_{r(i,j)}$ and $i^-_{r(i,j)}$: $S_{R_\infty} \to S_{R_\infty + 1}$ are embeddings given by the compositions

$$\begin{tikzcd} S_{R_\infty} \arrow[r, "\tau_{ij}"] & S_{R_\infty} \arrow[r, "\widehat{R_i+1}"] & S_{R_\infty+1} \arrow[r, "\lambda(R_i+1\, R_j+2)"] & S_{R_\infty+1} \end{tikzcd}$$

and

$$\begin{tikzcd} S_{R_\infty} \arrow[r, "\tau_{ij}"] & S_{R_\infty} \arrow[r, "\widehat{R_i+1}"] & S_{R_\infty+1} \arrow[r, "\varphi(R_i+1\, R_j+2)"] & S_{R_\infty+1}. \end{tikzcd}$$

$\tau_{ij}(\sigma) = (R_j + 1 \dots R_{i+j} + 1)\sigma(R_{i+j} + 1 \dots R_j + 1)$, $\widehat{R_i + 1}$ is the embedding onto permutations fixing $R_i + 1$, and $\lambda(R_i + 1\ R_j + 2)$ and $\rho(R_i + 1\ R_j + 2)$ are the left and right translations by $(R_i + 1\ R_j + 2)$. (If $i = j$, the only change is to write $\frac{(r_i + 2)(r_i + 1)}{r_k}$ instead of $\frac{(r_i + 1)(r_j + 1)}{r_k}$.)

This example indicates what bookkeeping means in general. One must keep track of the components of δ_2 and the signs. The first problem becomes easier if K is assumed to be free of relations except possibly relations expressing graded commutativity; the second requires a choice of a normal form for writing down elements of $\mathscr{C}(M_n(\bar{K}))$. We used in both examples the obvious forms requiring monomials of K to appear in order of increasing degree.

We have yet to comment on the differential δ_1. In view of 1.3(b) one would need to know δ_1 explicitly only for the sake of determining the E_2 term of the second spectral sequence. We will make no further comment here.