

BOUSFIELD LOCALIZATIONS OF CLASSIFYING SPACES OF NILPOTENT GROUPS

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1. INTRODUCTION

Let G be a finitely generated nilpotent group. The object of this paper is to identify the Bousfield localization $L_h BG$ of the classifying space BG with respect to a multiplicative complex oriented homology theory h_* . We show that $L_h BG$ is the same as the localization of BG with respect to the ordinary homology theory determined by the ring h_0 . This is similar to what happens when one localizes a space X with respect to a connected ring theory E : it follows from results of Bousfield [Bou79, Theorem 3.1] that $L_E X$ is the localization of X with respect to ordinary homology with coefficients in the ring E_0 . The point in this paper is that we do not require the spectrum h to be connected.

Our main result is

Theorem 1. *Let G be a finitely generated nilpotent group, and let h_* be a multiplicative complex oriented homology theory. Then $L_h BG = L_R BG$, where R is the ring h_0 and L_R is localization with respect to the ordinary homology theory determined by R .*

The hypothesis that h_* be multiplicative is not essential to any of our arguments. We include it mainly to avoid cumbersome statements, and because most complex oriented theories of interest, such as Morava K-theory, are multiplicative. By modifying somewhat the results it seems that one could remove the assumption that h_* is multiplicative. However, do not consider this case. Complex orientability *is* used in an essential way, in the proof of Theorem 3.

Our method of proof is to begin with finite p -groups and proceed by induction on the order of the group. We show that if G has a normal subgroup H such that BH is h_* -local, $G/H = \mathbf{Z}/p$, and $B(G/H)$ is R -local for $R = h_0$, then BG is h_* -local. We do this by studying the fibration

$$BH \longrightarrow BG \longrightarrow B(G/H) = B\mathbf{Z}/p$$

To pass to arbitrary finitely generated nilpotent groups, we use the arithmetic square decomposition of $L_h X$ due to Mislin and Bousfield.

Our main task is to study the fibration displayed above. In general it is not true that if the base and fibre of a fibration are local with respect to some homology theory, then the total space is also local. For example in the fibration

$$S^2 \longrightarrow \mathbf{R}P^2 \longrightarrow B\mathbf{Z}/(2)$$

both fiber and base are local with respect to ordinary integral homology, but the total space is not [BK72], [DDK77].

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Our technique for dealing with this problem is to use the following lemma. If B is a space and \mathcal{C} is a class of fibrations over B , say that \mathcal{C} has h_* -accessible fibres if any h_* -equivalence (over B) between fibrations in \mathcal{C} induces an h_* -equivalence on fibres.

Lemma 2. *Let h_* be an arbitrary homology theory, and consider the diagram*

$$\begin{array}{ccccc} F & \longrightarrow & E & \xrightarrow{\pi} & B \\ f \downarrow & & g \downarrow & & \parallel \\ F' & \longrightarrow & L_h E & \xrightarrow{L_h \pi} & B \end{array}$$

in which each row is a fiber sequence. Suppose that B and F are both h_* -local, and that there exists some class \mathcal{C} of fibrations over B which has h_* -accessible fibres and contains both π and $L_h \pi$. Then E is h_* -local.

Proof. Since \mathcal{C} has h_* -accessible fibres and $h_*(g)$ is an equivalence, $h_*(f)$ is also an equivalence. The space F is h_* -local by assumption, and F' is h_* -local since it is the homotopy fiber of a map between h_* -local spaces. Therefore f , being an h_* -equivalence between h_* -local spaces, is an equivalence. It follows that g is also an equivalence and E is h_* -local. \square

In order to use this lemma, we show in the next section that if h_* is a multiplicative complex oriented homology theory with the property that h_0 is a vector space over \mathbf{Z}/p , then the class of all fibrations over $B\mathbf{Z}/p$ has h_* -accessible fibres.

2. FIBRATIONS OVER $B\mathbf{Z}/p$

In this section we prove the following theorem. The results and arguments are inspired by the work of Kriz [Kri].

Theorem 3. *Let h_* be a multiplicative complex oriented homology theory such that the ring h_0 is a mod p vector space. Suppose that E and E' are fibrations over $B\mathbf{Z}/p$ with fibres F and F' , respectively, and that $f : E \rightarrow E'$ is a map over $B\mathbf{Z}/p$ which induces an isomorphism $h_* E \cong h_* E'$. Then f also induces an isomorphism $h_* F \cong h_* F'$.*

Recall that the *homotopy coequalizer* of a pair of maps $f, g : X \rightarrow Y$ is obtained by taking the cylinder $X \times [0, 1]$ and gluing one end to Y by f and the other end to Y by g . This construction is sometimes also called the *double mapping cylinder* of f and g . Given maps of pairs $f, g : (X, A) \rightarrow (Y, B)$, the homotopy coequalizer of f and g is the pair (Z, C) , where Z is the homotopy coequalizer of the two maps $X \rightarrow Y$ and C is the homotopy coequalizer of the two maps $B \rightarrow C$. A diagram of pairs equivalent to

$$(X, A) \xrightarrow{f, g} (Y, B) \longrightarrow (Z, C)$$

is said to be a *homotopy coequalizer diagram*. The following lemma is elementary.

Lemma 4. *Let*

$$(X, A) \xrightarrow{f, g} (Y, B) \longrightarrow (Z, C)$$

be a homotopy coequalizer diagram. Suppose that h_* is a homology theory. Then there is a natural long exact sequence

$$\cdots \longrightarrow h_i(X, A) \xrightarrow{f_* - g_*} h_i(Y, B) \longrightarrow h_i(Z, C) \longrightarrow h_{i-1}(X, A) \longrightarrow \cdots$$

Lemma 5. *Let G be a group of order p with generator g , and V a mod p vector space with an action of G . Then the endomorphism $(1 - g)$ of V is nilpotent (in the sense that for some integer k , $(1 - g)^k = 0$). In particular, the kernel of $(1 - g) : V \rightarrow V$ is nontrivial.*

Proof. It is possible to choose $k = p$, since, in view of the fact that we are working mod p , $(1 - g)^p = 1 - g^p = 0$. \square

Lemma 6. *Suppose that h_* is a multiplicative complex oriented homology theory. Consider a homotopy fibre square*

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{g} & \tilde{E}' \\ q \downarrow & & \downarrow q' \\ E & \xrightarrow{f} & E' \end{array}$$

in which q and q' are principal S^1 -bundles. If f induces an isomorphism on h_* , then so does g .

Proof. Let ξ' be the complex line bundle over E' associated to q' and ξ the complex line bundle over E associated to q . Denote the Thom spaces of these bundles by $M(\xi)$ and $M(\xi')$ respectively. There is a map of cofibration sequences

$$\begin{array}{ccccc} \tilde{E} & \xrightarrow{q} & E & \longrightarrow & M(\xi) \\ g \downarrow & & \downarrow f & & \downarrow M(f) \\ \tilde{E}' & \xrightarrow{q'} & E' & \longrightarrow & M(\xi') \end{array}$$

The map $h_*(f)$ is an isomorphism by hypothesis. By the Thom isomorphism for h_* , the map $h_*(M(f))$ can be identified with $h_*(f)$ and so it too is an isomorphism. The fact that $h_*(g)$ is an isomorphism follows from looking at long exact homology sequences and using the five lemma. \square

Proof of Theorem 3. Let G denote the group \mathbf{Z}/p , and $g \in G$ some chosen generator. We can assume that F and F' are G -spaces, and that f is obtained up to homotopy by taking the Borel construction $\beta(f)$ on a G -map $F \rightarrow F'$. (One way to obtain a suitable G -space equivalent to F , for instance, is to take the pullback over $E \rightarrow BG$ of the universal cover of BG .)

Suppose that X is a G -space (in our case either F or F'). Note that the homotopy coequalizer of the G -maps $1, g : G \rightarrow G$ is the circle S^1 with the usual rotation action of G . More generally, the homotopy coequalizer of $1, g : X \times G \rightarrow X \times G$ is the product G -space $X \times S^1$. Taking Borel constructions gives a homotopy coequalizer diagram

$$\beta(X \times G) \xrightarrow{u, v} \beta(X \times G) \longrightarrow \beta(X \times S^1)$$

where u and v are the appropriate induced Borel construction maps. It is clear that $\beta(X \times S^1)$ is the total space of a principal S^1 -bundle over $\beta(X)$, in fact, the total space of the pullback along the map $\beta(X) \rightarrow BG$ of the usual principal S^1 -bundle $\beta(S^1) \rightarrow BG$. The Borel construction $\beta(X \times G)$, on the other hand, can be identified up to homotopy with X itself in such a way that u and v can be identified with the original maps 1 and g .

Let \tilde{E} denote the Borel construction $B(G, F \times S^1)$ and \tilde{E}' the Borel construction $B(G, F' \times S^1)$. According to the above considerations we have a homotopy coequalizer diagram

$$(F', F) \xrightarrow{1, g} (F', F) \longrightarrow (\tilde{E}', \tilde{E})$$

Since $h_*(E', E)$ vanishes by assumption, $h_*(\tilde{E}', \tilde{E})$ vanishes by Lemma 6. Lemma 4 then implies that the endomorphism $(1 - g_*)$ of $h_*(F', F)$ is an automorphism, but by Lemma 5 this can happen only if $h_*(F', F) = 0$, in other words, only if the map $F \rightarrow F'$ induces an h_* -isomorphism. \square

3. LOCALIZATION OF CLASSIFYING SPACES

We begin by recalling a result of Bousfield [Bou82] about localizations of $B\mathbf{Z}/p$. (In that paper he actually determines $L_h K(A, n)$ for any homology theory h_* and any abelian group A .) We say that a space is h_* -acyclic if the reduced homology $\tilde{h}_*(X)$ vanishes, or equivalently if $L_h(X)$ is contractible.

Lemma 7. *If h_* is a multiplicative homology theory, then the space $B\mathbf{Z}/p$ is h_* -acyclic if p is invertible in h_0 and h_* -local otherwise. Equivalently, $L_h B\mathbf{Z}/p = L_R B\mathbf{Z}/p$, where $R = h_0$ and L_R denotes localization with respect to $H_*(-; R)$.*

With Theorem 3 in hand we can prove the following.

Theorem 8. *Suppose that h_* is a multiplicative complex oriented homology theory, and that G is a finite p -group. The space BG is h_* -acyclic if p is invertible in h_0 and h_* -local otherwise.*

Proof. If p is invertible in h_0 it is obvious from the Atiyah-Hirzebruch spectral sequence that BG is h_* -acyclic, so we assume that p is not invertible in h_0 and prove that BG is h_* -local. Suppose first that h_0 is a \mathbf{Z}/p -vector space. We argue by induction on the order of G . Let $H \subset G$ be a normal subgroup of index p . The space $B\mathbf{Z}/p$ is h_* -local by Lemma 7, and so there is a diagram of fibration sequences:

$$\begin{array}{ccccc} BH & \longrightarrow & BG & \longrightarrow & B\mathbf{Z}/p \\ \downarrow f & & \downarrow g & & \parallel \\ F' & \longrightarrow & L_h BG & \longrightarrow & B\mathbf{Z}/p. \end{array}$$

It thus follows immediately from Lemma 2 and Theorem 3 that BG is h_* -local.

Now consider a general h of the specified type. For a prime q , let h/q denote the smash product of the spectrum h representing h_* with a mod q Moore spectrum

denoted here by M . Clearly h/q is still complex orientable: This is true since a complex orientation for a spectrum E is a class $x \in E^2(CP^\infty)$ with certain

properties. One has a map $E \rightarrow E \wedge M$ induced by the unit in M , and one can use the image of x under this map as a complex orientation for $E \wedge M$.

Alternatively, E is complex orientable iff it is an MU -module spectrum. If E is an MU -module spectrum, so is $E \wedge M$.

If X is a space, let $X_{\mathbf{Q}}$ denote the localization of X with respect to rational homology. Since $(BG)_{\mathbf{Q}} = *$, it follows from Proposition 7.2 of [Bou82] that we have a fibration sequence

$$L_h BG \longrightarrow \prod_q L_{h/q} BG \longrightarrow \left(\prod_q L_{h/q} BG \right)_{\mathbf{Q}} = \text{pt.}$$

where q runs through the primes not invertible in h_0 . We know from above that $L_{h/p} BG = BG$ and that $L_{h/q} BG = \text{pt.}$ for $q \neq p$. It follows that $L_h BG = BG$ as claimed. \square

Slightly more generally we have

Theorem 9. *Suppose that G be a finite nilpotent group and that h is a multiplicative complex oriented homology theory. Then $L_h BG = L_R BG$, where L_R is as in Lemma 7. In particular, if no prime dividing the order of G is invertible in h_0 , then BG is h_* -local.*

Proof. The group G is the direct product of its Sylow p -subgroups G_p , so we have $BG \simeq \prod_p BG_p$ and $L_h BG \simeq \prod_p L_h BG_p$. The factors in this second product can be identified with the help of Theorem 8. There is a similar product formula for $L_R BG$. \square

We now turn to the proof of the main theorem.

Proof of Theorem 1. It is shown by Bousfield in [Bou82] that for any space X , $L_h X \simeq L_h L_R X$ where L_R is localization with respect to $H_*(-; R)$. It is easy to check that a map of spaces is an isomorphism on $H_*(-; R)$ if and only if it is an isomorphism on $\oplus_p H_*(-; \mathbf{Z}/p \otimes R)$ as well as an isomorphism on $H_*(-; \mathbf{Q} \otimes R)$. Let \mathcal{P} be the set of all primes which are not invertible in R . It follows that a map of spaces is an isomorphism on $H_*(-; R)$ if and only if it is an isomorphism on $H_*(-; \oplus_{p \in \mathcal{P}} \mathbf{Z}/p)$, as well as, if $\mathbf{Q} \otimes R \neq 0$, an isomorphism on $H_*(-; \mathbf{Q})$. Since BG is a nilpotent space, the results of [DDK77] imply that if $\mathbf{Q} \otimes R = 0$ there is an equivalence

$$L_R BG \simeq \prod_{p \in \mathcal{P}} L_{\mathbf{Z}/p} BG$$

while if $\mathbf{Q} \otimes R \neq 0$ there is a homotopy fibre square

$$\begin{array}{ccc} L_R BG & \longrightarrow & \prod_{p \in \mathcal{P}} L_{\mathbf{Z}/p} BG \\ \downarrow & & \downarrow \\ (BG)_{\mathbf{Q}} & \longrightarrow & \left(\prod_{p \in \mathcal{P}} L_{\mathbf{Z}/p} BG \right)_{\mathbf{Q}}. \end{array}$$

We will carry out the proof by showing that $L_R BG$ is h_* -local, so that $L_h BG \simeq L_h L_R BG \simeq L_R BG$. To do this we will show that all of the constituents in the above formulas for $L_R BG$ are h_* -local, and then appeal to the fact that the class of h_* -local spaces is closed under homotopy inverse limit constructions.

Now according to [BK72, VI 2.6, 2.2 and IV §2] the space $L_{\mathbf{Z}/p}BG \simeq (\mathbf{Z}/p)_{\infty}BG$ can be identified as $B(\hat{G}_p)$, where $\hat{G}_p = \lim G/\Gamma_s^p G$ is the p -lower-central-series completion of G . In particular $L_{\mathbf{Z}/p}BG$ is equivalent to the homotopy inverse limit of the tower $\{B(G/\Gamma_s^p G)\}_s$. If $p \in \mathcal{P}$ then each space in this tower is h_* -local (Theorem 8), and so $L_{\mathbf{Z}/p}BG$ is h_* -local by homotopy inverse limit closure. By the same principle, $\prod_{p \in \mathcal{P}} L_{\mathbf{Z}/p}BG$ is h_* -local.

We can complete the proof by showing that if $\mathbf{Q} \otimes R \neq 0$ then any space W local with respect to rational homology is also local with respect to h_* . Given the definition of what it means for a space to be h_* -local, we have to show that any h_* -equivalence $f : X \rightarrow Y$ induces a bijection $f^{\#} : [Y, W] \rightarrow [X, W]$ (where the brackets indicate homotopy classes of maps). However, by [Bou82, 3.3], such an f is a rational equivalence, so the fact that $f^{\#}$ is a bijection follows from the fact that W is local with respect to rational homology. \square

4. POSSIBLE EXTENSIONS AND RELATED PROBLEMS

It was shown above that the Bousfield localization with respect to certain homology theories of the classifying space BG of a finitely generated nilpotent group G is the same as the localization with respect to a classical homology theory with appropriate coefficients. The question remains open for other (non finitely generated) nilpotent groups and other localization functors. Using the fact that $K(F, 2)$, where F is any free abelian group, is local with respect to complex K -theory it is not hard to see that so is $K(G, 1)$ for any abelian group G and in fact one can show that theorem 1 holds for any abelian group.

To go beyond Eilenberg-MacLane spaces the following is a natural possible extension of the main results above.

Let N be a nilpotent space whose homotopy groups vanish above certain dimension n . Is it true that any Bousfield homological localization of N is equivalent to its localization with respect to a well chosen classical homology theory?

Similar questions arise beyond the realm of Bousfield homological localization. Namely, one may ask for analogues of the above questions for an arbitrary homotopical localization L_f with respect to an arbitrary map f . In that case it is not true that the localization will be the same as the localization with respect to a well chosen classical homology. This is because the map $B\mathbf{Z}/p^2 \rightarrow B\mathbf{Z}/p$, induced by the quotient group map, is in fact a homotopy localization map, but it is not an homological localization map. But one does expect that an arbitrary localization $L_f N$ of a nilpotent space N as above will also be a nilpotent space with vanishing homotopy groups above a certain dimension that depends only on n .

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