LOCAL COHOMOLOGY IN COMMUTATIVE ALGEBRA, HOMOTOPY THEORY, AND GROUP COHOMOLOGY

The notion of local cohomology was prominent in the summer school; in this volume it puts in explicit appearances in the papers of Greenlees and Huneke/Taylor, and figures at least peripherally in material related to the paper of Adem. The aim of this short account is to introduce local cohomology and, in a very sketchy way, tie together some of its various manifestations. There are some real differences to accommodate, or more accurately, to ignore: in commutative algebra the interest tends to be in local cohomology groups and their relationships to structural properties of the ambient ring, while in topology the interest is often in more abstract properties of local cohomology chain complexes (or even local cohomology spectra).

Local Cohomology as Approximation. The transition from local cohomology groups (algebra) to local cohomology chain complexes (topology) is simple; local cohomology groups are homology groups of the local cohomology chain complexes. Taking the chain complex point of view has the advantage of permitting a simple conceptual description of local cohomology. Let R be a ring, and given two chain complexes A and B of R-modules, say that B is *built from* A if B lies in the smallest class of chain complexes over R which contains A and is closed under shifts (i.e. suspensions), coproducts, exact triangles, quasi-isomorphisms, and retracts. Then there is a widely applicable principle which guarantees that any chain complex X over R has a "best approximation" $\mathrm{Cell}_A(X) \to X$ by a chain complex $\mathrm{Cell}_A(X)$ which is built from A. This is sometimes called an "A-cellular" approximation to X. What makes the approximation "best" is that the approximation map induces a quasi-isomorphism

$$\mathrm{RHom}(A, \mathrm{Cell}_A(X)) \to \mathrm{RHom}(A, X)\,. \tag{1}$$

If R happens to be commutative, $I \subset R$ is a finitely generated ideal, A is the chain complex $\langle R/I, 0\rangle$ which has R/I in degree 0 and vanishes in all other degrees, and X is $\langle M, 0\rangle$, then there are formulas

$$H_i \,\mathrm{Cell}_A(X) = \begin{cases} 0 & i > 0 \\ H_I^{-i}(M) & i < 0 \end{cases}$$

The groups on the left hand side are the homology groups of the chain complex $\mathrm{Cell}_A(X)$, while those on the left are the local cohomology groups of M at I. In other words, local cohomology is just cellular approximation turned upside down. It is the cellular approximation form which most often shows up in topology; see Greenlees, §8.

N.B. From now on we won't distinguish in notation between $\langle M, 0\rangle$ and M.

Spectra. The above discussion can be repeated almost verbatim with the ring R replaced by a general differential graded algebra. (Note that a ring is just a differential graded algebra concentrated in degree 0.) This generalization arises naturally even in purely algebraic contexts; for instance, if R is a ring and A is an R-module, the derived endomorphism complex $\mathrm{RHom}(A, A)$ of A is usefully treated as a differential graded algebra. See (Greenlees, 9.3) for a situation in which using this endomorphism complex leads to a simple formula for $\mathrm{Cell}_A(-)$ and hence for the ordinary algebraic local cohomology complex.

More ambitiously, the above discussion can be repeated with "chain complex" replaced by "spectrum" (in the sense of stable homotopy theory), "ring" replaced by "ring spectrum", and "chain complex over a ring" by "module spectrum over a ring spectrum". This allows for examples that are tied to stable stable homotopy theory or even to the geometry of finite complexes. A linguistic change is that in this more general setting the term "quasi-isomorphism" is usually replaced by "weak equivalence" or just "equivalence."

Gorenstein rings and their generalizations. According to (Huneke/Taylor, 4.3), a (commutative) local ring R of dimension d with maximal ideal $\mathfrak{m}$ is Gorenstein if there are isomorphisms

$$H^i_{\mathfrak{m}}(R) = \begin{cases} 0 & i \neq d \\ E & i = d \end{cases} \tag{2}$$

where E is the injective hull of the residue class field $R/\mathfrak{m}$. This amounts to the statement that $\mathrm{Cell}_{R/\mathfrak{m}}(R)$ is a chain complex whose homology groups vanish except for a copy of E in degree $-d$, so that there is a quasi-isomorphism

$$\mathrm{Cell}_{R/\mathfrak{m}}(R) \sim \Sigma^{-d} E. \tag{3}$$

This leads to an interesting phenomenon. As in (1) there is a quasi-isomorphism

$$\mathrm{RHom}(B, \mathrm{Cell}_{R/\mathfrak{m}}(R)) \xrightarrow{\sim} \mathrm{RHom}(B, R) \tag{4}$$

for $B = R/\mathfrak{m}$ and hence for any chain complex B over over R built from $R/\mathfrak{m}$, i.e., for any chain complex whose homology groups are $\mathfrak{m}$-primary torsion modules (Greenlees, 8.C). Now there are two ways to dualize a chain complex B over R; one is to take the ordinary (or "Spanier-Whitehead") dual $B^{\#} = \mathrm{RHom}(B, R)$ and the other is to take the Matlis dual $B^{\vee} = \mathrm{RHom}(B, E)$. The upshot of (3) and (4) is that if R is Gorenstein and B has $\mathfrak{m}$-primary torsion homology groups, these two duals agree up to a shift: $B^{\#} = \Sigma^{-d} B^{\vee}$. There is a homological reflection of this in Huneke/Taylor, 4.4(1).

The above definition of Gorenstein ring doesn't generalize easily, but there is another more tractable characterization, namely

$$(R/\mathfrak{m})^{\#} = \Sigma^{-d}(R/\mathfrak{m})^{\vee} = \Sigma^{-d}(R/\mathfrak{m})\,.$$

This is just the statement $B^{\#} = \Sigma^{-d} B^{\vee}$ in the special case $B = R/\mathfrak{m}$, but it turns out that it implies the equivalence (3) (see Greenlees, §11). This suggests defining a general map of ring objects $R \to k$ to be Gorenstein if there is an integer d such that there is an equivalence

$$\mathrm{RHom}(k, R) \sim \Sigma^{-d} k\,.$$

See (Greenlees, 11.1). The RHom here is to calculated in the category of module objects over R (chain complexes over R if R is a ring or differential graded algebra, module spectra over R if R is a ring spectrum). In practice, it is also useful to ask for another technical condition (Greenlees, 11.A). Note the disappearance of the ideal $\mathfrak{m}$. In topological situations it is difficult to talk of ideals and quotient rings, and so it is convenient to give up the idea of a quotient homomorphism $R \to R/\mathfrak{m}$ in favor of an arbitrary homomorphism $R \to k$.

It's natural to wonder about the topological fate of the injective hull E of the residue class field, which in the commutative ring case gave rise to Matlis duality. In this algebraic case, E is characterized two conditions: E is built from $R/\mathfrak{m}$, and $\mathrm{RHom}(R/\mathfrak{m}, E)$ is quasi-isomorphic to $R/\mathfrak{m}$ itself. This suggests in general considering R-module objects I such that I is built from k and $\mathrm{RHom}(k, I)$ is equivalent to k. It turns out that there can be *many* such I, and that they are determined in a natural way by right actions of the endomorphism complex $\mathrm{RHom}(k, k)$ on k. Each such I gives rise to a different notion of Matlis duality for R-modules. If $R \to k$ is Gorenstein, then the Gorenstein condition picks out a distinguished I_0, and for R-modules built from k, Spanier-Whitehead duality agrees up to a shift with the version of Matlis duality given by I_0.

Group rings and group cohomology. Suppose that Γ is a group with the property that the trivial Γ-module $\mathbb{Z}$ has a resolution of finite length by finitely generated free $\mathbb{Z}[\Gamma]$-modules. Then $\mathbb{Z}[\Gamma] \to \mathbb{Z}$ is Gorenstein in the above sense if and only if Γ is what is traditionally called a Poincaré duality group. Assume for simplicity that a suitable orientability condition holds. In this case the classifying space $B\Gamma$ satisfies ordinary Poincaré duality, which along the lines of (Huneke/Taylor, §4) guarantees that the cochain complex $C^*(B\Gamma;\mathbb{Z})$ is Gorenstein. (Note that the Poincaré duality amounts to the statement that $C_*(B\Gamma;\mathbb{Z}) = \mathrm{Hom}(C^*(B\Gamma;\mathbb{Z}),\mathbb{Z})$ is quasi-isomorphic to $C^*(B\Gamma;\mathbb{Z})$ as a module over $C^*(B\Gamma;\mathbb{Z})$, and compare with the discussion following (Huneke/Taylor, 4.8).)

Suppose for simplicity that k is a field. The above suggests the possibility of some odd connection between the question of whether $k[\Gamma] \to k$ is Gorenstein and the question of whether $C^*(B\Gamma;k) \to k$ is Gorenstein. In fact in many cases there is such a connection (Greenlees, 11.D); it is most easily stated if $k = \mathbb{F}_p$ and Γ is a finite group (and more easily proved if in addition Γ is a finite p-group). In this case $k[\Gamma] \to k$ is always Gorenstein, because $k[\Gamma]$ is injective as a module over itself, and the conclusion is that $C^*(B\Gamma;k) \to k$ is Gorenstein too. The associated cellular approximation (shifted dualizing module) is

$$\mathrm{Cell}_k(C^*(B\Gamma;k)) = \mathrm{Hom}_k(C^*(B\Gamma;k),k) = C_*(B\Gamma;k)\,, \tag{5}$$

which fits with (Huneke/Taylor, 4.7). Let $\mathfrak{m}$ be the ideal in $H^*(B\Gamma;k)$ given by the elements of strictly positive degree. Formula (5) gives rise to a local cohomology spectral sequence

$$H^*_{\mathfrak{m}}(H^*(B\Gamma;k)) \Rightarrow H_*(B\Gamma;k) \tag{6}$$

which has been studied extensively by Greenlees and reveals various commutative algebra properties of the cohomology ring $H^*(B\Gamma;k)$. Note what has happened here: a Gorenstein condition on the non-commutative ring $k[\Gamma]$ has led to a Gorenstein condition on the differential graded algebra $C^*(B\Gamma;k)$, which in turn has led to a "hyper-Gorenstein" condition on the group cohomology ring $H^*(B\Gamma;k)$, reflected in a spectral sequence (6) involving ordinary local cohomology.

There are similar sorts of results for some other cohomology theories (Greenlees, 7.1).

There are many more paths to follow. For example, it would be possible to take the ring R in question to be the suspension spectrum of the loop space of a finite complex X. In this case the augmentation map from R to the sphere spectrum is Gorenstein if and only if X satisfies Poincaré duality. The possible dualizing modules I mentioned above correspond to stable spherical fibrations over X, and the one distinguished by the Gorenstein condition is the Spivak normal bundle of X. Local cohomology also has a very interesting role to play in studying the layers of the chromatic filtration of stable homotopy theory alluded to in Jack Morava's talk at the summer school. The fact that the ideas of local cohomology, cellular approximation, Matlis duality, and Gorenstein conditions arise in what are apparently radically different contexts is part of what makes them intriguing.