Hochschild-Mitchell cohomology of simplicial categories and the cohomology of simplicial diagrams of simplical sets*

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1. INTRODUCTION

1.1. Summary. The aim of this note is to tie together three different kinds of cohomology:

I the algebraic *Hochschild-Mitchell cohomology of small categories* [1],

II the internal category-theoretic *cohomology of small simplicial categories* (with a fixed set of objects) [7], and

III the homotopy theoretic *cohomology of diagrams of simplicial sets* [5].

This is done by

(i) extending the definition of cohomology for diagrams of simplicial sets of [5] to simplicial diagrams of simplicial sets,

(ii) defining the Hochschild-Mitchell cohomology of a small simplicial category $\mathbf{C}$ as the cohomology (in the sense of (i)) of an associated $(\mathbf{C}^{op}\times\mathbf{C})$-diagram of simplicial sets (and observing that this definition indeed generalizes the one of [1]), and

(iii) proving (and this is, in some sense, our main result) that *the cohomology of a small simplicial category* [7] *is its Hochschild-Mitchell cohomology* (in the sense of (ii)) *with a shift in dimension.*

Of course, composition with the singular functor yields the corresponding notions of Hochschild-Mitchell cohomology of small topological categories (with discrete object sets) and cohomology of (see [8]) topological diagrams of topological spaces.

1.2. Remark. That (see 1.1(iii)) one can approach the cohomology of a small simplicial category in two ways, which differ by a shift in dimension, is already apparent in the case of a *group* (i.e. a category with only one object, in which every map is invertible). The Hochschild-Mitchell approach then reduces to the

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usual notion of group cohomology. One can, however, also consider a group G as an object in the category of simplicial groups and define cohomology classes as (loop) homotopy classes of homomorphisms to an Eilenberg-MacLane object [10] from a free simplicial group which is weakly (loop homotopy) equivalent to G. This second approach readily generalizes to the notion of cohomology of small simplicial categories of [7], which is (see [8]) the natural place for obstructions to realizing diagrams in the homotopy category by means of simplicial diagrams of simplicial sets or, equivalently, topological diagrams of topological spaces.

2. PRELIMINARIES

We will freely use the following notation, terminology and results.

2.1. Simplicial categories. A simplicial category is always assumed to have *the same objects in each dimension*; it thus is a category *enriched* [9, p. 180] over $\mathbf{S}$, the category of simplicial sets. If $\mathbf{C}$ is a simplicial category, then the function complex (i.e. simplicial hom-set) between any two objects $x, y \in \mathbf{C}$ will usually be denoted by $\mathbf{C}(x, y)$.

2.2. The category G of groupoids. This is the category with as objects the small categories in which all maps are invertible and as maps the functors between them.

2.3. The category M of modules over groupoids. An object of $\mathbf{M}$ is a functor $M: G \to$ **(abelian groups)**, in which G is a groupoid, and a map $M_1 \to M_2 \in \mathbf{M}$ (where $M_i: G_i \to$ **(abelian groups)**, $i = 1, 2$) consists of a pair (g, m), where $g: G_1 \to G_2$ is a functor and $m: M_1 \to M_2 g$ is a natural transformation. There is an obvious *forgetful functor* $\gamma: \mathbf{M} \to \mathbf{G}$.

2.4. The nerve functor $N: \mathbf{G} \to \mathbf{S}$. This functor sends a groupoid G to the simplicial set NG, which has as n-simplices ($n \geq 0$) the sequences $G_0 \to \cdots \to G_n$ of composable maps in G.

2.5. The n-th Eilenberg-MacLane object functor $K(-, n): \mathbf{M} \to \mathbf{S}$ ($n \geq 0$). This is the functor which sends an object $M \in \mathbf{M}$ to the simplicial set $K(M, n)$ which has as k-simplices the pairs (u, v) such that u is a k-simplex $G_0 \to \cdots \to G_k$ of $N\gamma M$ (see 2.3 and 2.4) and v is a k-simplex of the Eilenberg-MacLane complex $K(MG_0, n)$ [10, §23]. Clearly *the forgetful map* $j: K(M, n) \to N\gamma M \in \mathbf{S}$ *is a fibration and has a zero cross section* $i: N\gamma M \to K(M, n)$.

2.6. Simplicial structures on S, G and M. *The categories* $\mathbf{S}$, $\mathbf{G}$ *and* $\mathbf{M}$ *are simplicial categories and the above functors* $\gamma: \mathbf{M} \to \mathbf{G}$, $N: \mathbf{G} \to \mathbf{S}$ *and* $K(-, n): \mathbf{M} \to \mathbf{S}$ *are simplicial functors.* The simplicial structure on $\mathbf{S}$ is the usual one and the one on $\mathbf{G}$ assigns to every two groupoids G_1 and G_2 the nerve of the category which has as objects the functors $G_1 \to G_2$ and as maps the natural transformations between them. Similarly the simplicial structure on $\mathbf{M}$ assigns to every two objects $M_i: G_i \to$ (abelian groups) $(i=1, 2)$ of $\mathbf{M}$ the nerve of the category which has as objects the maps $(g, m): M_1 \to M_2 \in \mathbf{M}$ and as maps $(g, m) \to (g', m')$ between such objects the natural transformations $h: g \to g'$ such that $m' = (M_2 h)m$.

2.7. Simplicial diagrams of simplicial sets. If $\mathbf{C}$ is a small simplicial category (2.1), we denote by $\mathbf{S}^{\mathbf{C}}$ the category of $\mathbf{C}$-*diagrams of simplicial sets* (which has as objects the simplicial functors $\mathbf{C} \to \mathbf{S}$ and as maps the natural transformations between them) and recall from [4,1.3] that $\mathbf{S}^{\mathbf{C}}$ admits a *closed simplicial model category structure* in which the simplicial structure is the obvious one and in which a map $U \to V \in \mathbf{S}^{\mathbf{C}}$ is a fibration or a weak equivalence whenever, for every object $x \in \mathbf{C}$, the restriction $Ux \to Vx \in \mathbf{S}$ is a fibration or a weak equivalence.

2.8. A pair of adjoint functors $\mathbf{S}^{\mathbf{B}} \leftrightarrow \mathbf{S}^{\mathbf{C}}$. Let $f: \mathbf{B} \to \mathbf{C}$ be a simplicial functor between small simplicial categories. Then *the induced functor* $f^*: \mathbf{S}^{\mathbf{C}} \to \mathbf{S}^{\mathbf{B}}$ *preserves fibrations and weak equivalences* (2.7) *and hence its left adjoint* $f_*: \mathbf{S}^{\mathbf{B}} \to \mathbf{S}^{\mathbf{C}}$ *preserves cofibrations and* [2,1.2 and 1.3] *weak equivalences between cofibrant objects.* Moreover, *the functors* f_* *and* f^* *are simplicially adjoint,* i.e. their adjointness induces, for every pair of objects $U \in \mathbf{S}^{\mathbf{B}}$ and $V \in \mathbf{S}^{\mathbf{C}}$, a natural isomorphism of function complexes $\mathbf{S}^{\mathbf{C}}(f_* U, V) \approx \mathbf{S}^{\mathbf{B}}(U, f^* V)$.

2.9. Abelian group objects. Let $\mathbf{T}$ be a category with finite inverse limits, let $L \in \mathbf{T}$ be an object and let $\mathbf{T}/L$ denote the resulting over category (which has as objects the maps $K \to L \in \mathbf{T}$). An *abelian group object* over L then consists of a map $f: K \to L \in \mathbf{T}$ together with a *multiplication* map m, a *unit* map u and an *inverse* map i in $\mathbf{T}/L$

$$\begin{array}{ccc} K \times_L K & \xrightarrow{\ m\ } & K \\ & \searrow \quad \swarrow & \\ & L & \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{\ u\ } & K \\ & \searrow \quad \swarrow & \\ & L & \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{\ i\ } & K \\ & \searrow \quad \swarrow & \\ & L & \end{array}$$

satisfying the usual abelian group axioms. These abelian group objects over L form a category which we will denote by $\mathbf{ab}/L$. It often is an *abelian category*, for instance when $\mathbf{T} = \mathbf{S}$.

3. THE COHOMOLOGY OF SIMPLICIAL DIAGRAMS OF SIMPLICIAL SETS

We start with extending the definition of cohomology groups of diagrams of simplicial sets of [5] to simplicial diagrams of simplicial sets.

3.1. The cohomology of simplicial diagrams of simplicial sets. Given (see §2) a small simplicial category $\mathbf{C}$, a cofibrant object $U \in \mathbf{S}^{\mathbf{C}}$, a cofibration $U \to V \in \mathbf{S}^{\mathbf{C}}$, a simplicial functor $W: \mathbf{C} \to \mathbf{M}$ and a 'twisting map' $t: V \to$ $\to N\gamma W \in \mathbf{S}^{\mathbf{C}}$, the *relative cohomology group* $H^n(V, U; W)$ $(n \geq 0)$ *with local* *coefficients induced by t* will be the abelian group of the homotopy classes of 'liftings' (i.e. dotted arrows which make the diagram commutative) in the diagram

$$\begin{array}{ccc} U & \xrightarrow{\text{zero section}} & K(W, n) \\ \downarrow & \nearrow & \downarrow j \\ V & \xrightarrow{t} & N\gamma W \end{array}$$

or, equivalently, in the induced diagram

$$\begin{array}{ccc} U & \longrightarrow & K(W, n) \times_{N\gamma W} V \\ \downarrow & \nearrow & \downarrow \\ V & \xrightarrow{id} & V \end{array}$$

If U is the initial object of $\mathbf{S}^{\mathbf{C}}$ (i.e. Ux is empty for every object $x \in \mathbf{C}$), one often writes $H^n(V; W)$ instead of $H^n(V, U; W)$.

If the object $U \in \mathbf{S}^{\mathbf{C}}$ is not cofibrant and/or the map $U \to V \in \mathbf{S}^{\mathbf{C}}$ is not a cofibration, one chooses a weak equivalence $U' \to U \in \mathbf{S}^{\mathbf{C}}$ such that U' is cofibrant and a factorization $U' \to V' \to V$ of the composition $U' \to U \to V$ into a cofibration $U' \to V'$ followed by a weak equivalence $V' \to V$ and defines $H^n(V, U; W)$ as $H^n(V', U', W)$. This is permissible because, by a standard homotopical algebra argument [11], any two such choices give rise to the same group, up to a canonical isomorphism.

As usual this definition implies:

3.2. Proposition. *Given a map $U \to V \in \mathbf{S}^{\mathbf{C}}$ and a twisting map $t: V \to \to N\gamma W \in \mathbf{S}^{\mathbf{C}}$, there is a natural long exact sequence*

$$\cdots \to H^n(V, U; W) \to H^n(V; W) \to H^n(U; W) \to H^{n+1}(V, U; W) \to \cdots.$$

It turns out to be convenient (see §4) to further extend this definition of cohomology to

3.3. The doubly relative case. Given (see §2) a simplicial functor $f: \mathbf{B} \to \mathbf{C}$ between small simplicial categories, a cofibrant object $U \in \mathbf{S}^{\mathbf{B}}$, a cofibration $f_* U \to V \in \mathbf{S}^{\mathbf{C}}$, a simplicial functor $W: \mathbf{C} \to \mathbf{M}$ and a twisting map $t: V \to \to N\gamma W \in \mathbf{S}^{\mathbf{C}}$, we define the *doubly relative cohomology group* $H^n(V, U; W)$ as $H^n(V, f_* U; W)$.

Similarly, if the object $U \in \mathbf{S}^{\mathbf{B}}$ is not cofibrant and/or the map $f_* U \to V \in \mathbf{S}^{\mathbf{C}}$ is not a cofibration, one chooses a weak equivalence $U' \to U \in \mathbf{S}^{\mathbf{B}}$ such that U' is cofibrant and a factorization $f_* U' \to V' \to V$ of the composition $f_* U' \to f_* U \to \to V$ into a cofibration $f_* U' \to V'$ followed by a weak equivalence $V' \to V$, and defines $H^n(V, U; W)$ as $H^n(V', U'; W)$.

Of course one has, as an immediate consequence of 3.2:

3.4. Proposition. *Given an object $U \in \mathbf{S}^{\mathbf{B}}$, a map $f_* U \to V \in \mathbf{S}^{\mathbf{C}}$ and a twisting map $t: V \to N\gamma W \in \mathbf{S}^{\mathbf{C}}$, there is a natural long exact sequence*

$$\cdots \to H^n(V, U; W) \to H^n(V; W) \to H^n(U; W) \to H^{n+1}(V, U; W) \to \cdots.$$

4. HOCHSCHILD-MITCHELL COHOMOLOGY OF SIMPLICIAL CATEGORIES

Next we use the results of the previous section to generalize the definition of Hochschild-Mitchell cohomology of small categories of [1] to small simplicial categories.

4.1. Hochschild-Mitchell cohomology of simplicial categories. Let $\mathbf{C}$ be a small simplicial category. Then *the function complexes $\mathbf{C}(x, y)$ give rise to a simplicial diagram of simplicial sets $\mathbf{C}^{\#} \in \mathbf{S}^{\mathbf{C}^{op} \times \mathbf{C}}$.* Given a 'bi-module' W over $\mathbf{C}$, i.e. a simplicial functor $W: \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{M}$, and a twisting map t: $: \mathbf{C}^{\#} \to N\gamma W \in \mathbf{S}^{\mathbf{C}^{op} \times \mathbf{C}}$, we now define the Hochschild-Mitchell cohomology $\mathrm{Hoch}^n(\mathbf{C}; W)$ $(n \geq 0)$ of $\mathbf{C}$ as $H^n(\mathbf{C}^{\#}; W)$ and the *relative Hochschild-Mitchell cohomology* $\mathrm{Hoch}^n(\mathbf{C}, \mathbf{B}; W)$ $(n \geq 0)$ of a simplicial functor $\mathbf{B} \to \mathbf{C}$ as $H^n(\mathbf{C}^{\#}, \mathbf{B}^{\#}; W)$.

Proposition 3.4 then implies:

4.2. Proposition. *Given a simplicial functor $\mathbf{B} \to \mathbf{C}$ between small simplicial categories and a twisting map $t: \mathbf{C}^{\#} \to N\gamma W \in \mathbf{S}^{\mathbf{C}^{op} \times \mathbf{C}}$, there is a natural long exact sequence*

$$\cdots \to \mathrm{Hoch}^n(\mathbf{C}, \mathbf{B}; W) \to \mathrm{Hoch}^n(\mathbf{C}; W) \to$$

$$\to \mathrm{Hoch}^n(\mathbf{B}; W) \to \mathrm{Hoch}^{n+1}(\mathbf{C}, \mathbf{B}; W) \to \cdots$$

Moreover, if $\mathbf{C}^0 \subset \mathbf{C}$ denotes the subcategory of $\mathbf{C}$ which consists of the identity maps only, one readily verifies:

4.3. Proposition. $\mathrm{Hoch}^n(\mathbf{C}^0; W) = 0$ for $n > 0$.

4.4. Corollary. *The natural map* (4.2)

$$\mathrm{Hoch}^n(\mathbf{C}, \mathbf{C}^0; W) \to \mathrm{Hoch}^n(\mathbf{C}; W)$$

is an isomorphism for $n > 1$ and is onto for $n = 1$.

4.5. Remark. In general, for a simplicial functor $f: \mathbf{B} \to \mathbf{C}$, the diagram $\mathbf{B}^{\#} \in \mathbf{S}^{\mathbf{B}^{op} \times \mathbf{B}}$ is not cofibrant, nor is the map $(f^{op} \times f)_* \mathbf{B}^{\#} \to \mathbf{C}^{\#} \in \mathbf{S}^{\mathbf{C}^{op} \times \mathbf{C}}$ a cofibration. In order to give an explicit description of $\mathrm{Hoch}^n(\mathbf{C}, \mathbf{B}; W)$ one thus (§3) has to replace $\mathbf{B}^{\#}$ by a suitable cofibrant object and $(f^{op} \times f)_* \mathbf{B}^{\#} \to$ $\to \mathbf{C}^{\#}$ by a suitable cofibration. If the functor $f: \mathbf{B} \to \mathbf{C}$ is $1-1$, this can be done in a rather simple canonical manner using the following

4.6. Generalized nerve construction. The *generalized nerve* of a small category $\mathbf{E}$ will be the diagram $N^{\#}\mathbf{E} \in \mathbf{S}^{\mathbf{E}^{op}\times\mathbf{E}}$ such that, for every pair of objects $x, y \in \mathbf{E}$, the simplicial set $N^{\#}\mathbf{E}(x, y)$ has as n-simplices ($n \geq 0$) the sequences $x \to z_0 \to \cdots \to z_n \to y$ of composable maps in $\mathbf{E}$, with the obvious faces and degeneracies. Clearly composition of these maps gives rise to a *canonical map* $N^{\#}\mathbf{E} \to \mathbf{E}^{\#} \in \mathbf{S}^{\mathbf{E}^{op}\times\mathbf{E}}$.

One readily verifies

4.7. Proposition. *Let* $\mathbf{B}$ *be a small simplicial category. Then the object* $\operatorname{diag} N^{\#}\mathbf{B} \in \mathbf{S}^{\mathbf{B}^{op}\times\mathbf{B}}$ *is free and hence cofibrant and the canonical map* $\operatorname{diag} N^{\#}\mathbf{B} \to \mathbf{B}^{\#} \in \mathbf{S}^{\mathbf{B}^{op}\times\mathbf{B}}$ *is a weak equivalence.*

4.8. Proposition. *Let* $f: \mathbf{B} \to \mathbf{C}$ *be a simplicial functor between small simplicial categories, which is* $1-1$. *Then the induced map* $(f^{op}\times f)_* \operatorname{diag} N^{\#}\mathbf{B} \to$ $\to \operatorname{diag} N^{\#}\mathbf{C} \in \mathbf{S}^{\mathbf{C}^{op}\times\mathbf{C}}$ *is free and hence a cofibration.*

These propositions imply

4.9. Corollary. *If* $f: \mathbf{B} \to \mathbf{C}$ *is as in* 4.8 *and* W *and* t *are as in* 4.1, *then* $\operatorname{Hoch}^n(\mathbf{C}, \mathbf{B}; W)$ *can be identified with the abelian group of the homotopy classes of liftings in the diagram*

$$\begin{array}{ccccc}
(f^{op}\times f)_* \operatorname{diag} N^{\#}\mathbf{B} & & \xrightarrow{\text{zero section}} & & K(W, n) \\
\downarrow & & \nearrow & & \downarrow j \\
\operatorname{diag} N^{\#}\mathbf{C} & \longrightarrow & \mathbf{C}^{\#} & \xrightarrow{t} & N\gamma W
\end{array}$$

or, equivalently, in the induced diagram

$$\begin{array}{ccc}
(f^{op}\times f)_* \operatorname{diag} N^{\#}\mathbf{B} & \longrightarrow & K(W, n) \times_{N\gamma W} \mathbf{C}^{\#} \\
\downarrow & \nearrow & \downarrow \\
\operatorname{diag} N^{\#}\mathbf{C} & \longrightarrow & \mathbf{C}^{\#}
\end{array}$$

5. THE MAIN RESULT

Now we state our main result, that *the cohomology of small simplicial categories of* [7] *is just Hochschild-Mitchell cohomology with a shift in dimension.*

First we recall from [7]:

5.1. Cohomology of simplicial categories. Let O be a set (of objects) and let $\mathbf{S}O$-**Cat** denote the category of simplicial O-categories (i.e. categories with object set O, which are enriched over $\mathbf{S}$). Then $\mathbf{S}O$-**Cat** admits a *closed simplicial model category structure* in which the simplicial structure is the obvious one, in which a map $\mathbf{B}\to\mathbf{C}$ is a *fibration* or a *weak equivalence* whenever, for every pair of objects $x, y\in O$, the restriction $\mathbf{B}(x, y)\to\mathbf{C}(x, y)\in\mathbf{S}$ is a fibration or a weak equivalence and in which a map is a *cofibration* iff it is a strong retract [3, §7] of a free map.

Given a cofibration $\mathbf{B}\to\mathbf{C}\in\mathbf{S}O\text{-}\mathbf{Cat}$ an O-category $\mathbf{W}$ enriched over $\mathbf{M}$ and a twisting map $\mathbf{t}:\mathbf{C}\to N\gamma\mathbf{W}\in\mathbf{S}O\text{-}\mathbf{Cat}$, the *relative cohomology group* $H^n(\mathbf{C}, \mathbf{B}; \mathbf{W})$ $(n\geq 0)$ then is the abelian group of the homotopy classes of liftings in the diagram

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\text{zero section}} & K(\mathbf{W}, n) \\ \downarrow & \nearrow & \downarrow j \\ \mathbf{C} & \xrightarrow{\mathbf{t}} & N\gamma\mathbf{W} \end{array}$$

or, equivalently, in the induced diagram

$$\begin{array}{ccc} \mathbf{B} & \longrightarrow & K(\mathbf{W}, n)\times_{N\gamma\mathbf{W}}\mathbf{C} \\ \downarrow & \nearrow & \downarrow \\ \mathbf{C} & \xrightarrow{id} & \mathbf{C} \end{array}$$

If $\mathbf{B}\to\mathbf{C}\in\mathbf{S}O\text{-}\mathbf{Cat}$ is a map which is not a cofibration, one chooses a factorization $\mathbf{B}\to\mathbf{C}'\to\mathbf{C}$ of $\mathbf{B}\to\mathbf{C}$ into a cofibration $\mathbf{B}\to\mathbf{C}'$ followed by a weak equivalence $\mathbf{C}'\to\mathbf{C}$ and defines $H^n(\mathbf{C}, \mathbf{B}; \mathbf{W})$ as $H^n(\mathbf{C}'\mathbf{B}; \mathbf{W})$. If $\mathbf{B}$ is the initial object of $\mathbf{S}O\text{-}\mathbf{Cat}$ (i.e., in the notation of 4.2–4, $\mathbf{B}=\mathbf{C}^0$), one often writes $H^n(\mathbf{C}; \mathbf{W})$ instead of $H^n(\mathbf{C}, \mathbf{C}^0; W)$.

Now we can formulate

5.2. The main result. Given a set O, a map $\mathbf{B}\to\mathbf{C}\in\mathbf{S}O\text{-}\mathbf{Cat}$, an O-category $\mathbf{W}$ enriched over $\mathbf{M}$ and a twisting map $\mathbf{t}:\mathbf{C}\to N\gamma\mathbf{W}\in\mathbf{S}O\text{-}\mathbf{Cat}$, a straightforward calculation (using 2.6) yields that $\mathbf{W}$ *and* $\mathbf{t}$ *give rise to a bi-module* $\mathbf{W}^{\#}$: $:\mathbf{C}^{op}\times\mathbf{C}\to\mathbf{M}$ such that $\mathbf{W}^{\#}(x, y)=\mathbf{W}(x, y)$ for every two objects $x, y\in O$, *and a twisting map* $\mathbf{t}^{\#}:\mathbf{C}^{\#}\to N\gamma\mathbf{W}\in\mathbf{S}^{\mathbf{C}^{op}\times\mathbf{C}}$. One then has:

5.3. Theorem. *There are natural isomorphisms*

$$H^n(\mathbf{C}, \mathbf{B}; \mathbf{W})\approx \mathrm{Hoch}^{n+1}(\mathbf{C}, \mathbf{B}; \mathbf{W}^{\#}) \qquad n\geq 0.$$

This will be proven in §6.

In view of 4.4 this theorem implies

5.4. Corollary. *There are natural isomorphisms*

$$H^n(\mathbf{C}; \mathbf{W})\approx \mathrm{Hoch}^{n+1}(\mathbf{C}, \mathbf{C}^0; \mathbf{W}^{\#}) \qquad n\geq 0$$

$$H^n(\mathbf{C}; \mathbf{W})\approx \mathrm{Hoch}^{n+1}(\mathbf{C}; \mathbf{W}^{\#}) \qquad n\geq 2.$$

We end with some

5.5. Remarks. The arguments used in the proof of theorem 5.3 (§6) readily imply:

(i) *Given an object* $\mathbf{C} \in SO\text{-}\mathbf{Cat}$, *a bi-module* $W' : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{M}$ and a twisting map $t' : \mathbf{C}^{\#} \to N\gamma W' \in \mathbf{S}^{\mathbf{C}^{op} \times \mathbf{C}}$, *there exists an O-category* $\mathbf{W}$ *enriched over* $\mathbf{M}$ *and a twisting map* $\mathbf{t} : N\gamma\mathbf{W} \in SO\text{-}\mathbf{Cat}$ *such that, for every map* $\mathbf{B} \to \mathbf{C} \in SO\text{-}\mathbf{Cat}$, *there are natural isomorphisms*

$$\mathrm{Hoch}^n(\mathbf{C}, \mathbf{B}; W') \approx \mathrm{Hoch}^n(\mathbf{C}, \mathbf{B}; \mathbf{W}^{\#}) \qquad n \geq 0.$$

(ii) Theorem 5.3 remains valid if one takes coefficients in a *generalized Eilenberg-MacLane object* in the sense of Quillen [11, Ch. II, §5].

6. PROOF OF THEOREM 5.3

We begin with observing that, given a set O and an object $\mathbf{C} \in SO\text{-}\mathbf{Cat}$, the categories (2.9) $\mathbf{ab}/\mathbf{C}$ and $\mathbf{ab}/\mathbf{C}^{\#}$ are both abelian categories. Moreover the arguments of [6] readily yield

6.1. Proposition. *The categories* $\mathbf{ab}/\mathbf{C}$ *and* $\mathbf{ab}/\mathbf{C}^{\#}$ *admit closed simplicial model category structures* [11] *in which the simplicial structures are the obvious ones and in which a map* $Z_1 \to Z_2$ *is a weak equivalence or a trivial fibration whenever, for every pair of objects* x, $y \in O$, *the restriction* $Z_1(x, y) \to Z_2(x, y) \in \mathbf{ab}/C(x, y)$ *is a weak equivalence or a trivial fibration* [6, §4].

6.2. Proposition. *The forgetful functors* $\mathbf{ab}/\mathbf{C} \to SO\text{-}\mathbf{Cat}/\mathbf{C}$ *and* $\mathbf{ab}/\mathbf{C}^{\#} \to \mathbf{S}^{\mathbf{C}^{op} \times \mathbf{C}}/\mathbf{C}^{\#}$ *have left adjoints* $A : SO\text{-}\mathbf{Cat}/\mathbf{C} \to \mathbf{ab}/\mathbf{C}$ *and* $A : \mathbf{S}^{\mathbf{C}^{op} \times \mathbf{C}}/\mathbf{C}^{\#} \to \mathbf{ab}/\mathbf{C}^{\#}$ *respectively. Moreover these forgetful functors both preserve fibrations and weak equivalences between fibrant objects and their left adjoints preserve cofibrations and weak equivalences between cofibrant objects.*

Next, let $J : \mathbf{ab}/\mathbf{C} \to \mathbf{ab}/\mathbf{C}^{\#}$ denote the functor which sends an object $(D \to C) \in \mathbf{ab}/\mathbf{C}$ to the object (2.9) $((u^{op} \times u)^* \mathbf{D}^{\#} \to \mathbf{C}^{\#}) \in \mathbf{ab}/C^{\#}$. A straightforward calculation then yields the somewhat surprising

6.3. Proposition. *The functor* $J : \mathbf{ab}/\mathbf{C} \to \mathbf{ab}/\mathbf{C}^{\#}$ *is an equivalence of categories which, moreover, is compatible with the closed simplicial model category structures* (6.1)

Now we are ready for a

6.4. Proof of theorem 5.3. We clearly may assume that the map $f : \mathbf{B} \to \mathbf{C} \in SO\text{-}\mathbf{Cat}$ is a cofibration (5.1). Then 6.2 and [11, Ch.I, §4] readily imply that $H^n(\mathbf{C}, \mathbf{B}; \mathbf{W}) \approx \pi_0 \hom(X, Y)$, where hom denotes the function complex in $\mathbf{ab}/\mathbf{C}$ and X, $Y \in \mathbf{ab}/\mathbf{C}$ are the cofibrant and the fibrant objects given by (6.2) $X = Ai_{\mathbf{C}}/Af$ and $Y = (K(\mathbf{W}, n) \times_{N\gamma\mathbf{W}} \mathbf{C} \to \mathbf{C})$. Similarly 4.9 yields an isomorphism $\mathrm{Hoch}^{n+1}(C, B; W^{\#}) \approx \pi_0 \hom(X', Y')$, where hom denotes the function complex in $\mathbf{ab}/\mathbf{C}^{\#}$ and X', $Y' \in \mathbf{ab}/\mathbf{C}^{\#}$ are the cofibrant and the fibrant objects given by $X' = A(\operatorname{diag} N^{\#}\mathbf{C} \to \mathbf{C}^{\#})/A((f^{op} \times f)_{*} \operatorname{diag} N^{\#}\mathbf{B} \to \mathbf{C}^{\#})$ and $Y' = (K(\mathbf{W}^{\#}, n+1) \times_{N\gamma W} \mathbf{C}^{\#} \to \mathbf{C}^{\#})$. As (6.3) $\hom(X, Y) \approx \hom(JX, JY)$, it thus remains to show that $\pi_0 \hom(JX, JY)$ is isomorphic to $\pi_0 \hom(X', Y')$. But this follows [11, Ch. I, §2] from the fact that

(i) *JY has the homotopy type of the loops on Y'*, i.e. there exists a pull back diagram in $\mathbf{ab}/\mathbf{C}^{\#}$

$$\begin{array}{ccc} JY & \longrightarrow & Y'' \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y' \end{array}$$

in which the map $JY'' \to Y'$ is a fibration, $* = (\mathbf{C}^{\#} \xrightarrow{id} \mathbf{C}^{\#})$ and Y'' is weakly equivalent to $*$, and

(ii) *X' has the homotopy type of the suspension of JX*, i.e. there exists a push out diagram in $\mathbf{ab}/\mathbf{C}^{\#}$

$$\begin{array}{ccc} JX & \longrightarrow & * \\ \downarrow & & \downarrow \\ X'' & \longrightarrow & X' \end{array}$$

in which the map $JX \to X''$ is a cofibration and X'' is weakly equivalent to $*$.

That (i) holds is easy to see, but (ii) is less obvious. To prove (ii) let, for every integer $n \geq 0$, $f_n : \mathbf{B}_n \to \mathbf{C}_n$ denote the restriction of f to dimension n, let $X_n = Ai_{\mathbf{C}_n}/Af_n$ and $X'_n = A(N^{\#}\mathbf{C}_n \to \mathbf{C}_n^{\#})/A((f_n^{op} \times f_n)_{*} N^{\#}\mathbf{B}_n \to \mathbf{C}_n^{\#}))$ and let $\tilde{X}'_n \in ab/\mathbf{C}_n^{\#}$ be the 'universal covering' of X'_n (i.e., for every map $c \in \mathbf{C}_n$, $(\tilde{X}'_n)^{-1}c$ is the universal covering of $(X'_n)^{-1}c$ and, for every integer $k \geq 0$, let $\pi_k X'_n$ be the 'k-th homotopy group' of X'_n (i.e., for every map $c \in \mathbf{C}_n$, $(\pi_k X'_n)^{-1}c$ is the k-th homotopy group of $(X'_n)^{-1}c$). A lengthy but straightforward calculation then yields a canonical isomorphism $JX_n \approx \pi_1 X'_n$ and a canonical push out diagram in $\mathbf{ab}/\mathbf{C}_n^{\#}$

$$\begin{array}{ccc} JX_n & \longrightarrow & * = (\mathbf{C}_n^{\#} \xrightarrow{id} \mathbf{C}_n^{\#}) \\ \downarrow & & \downarrow \\ \tilde{\mathbf{X}}'_n & \longrightarrow & X'_n \end{array}$$

in which the map $JX_n \to \tilde{X}'_n$ is a cofibration. If the map $f_n : \mathbf{B}_n \to \mathbf{C}_n \in O\text{-}\mathbf{Cat}$ is free [3, §7], one readily verifies, using [3, 2.9], that $\pi_k X'_n \approx *$ for $k \neq 1$ and that therefore $\tilde{X}'_n$ is weakly equivalent to $*$, and the same conclusion obviously holds if the map f_n is a strong retract [3, §7] of a free map. A standard diagonal argument now yields (ii).

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