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Function complexes for diagrams of simplicial sets.*

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SUMMARY

Let $\mathbf{S}$ be the category of simplicial sets, let $\mathbf{D}$ be a small category and let $\mathbf{S}^{\mathbf{D}}$ denote the category of $\mathbf{D}$-diagrams of simplicial sets. Then $\mathbf{S}^{\mathbf{D}}$ admits a closed simplicial model category structure and the aim of this note is to show that, *for every cofibrant diagram $X \in \mathbf{S}^{\mathbf{D}}$ and every fibrant diagram $Y \in \mathbf{S}^{\mathbf{D}}$, the homotopy type of the function complex* $\hom(X, Y)$ *can be computed as a homotopy inverse limit involving function complexes in* $\mathbf{S}$ *between the simplicial sets that appear in X and Y.*

1. INTRODUCTION

1.1 THE MAIN RESULT. Let $\mathbf{S}$ denote the category of simplicial sets and let $\mathbf{D}$ be an arbitrary but fixed small category. The results of Quillen [8, Ch. II] then readily imply that the category $\mathbf{S}^{\mathbf{D}}$ of $\mathbf{D}$-diagrams of simplicial sets (i.e. functors $\mathbf{D} \to \mathbf{S}$) admits a closed simplicial model category structure, i.e. the category $\mathbf{S}^{\mathbf{D}}$ admits notions of weak equivalences, fibrations, cofibrations and function complexes which are related in the usual manner. In particular, if $X \in \mathbf{S}^{\mathbf{D}}$ is a diagram which is cofibrant with respect to this model category structure and $Y \in \mathbf{S}^{\mathbf{D}}$ is fibrant, then the function complex $\hom(X, Y)$ has "homotopy meaning", i.e. its homotopy type depends only on the weak equivalence classes of X and Y.

The aim of this note now is to show that, *for every cofibrant diagram $X \in \mathbf{S}^{\mathbf{D}}$ and every fibrant diagram $Y \in \mathbf{S}^{\mathbf{D}}$, the homotopy type of the function complex* $\hom(X, Y)$ *can be computed as a homotopy inverse limit involving function*

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complexes in **S** *between the simplicial sets that appear in X and Y.* The indexing category for this homotopy inverse limit is the twisted arrow category $a\mathbf{D}$ which has as objects the maps $D_0 \to D_1$ of $\mathbf{D}$ and has as maps the commutative diagrams of the form

$$\begin{array}{ccc} D_0 & \longleftarrow & D_0' \\ \downarrow & & \downarrow \\ D_1 & \longrightarrow & D_1' \end{array}$$

1.2 REMARK. Of course, diagrams are worth studying in their own right, but the motivation for our work is the fact that a good number of apparently unrelated problems in topology can be directly reduced to questions about the homotopy theory of $\mathbf{S}^{\mathbf{D}}$, for appropriate choice of $\mathbf{D}$ ([3], [4]). Our main result allows some of these questions about $\mathbf{S}^{\mathbf{D}}$ to be reduced in turn to questions about ordinary homotopy theory.

1.3 ORGANIZATION OF THE PAPER. After a brief description of the closed simplicial model category structure on $\mathbf{S}^{\mathbf{D}}$ (in § 2), we state our main result (in § 3) and (in § 4) prove it under the assumption that $\mathbf{D}$ is a direct category (4.1). Next we discuss (in § 5) the notion of subdivision of a small category and then (in § 6) use this to prove our result in general.

1.4 NOTATION, TERMINOLOGY, ETC. (i) Apart from some familiarity with *simplicial sets*, the paper requires some knowledge of *model categories* and *homotopy limits* as can be found in [8] and [2, Ch. XI and Ch. XII] respectively.

(ii) If $\mathbf{D}$ is a small category, then we denote by the *same* symbol its *nerve*, i.e. [2, Ch. XI, § 2] the simplicial set which has as n-simplices the sequence $D_0 \to \ldots \to D_n$ of maps in $\mathbf{D}$.

(iii) A map in $\mathbf{S}$ will be called a *weak equivalence* if it is a weak homotopy equivalence, i.e. if its geometric realization is a homotopy equivalence. Similarly, two objects $X, Y \in \mathbf{S}$ will be called *weakly equivalent* if they can be connected by a finite string of weak equivalences, i.e. if their geometric realizations have the same homotopy type.

(iv) For any two objects $X, Y \in \mathbf{S}$, we denote by $\hom(X, Y)$ the usual *function complex*, i.e. the simplicial set which has as its n-simplices the maps $X \times \Delta[n] \to Y \in \mathbf{S}$.

2. THE MODEL CATEGORY STRUCTURE

We start with a brief discussion of the closed simplicial model category structure on the category $\mathbf{S}^{\mathbf{D}}$.

2.1 CATEGORIES OF DIAGRAMS OF SIMPLICIAL SETS. Let $\mathbf{D}$ be a small category and let $\mathbf{S}$ denote the category of simplicial sets. Then we denote by $\mathbf{S}^{\mathbf{D}}$ the *category of* **D***-diagrams of simplicial sets*, i.e. the category which has as objects the functors $\mathbf{D} \to \mathbf{S}$ and as maps the natural transformations between them, and note that ([8, Ch. II, 4] and [2, Ch. XI, § 8]) the category $\mathbf{S}^{\mathbf{D}}$, with *weak equivalences, fibrations, cofibrations* and *function complexes* as defined below, is a *closed simplicial model category* in the sense of Quillen [8, Ch. II].

2.2 WEAK EQUIVALENCES IN $\mathbf{S}^{\mathbf{D}}$. A map $f: X \to Y \in \mathbf{S}^{\mathbf{D}}$ is a *weak equivalence* if, for every object $D \in \mathbf{D}$, the map $fD: XD \to YD \in \mathbf{S}$ is a weak equivalence (1.4 (iii)). Similarly two objects $X, Y \in \mathbf{S}^{\mathbf{D}}$ will be called *weakly equivalent* if they can be connected by a finite string of weak equivalences.

2.3 FIBRATIONS IN $\mathbf{S}^{\mathbf{D}}$. A map $f: X \to Y \in \mathbf{S}^{\mathbf{D}}$ is a *fibration* if, for every object $D \in \mathbf{D}$, the map $fD: XD \to YD \in \mathbf{S}$ is a fibration. In particular, *an object $X \in \mathbf{S}^{\mathbf{D}}$ is fibrant if, for every object $D \in \mathbf{D}$, the object $XD \in \mathbf{S}$ is fibrant* (i.e. satisfies the extension condition [7, § 1]).

2.4 COFIBRATIONS IN $\mathbf{S}^{\mathbf{D}}$. A map $f: X \to Y \in \mathbf{S}^{\mathbf{D}}$ is a *cofibration* if it has the left lifting property [8, Ch. I, § 5] with respect to the class of trivial fibrations (i.e. fibrations which are weak equivalences).

Call a map $f: X \to Y \in \mathbf{S}^{\mathbf{D}}$ *free* if, for every object $D \in \mathbf{D}$, the map $fD: XD \to YD \in \mathbf{S}$ is a cofibration (i.e. injection) and if there exists a set B of simplices of Y such that

(i) no simplex of B is in the image of f,
(ii) B is closed under degeneracy operators, and
(iii) for every object $D \in \mathbf{D}$ and every simplex $y \in YD$ which is not in the image of fD, there is a unique simplex $b \in B$ and a unique map $d \in \mathbf{D}$, such that $(Yd)b = y$.

Then it is not difficult to see that *the cofibrations of $\mathbf{S}^{\mathbf{D}}$ are exactly the free maps and their retracts.*

2.5 FUNCTION COMPLEXES IN $\mathbf{S}^{\mathbf{D}}$. These are induced by the simplicial structure of $\mathbf{S}$, i.e. for every two diagrams $X, Y \in \mathbf{S}^{\mathbf{D}}$, the *function complex* $\hom(X, Y)$ is the simplicial set which has as its n-simplices the maps $X \times \Delta[n] \to$ $\to Y \in \mathbf{S}^{\mathbf{D}}$.

We end with considering

2.6 NATURALITY WITH RESPECT TO D. A functor $j: \mathbf{D}' \to \mathbf{D}$ between two small categories clearly induces (by composition) a functor $j^*: \mathbf{S}^{\mathbf{D}} \to \mathbf{S}^{\mathbf{D}'}$ which is *compatible with the function complexes* and which *preserves weak equivalences and fibrations*. It should be noted however that j^* need *not* preserve cofibrations.

3. THE MAIN RESULT

In order to formulate our main result (3.3) we need the notion of

3.1 THE TWISTED ARROW CATEGORY. Let $\mathbf{D}$ be a small category. Then its *twisted arrow category* $a\mathbf{D}$ is the category which has as objects the maps of D and as maps $(D_0 \to D_1) \to (D_0' \to D_1')$ the commutative diagrams (note that the horizontal maps go in *opposite* directions).

$$\begin{array}{ccc} D_0 & \longleftarrow & D_0' \\ \downarrow & & \downarrow \\ D_1 & \longrightarrow & D_1' \end{array}$$

Clearly $a\mathbf{D}$ comes with obvious functors $a\mathbf{D} \to \mathbf{D}$ and $a\mathbf{D} \to \mathbf{D}^{\mathrm{op}}$ obtained by restriction to the range and domain respectively.

Given two diagrams $X, Y \in \mathbf{S}^{\mathbf{D}}$ one can form an $a\mathbf{D}$-diagram $\hom_a(X, Y) \in \mathbf{S}^{a\mathbf{D}}$ by putting (1.3 (iv)) $(D_0 \to D_1) \to \hom(XD_0, YD_1)$ and note that

3.2 PROPOSITION. For every two objects $X, Y \in \mathbf{S}^{\mathbf{D}}$ there is an obvious (natural) isomorphism

$$\hom(X, Y) \approx \underleftarrow{\lim}^{a\mathbf{D}} \hom_a(X, Y)$$

Our main result now is

3.3 THEOREM. *Let* $X \in \mathbf{S}^{\mathbf{D}}$ *be cofibrant and let* $Y \in \mathbf{S}^{\mathbf{D}}$ *be fibrant. Then the obvious* [2, Ch. XI] *map*

$$\hom(X, Y) \approx \underleftarrow{\lim}^{a\mathbf{D}} \hom_a(X, Y) \to \underleftarrow{\mathrm{holim}}^{a\mathbf{D}} \hom_a(X, Y)$$

is a weak equivalence.

4. PROOF OF THEOREM 3.3 FOR DIRECT CATEGORIES

In this section we prove theorem 3.3 for

4.1 DIRECT CATEGORIES. A small category $\mathbf{D}$ will be called *direct* if, for every object $D \in \mathbf{D}$, the (nerve of the) over category $\mathbf{D}\downarrow D$ [6, p. 47] is finite-dimensional (A simplicial set if finite-dimensional if all simplices of a sufficiently high dimension are degenerate. If X is a finite-dimensional simplicial set, dim X denotes the largest dimension in which a non-degenerate simplex of X occurs). For every integer $n \geq 0$ we then denote

(i) by $\mathbf{D}^n \in \mathbf{D}$ the full subcategory spanned by the objects $D \in \mathbf{D}$ such that (1.3 (ii)) $\dim(\mathbf{D}\downarrow D) \leq n$,

(ii) by $j_n : \mathbf{D}^n \to \mathbf{D}$ the inclusion functor, and

(iii) by $\mathbf{D}_n \subset \mathbf{D}$ the (discrete) subcategory consisting of the objects $D \in \mathbf{D}$ such that $\dim(\mathbf{D}\downarrow D) = n$.

4.2 SOME PROPERTIES OF DIAGRAMS OVER DIRECT CATEGORIES. Let $\mathbf{D}$ be a direct category. Then it is not difficult to verify:

(i) *An object* $X \in \mathbf{S}^{\mathbf{D}}$ *is cofibrant iff, for every integer* $n \geq 0$ *and every object* $D \in \mathbf{D}_n$, *the induced map*

$$\underset{\rightarrow}{\lim}^{\mathbf{D}^{n-1}\downarrow D} j^{*}X \rightarrow \underset{\rightarrow}{\lim}^{\mathbf{D}^{n}\downarrow D} j^{*}X = XD \in \mathbf{S}$$

(where j denotes the obvious forgetful functors) *is a cofibration.*

(ii) *An object* $X \in \mathbf{S}^{\mathbf{D}}$ *is cofibrant iff the induced objects* $j_n^{*}X \in \mathbf{S}^{\mathbf{D}^n}$ *are so for all* $n \geq 0$.

(iii) *If* $X \in \mathbf{S}^{\mathbf{D}}$ *is cofibrant, then the obvious* [2, Ch. XII] *map*

$$\operatorname{holim}_{\rightarrow}^{\mathbf{D}} X \rightarrow \underset{\rightarrow}{\lim}^{\mathbf{D}} X$$

is a weak equivalence.

Now we are ready for a

4.3 PROOF OF THEOREM 3.3 FOR DIRECT CATEGORIES. Let $\mathbf{D}$ be a direct category. Then the desired result follows readily from the fact that

(i) *the restrictions*

$$\hom(j_{n+1}^{*}X, j_{n+1}^{*}Y) \rightarrow \hom(j_n^{*}X, j_n^{*}Y) \quad n \geq 0$$

are fibrations and

$$\hom(X, Y) \approx \underset{\leftarrow}{\lim}^{n} \hom(j_n^{*}X, j_n^{*}Y)$$

(ii) *the restrictions*

$$\operatorname{holim}_{\leftarrow}^{a\mathbf{D}^{n+1}} \hom_a(j_{n+1}^{*}X, j_{n+1}^{*}Y) \rightarrow \operatorname{holim}_{\leftarrow}^{a\mathbf{D}^{n}} \hom_a(j_n^{*}X, j_n^{*}Y) \quad n \geq 0$$

are fibrations, and

$$\operatorname{holim}_{\leftarrow}^{a\mathbf{D}} \hom_a(X, Y) \approx \underset{\leftarrow}{\lim}^{n} \operatorname{holim}_{\leftarrow}^{a\mathbf{D}^{n}} \hom_a(j_n^{*}X, j_n^{*}Y)$$

(iii) *the obvious maps*

$$\hom(j_n^{*}X, j_n^{*}Y) \rightarrow \operatorname{holim}_{\leftarrow}^{a\mathbf{D}^{n}} \hom_a(j_n^{*}X, j_n^{*}Y) \quad n \geq 0$$

are weak equivalences.

Statements (i) and (ii) are easy to verify, as is statement (iii) for $n = 0$. To prove (iii) in general note the existence of the pull back diagram

$$\begin{array}{ccc}
\hom(j_{n+1}^{*}X, j_{n+1}^{*}Y) & \rightarrow & \prod_{D \in \mathbf{D}_{n+1}} \hom(\underset{\rightarrow}{\lim}^{\mathbf{D}^{n+1}\downarrow D} j^{*}X, YD) \\
\downarrow & & \downarrow \\
\hom(j_n^{*}X, j_n^{*}Y) & \rightarrow & \prod_{D \in \mathbf{D}_{n+1}} \hom(\underset{\rightarrow}{\lim}^{\mathbf{D}^{n}\downarrow D} j^{*}X, YD)
\end{array}$$

in which the vertical maps are fibrations, and the homotopy pull back diagram

$$\begin{array}{ccc}
\underset{\leftarrow}{\text{holim}}^{a\mathbf{D}^{n+1}} \hom_a(j_{n+1}^*X, j_{n+1}^*Y) & \rightarrow & \prod\limits_{D\in\mathbf{D}_{n+1}} \underset{\leftarrow}{\text{holim}}^{(\mathbf{D}^{n+1}\downarrow D)^{\text{op}}} \hom(j^*X, YD) \\
\downarrow & & \downarrow \\
\underset{\leftarrow}{\text{holim}}^{a\mathbf{D}^{n}} \hom_a(j_n^*X, j_n^*Y) & \rightarrow & \prod\limits_{D\in\mathbf{D}_{n+1}} \underset{\leftarrow}{\text{holim}}^{(\mathbf{D}^{n}\downarrow D)^{\text{op}}} \hom(j^*X, YD)
\end{array}$$

Then use 4.2 and the natural isomorphism [2, p. 234]

$$\underset{\leftarrow}{\text{holim}} \hom(-,-) \approx \hom(\underset{\rightarrow}{\text{holim}} -,-)$$

The fact that the second diagram above is a homotopy pull back diagram is not completely obvious. It can be proved by showing that there is a pull back diagram

$$\begin{array}{ccc}
\underset{\leftarrow}{\text{holim}}^{a\mathbf{D}^{n+1}} \hom_a(j_{n+1}^*X, j_{n+1}^*Y) & \rightarrow & \prod\limits_{D\in\mathbf{D}_{n+1}} \underset{\leftarrow}{\text{holim}}^{a(\mathbf{D}^{n+1}\downarrow D)} \hom_a(j^*X, \overline{YD}) \\
\downarrow & & \downarrow \\
\underset{\leftarrow}{\text{holim}}^{a\mathbf{D}^{n}} \hom_a(j_n^*X, j_n^*Y) & \rightarrow & \prod\limits_{D\in\mathbf{D}_{n+1}} \underset{\leftarrow}{\text{holim}}^{a(\mathbf{D}^{n}\downarrow D)} \hom_a(j^*X, \overline{YD})
\end{array}$$

in which the vertical maps are fibrations ($\overline{YD}$ denotes the constant functor with value YD). The desired result then follows from the fact that the functors called $\hom_a(j^*X, \overline{YD})$ above factor through the natural (3.1) left confinal [2, Ch. XI] functors

$$a(\mathbf{D}^{n+1}\downarrow D)\rightarrow(\mathbf{D}^{n+1}\downarrow D)^{\text{op}} \text{ and } a(\mathbf{D}^{n}\downarrow D)\rightarrow(\mathbf{D}^{n}\downarrow D)^{\text{op}}.$$

5. THE SUBDIVISION OF A CATEGORY

To complete the proof of theorem 3.3 (in § 6) we need the "subdivision of a category" ([1], [5]) which will be discussed below. An easy way of describing it is by first considering the somewhat larger

5.1 DIVISION OF A CATEGORY. For every $n\geq 0$, let $\mathbf{n}$ denote the category which has the integers $0,\ldots,n$ as objects and which has exactly one map $i\rightarrow j$ whenever $i\leq j$. The *division* $d\mathbf{D}$ of a small category $\mathbf{D}$ then is defined as the category which has as objects the functors $\mathbf{n}\rightarrow\mathbf{D}$ ($n\geq 0$) and which has as maps

$$(J_1:\mathbf{n}_1\rightarrow\mathbf{D})\rightarrow(J_2:\mathbf{n}_2\rightarrow\mathbf{D})$$

the commutative diagrams of the form

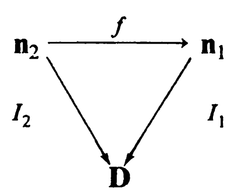
$$\begin{array}{ccc}
\mathbf{n}_2 & \longrightarrow & \mathbf{n}_1 \\
{\scriptstyle J_2}\searrow & & \swarrow{\scriptstyle J_1} \\
& \mathbf{D} &
\end{array}$$

5.2 SUBDIVISION OF A CATEGORY. The *subdivision* $sd\mathbf{D}$ of a small category $\mathbf{D}$ is the category obtained from the division $d\mathbf{D}$ by turning all the *"degeneracy maps"* (i.e. diagrams as in 5.1 in which the top map is onto) into *identity maps.* The subdivision comes with a functor $p: sd\mathbf{D} \to \mathbf{D}$ given by the formula $(J: \mathbf{n} \to \mathbf{D}) \to J(0)$.

Actually we will need (see 5.4) the opposite category of the subdivision which we will denote by $\overline{sd}\mathbf{D}$ and the corresponding functor $q: \overline{sd}\mathbf{D} \to \mathbf{D}$ given by the formula $(J: \mathbf{n} \to \mathbf{D}) \to J(n)$.

A straightforward calculation yields the following

5.3 OTHER DESCRIPTION OF THE SUBDIVISION. One can also describe the subdivision $sd\mathbf{D}$ as the category which has as objects the *"non-degenerate" functors* $\mathbf{n} \to \mathbf{D}$ $(n \geq 0)$ (i.e. the functors which send none of the maps $r \to r+1 \in \mathbf{n}$ $(0 \leq r < n)$ into an identity map of $\mathbf{D}$) and which has the following maps. Given two "non-degenerate" functors $I_1: \mathbf{n}_1 \to \mathbf{D}$ and $I_2: \mathbf{n}_2 \to \mathbf{D}$, consider all *"iterated face maps"* between them, i.e. all commutative diagrams of the form



in which f is $1-1$. The maps $I_1 \to I_2 \in sd\mathbf{D}$ then are the equivalence classes of such "iterated face maps", where two such maps f and g are *equivalent* iff, for every integer r with $0 \leq r \leq n_2$, the image under I_1 of the map

$$\min(f(r), g(r)) \to \max(f(r), g(r)) \in \mathbf{n}_1$$

is an identity map in $\mathbf{D}$.

This second description of $sd\mathbf{D}$ immediately implies

5.4 PROPOSITION. *The category $sd\mathbf{D}$ is direct.*

We also need the following properties of the functor $q: \overline{sd}\mathbf{D} \to \mathbf{D}$, of which the first two are readily verified.

5.5 PROPOSITION. *For every object $D \in \mathbf{D}$, the subcategory $q^{-1}D \subset \overline{sd}\mathbf{D}$ has an initial object and hence* (its nerve) *is contractible.*

5.6 PROPOSITION. *For every object $D \in \mathbf{D}$, the inclusion functor $q^{-1}D \to q \downarrow D$ has a left adjoint which is also a left inverse.*

5.7 PROPOSITION. *The functor $aq: a\overline{sd}\mathbf{D} \to a\mathbf{D}$ is left cofinal* [2, Ch. XI].

PROOF OF 5.7. It is not difficult to see from the definition that aq is left cofinal iff, for every pair of objects $D_0, D_1 \in \mathbf{D}$, the natural map from (the nerve of)

$a(D_0{\downarrow}q{\downarrow}D_1)$ to the discrete set $\hom_{\mathbf{D}}(D_0, D_1)$ is a weak equivalence, where $D_0{\downarrow}q{\downarrow}D_1$ is the category which fits into the obvious pull back diagram

$$\begin{array}{ccc} D_0{\downarrow}q{\downarrow}D_1 & \longrightarrow & q{\downarrow}D_1 \\ \Big\downarrow & & \Big\downarrow \\ D_0{\downarrow}q & \longrightarrow & \overrightarrow{sd}\mathbf{D} \end{array}$$

As, for any small category $\mathbf{C}$, the functor (3.1) $a\mathbf{C} \to \mathbf{C}$ is left cofinal and hence a weak equivalence, it suffices to show that $D_0{\downarrow}q{\downarrow}D_1$ is weakly equivalent to $\hom_{\mathbf{D}}(D_0, D_1)$. But this in turn follows easily from propositions 5.5 and 5.6.

6. COMPLETION OF THE PROOF OF THEOREM 3.3

Let $q_* : \mathbf{S}^{\overrightarrow{sd}\mathbf{D}} \to \mathbf{S}^{\mathbf{D}}$ be the left adjoint of the functor $q^* : \mathbf{S}^{\mathbf{D}} \to \mathbf{S}^{\overrightarrow{sd}\mathbf{D}}$ [6, Ch. X, § 3]. The fact that (2.6) *q^* preserves fibrations and weak equivalences* then readily implies that *q_* preserves cofibrations* and the desired result now follows by standard model category arguments from the two propositions below.

6.1 PROPOSITION. *Let $U \in \mathbf{S}^{\overrightarrow{sd}\mathbf{D}}$ be cofibrant and such that the adjunction map $i: U \to q^*q_*U \in \mathbf{S}^{\overrightarrow{sd}\mathbf{D}}$ is a weak equivalence and let $Y \in \mathbf{S}^{\mathbf{D}}$ be fibrant. Then there is an obvious commutative diagram*

$$\begin{array}{ccc} \hom(q_*U, Y) & \to & \underleftarrow{\mathrm{holim}}^{a\mathbf{D}} \hom_a(q_*U, Y) \\ q^* \Big\downarrow \approx & & \Big\downarrow (aq)^* \\ \hom(q^*q_*U, q^*Y) & \to & \underleftarrow{\mathrm{holim}}^{a\overrightarrow{sd}\mathbf{D}} \hom_a(q^*q_*U, q^*Y) \\ i^* \Big\downarrow \approx & & \Big\downarrow i^* \\ \hom(U, q^*Y) & \to & \underleftarrow{\mathrm{holim}}^{a\overrightarrow{sd}\mathbf{D}} \hom_a(U, q^*Y) \end{array}$$

in which

(i) *the maps on the left are isomorphisms,*
(ii) *the bottom map is a weak equivalence, and*
(iii) *the maps on the right are weak equivalences.*

PROOF. Part (i) is easy and part (ii) follows from 5.4. The lower map on the right is a weak equivalence in view of the homotopy invariance of homotopy inverse limits [2, p. 304] and the upper map on the right is so in view of the cofinality theorem for homotopy inverse limits [2, p. 317].

6.2 PROPOSITION. *Let $U \to q^*V \in \mathbf{S}^{\overrightarrow{sd}\mathbf{D}}$ be a weak equivalence such that U is cofibrant. Then its adjoint $q_*U \to V \in \mathbf{S}^{\mathbf{D}}$ is also a weak equivalence.*

PROOF. For every object $D \in \mathbf{D}$, consider the commutative diagram

$$\begin{array}{ccccc}
q^{-1}D \times U_0 = & \underset{\to}{\operatorname{holim}}^{q^{-1}D}\bar{U}_0 & \xrightarrow{\sim} & \underset{\to}{\operatorname{holim}}^{q^{-1}D} j^*U & \\
\text{proj.}\big\downarrow\sim & \big\downarrow & & \big\downarrow\sim & \\
U_0 = & \underset{\to}{\lim}^{q^{-1}D}\bar{U}_0 & \to & \underset{\to}{\lim}^{q^{-1}D} j^*U & \xrightarrow{\approx} \underset{\to}{\lim}^{q\downarrow D} j^*U = (q_*U)D
\end{array}$$

in which

(i) j denotes the forgetful functors,

(ii) U_0 denotes the image under U of the initial object of $q^{-1}D$.

(iii) $\bar{U}_0$ denotes the "constant" $q^{-1}D$-diagram which send all of $q^{-1}D$ to U_0 and its identity map and in which the maps are the obvious ones. As $q^{-1}D$ is contractible (5.5), the map on the left is a weak equivalence and, in view of the homotopy invariance of homotopy direct limits [2, p. 325], so is the top map. As $q^{-1}D$ is direct and $j^*U \in \mathbf{S}^{q^{-1}\mathbf{D}}$ is cofibrant (4.2 (ii)), the vertical map on the right is also a weak equivalence (4.2 (iii)) and the desired result is now immediate.

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