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An E^2 model category structure for pointed simplicial spaces *

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Abstract

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We find settings in which it is possible to resolve a topological space by simplicial spaces or cosimplicial spaces. We determine what such a resolution consists of, and study the sense in which any two resolutions are equivalent. As in ordinary homological algebra, these resolutions are useful for constructing spectral sequences.

1. Introduction

1.1. Summary. This paper is concerned with the construction and study of homotopy-theoretic resolutions.

The notion of a *resolution* is a basic one in several areas of algebra. It first arose in the "abelian" context of homological algebra [4], where (projective and injective) resolutions of a module by chain complexes were used to define

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and compute derived functors. The notion has also come up in "non-abelian" situations. For instance, Quillen [11] and André [1] used resolutions of a commutative ring by simplicial commutative rings to define and compute the cotangent complex. In fact, Quillen [9] developed a general mechanism to organize the study of resolutions. He showed that if **C** is any category of a particular *algebraic* nature the category **sC** of simplicial objects over **C** satisfies the axioms for a *closed model category* (1.5). An object X of **C** can then be treated as a constant simplicial object over **C**, and resolving X amounts to using the model category axioms to find a "cofibrant" (or, depending on the circumstances, "fibrant") object of **sC** weakly equivalent to X. The model category axioms imply that any two such resolutions are themselves weakly equivalent.

Our aim in this paper is to extend Quillen's result by obtaining a reasonable closed model category structure on **sC** for categories **C** of a particular *homotopy-theoretic* nature. There are two useful special cases:

(i) Let $\mathbf{T}_*$ denote the category of pointed topological spaces. We construct a closed model category structure on the category $\mathbf{sT}_*$ of simplicial objects over $\mathbf{T}_*$. In this model category structure, every cofibrant resolution of an object $X \in \mathbf{T}_*$ consists of spaces which have the homotopy type of a wedge of spheres of dimensions ≥ 1.

(ii) Let $\mathbf{S}_*$ denote the category of pointed simplicial sets. We construct, for every prime p, a closed model category structure on the category $\mathbf{cS}_*$ of cosimplicial objects over $\mathbf{S}_*$. In this model category structure, every fibrant resolution of an object $X \in \mathbf{S}_*$ consists of simplicial sets which have the homotopy type of a product of $K(\mathbf{Z}/p, n)$'s.

Certain model category structures on $\mathbf{sT}_*$ [12] and $\mathbf{cS}_*$ [3] have appeared in previous work, but these other structures, which we will refer to as *Reedy model category structures* (see Section 2), are too rigid for our purposes. Each Reedy weak equivalence in $\mathbf{sT}_*$ or $\mathbf{cS}_*$ is termwise a weak homotopy equivalence. It follows, for instance, that in the Reedy model category structure on $\mathbf{sT}_*$ any cofibrant resolution of a CW-complex $X \in \mathbf{T}_*$ is a simplicial space which in each simplicial dimension has the same homotopy type as X itself. Our new model category structures specify a larger class of weak equivalences than the Reedy structures do, and for this reason give rise to more interesting resolutions. In fact, for us a map between simplicial objects is a weak equivalence iff it induces an isomorphism on E^2-terms of a certain spectral sequence (3.6) and for this reason we will call our new model category structures *E^2 model category structures*.

1.2. Motivation. Resolutions as in 1.1(i) have already appeared in [14], where they were used to construct a *van Kampen spectral sequence*. Resolutions as in 1.1(ii) are familiar from the treatment of *unstable Adams spectral sequences* in [3]. However, both [14] and [3] used specific functorial resolutions of the appropriate type and did not address the possibility of working with other resolutions. This is like doing homological algebra by constructing a fixed functorial (projective or injective) resolution for each module and sidestepping (for instance) the question of what it would mean for two resolutions to be chain homotopy equivalent. The present paper is meant to provide, in the settings of [14] and [3], the same sort of flexibility in choosing "homotopy-theoretic" resolutions that one has always had available in the setting of [4] for constructing algebraic ones.

1.3. Applications. An application of the E^2 model category structure of 1.1(i) will be given in [6], where, for simplicial pointed topological spaces, we obtain a *Postnikov decomposition "in the simplicial direction"*. This Postnikov decomposition has the property that, for a cofibrant resolution of a space $X \in \mathbf{T}_*$, the associated Eilenberg–MacLane objects depend (up to E^2 weak equivalence) only on $\pi_* X$ as a Π-algebra (i.e. as a ≥ 1 graded group, together with an action of the primary homotopy operations), and thus gives rise to a sequence of obstructions to *realizing a Π-algebra.*

In a similar way one can use the E^2 model category structures of 1.1(ii) to attack the *realizability problem for unstable algebras over the Steenrod algebra.*

1.4. Organization of the paper and further details. After a brief review of *Reedy model category structures* (in Section 2), we formulate (in Section 3) and prove (in Section 4) our key result, the existence of the E^2 *model category structure* (mentioned in 1.1(i)) *on the category* $s\mathbf{T}_*$ of simplicial pointed topological spaces. We also show that the *simplicial structure* on $s\mathbf{T}_*$ which results from the fact that $s\mathbf{T}_*$ is a "category of simplicial objects over a category with finite limits" is compatible with this E^2 model category structure; in other words, the model category structure extends to a *closed simplicial model category structure* [9, Chapter II, Section 2].

In the remaining section (Section 5) we note that the E^2 model category structure on $s\mathbf{T}_*$ was obtained using only

(i) the fact that the category $\mathbf{T}_*$ is a *pointed* closed model category with *arbitrary colimits,* in which *all objects are fibrant,* and

(ii) the choice of a cofibrant co-grouplike object in $\mathbf{T}_*$ (namely the 1-sphere S^1).

We therefore conclude that for any closed model category $\mathbf{C}_*$ with the properties listed in (i) and every choice of a cofibrant co-grouplike object of $\mathbf{C}_*$, there is an associated E^2 model category structure on the category $s\mathbf{C}_*$ of simplicial objects over $\mathbf{C}_*$. An obvious example is obtained by taking $\mathbf{C}_* = \mathbf{T}_*$ and choosing, instead of S^1, any Moore space which is a suspension. One can, however, also take the category $\mathbf{S}_*$ of pointed simplicial sets (whose opposite has the desired properties) and choose a fibrant loop object in this category. In this case one gets an E^2 model category structure on the category $c\mathbf{S}_*$ of cosimplicial objects over $\mathbf{S}_*$. In particular, one obtains the model category structures on $c\mathbf{S}_*$ mentioned in 1.1 (ii) by choosing, for every prime p, the product of the $K(\mathbb{Z}/p, n)$'s. We end Section 5 with the observation that every E^2 model category structure on $s\mathbf{T}_*$ induces a corresponding E^2 model category structure on $s\mathbf{S}_*$ (in spite of the fact that some objects of $\mathbf{S}_*$ are not fibrant).

1.5. Notation, terminology, etc. We will freely use notation, terminology and results of [2], [9], and [12]. In particular, a *closed model category structure* on a category $\mathbf{C}$ consists of three classes of maps in $\mathbf{C}$, called *fibrations*, *cofibrations* and *weak equivalences*, satisfying axioms CM1–CM5 below. Note that axiom CM1 implies that $\mathbf{C}$ has an *initial* object as well as a *terminal* object. An object $U \in \mathbf{C}$ is called *fibrant* if the map $U \to$ (terminal object) $\in \mathbf{C}$ is a fibration and *cofibrant* if the map (initial object) $\to U \in \mathbf{C}$ is a cofibration. A map is called a *trivial* (co-)fibration if it is a weak equivalence as well as a (co)-fibration. A map $i : A \to B \in \mathbf{C}$ is said to have the *left lifting property* with respect to a map $p : X \to Y \in \mathbf{C}$ (and the map p is said to have the *right lifting property* with respect to the map i) if in every commutative solid arrow square in $\mathbf{C}$ of the shape

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

there exists a broken diagonal arrow such that the two resulting triangles are also commutative.

CM1 The category $\mathbf{C}$ has finite limits and colimits.

CM2 If f and g are maps such that gf is defined and two of f, g and gf are weak equivalences, then so is the third.

CM3 If f is a retract of g and g is a fibration, a cofibration or a weak equivalence, then so is f.

CM4 (i) Every cofibration has the left lifting property with respect to every trivial fibration.
(ii) Every fibration has the right lifting property with respect to every trivial cofibration.

CM5 Every map f can be factored
(i) $f = qj$, where j is a cofibration and q is a trivial fibration, and
(ii) $f = qj$, where q is a fibration and j is a trivial cofibration.

Of course *if* $\mathbf{C}$ *is a closed model category, then so is its opposite* $\mathbf{C}^{op}$ with as weak equivalences, cofibrations and fibrations the opposites of the weak equivalences, the fibrations and the cofibrations (respectively) of $\mathbf{C}$ itself.

2. Reedy model category structures

In this section we give a brief review of Reedy model category structures ([12] and [2, B.6]).

We start with some preliminaries.

2.1. Simplicial objects. Given a category **C**, we denote by s**C** the category of *simplicial objects* over **C**, i.e. [8, Chapter 1, Section 2] the category which has as objects the functors $\Delta^{\mathrm{op}} \to \mathbf{C}$ and as maps the natural transformations between them. (Here Δ is the category which has as objects the *finite ordered sets of integers* $[n] = (0, \ldots, n)$, $n \geq 0$, and as maps the weakly monotone functions between them). For an object $X \in \mathrm{s}\mathbf{C}$ one usually writes X_n instead of $X[n]$ $(n \geq 0)$. We will sometimes treat the category **C** as the subcategory of s**C** spanned by the constant simplicial objects.

As usual **S** will denote the category of *simplicial sets* (i.e. s**Sets**), $\Delta[n] \in \mathbf{S}$ $(n \geq 0)$ will be the *standard n-simplex* (which has as j-simplices $(j \geq 0)$ the maps $[j] \to [n] \in \Delta$), $\dot{\Delta}[n] \subset \Delta[n]$ $(n \geq 0)$ will be the subcomplex generated by the simplices of dimension $< n$ and $V[n,k] \subset \Delta[n]$ $(n > 0$, $0 \leq k \leq n)$ will be the subcomplex generated by all of the $(n-1)$-simplices except for the simplex given by the map $[n-1] \to [n]$ sending $(0, \ldots, n-1)$ to $(0, \ldots, \widehat{k}, \ldots, n)$.

2.2. A simplicial structure on s**C**. Let **C** be a category with *finite limits and colimits*. For every object $X \in \mathrm{s}\mathbf{C}$ and finite simplicial set K (i.e. simplicial set K with only a finite number of nondegenerate simplices), one can form the object $K \otimes X \in \mathrm{s}\mathbf{C}$ which, in each dimension $n \geq 0$, consists of the coproduct of as many copies of X_n as there are elements in K_n, and which has face and degeneracy maps induced by those of X and K [9, Chapter II, p. 1.8]. If K' is another finite simplicial set, there is clearly a *natural isomorphism* $K' \otimes (K \otimes X) \cong (K \times K') \otimes X$ and one can therefore define a *simplicial structure* [9, Chapter II] on s**C** by assigning to every pair of objects $X, Y \in \mathrm{s}\mathbf{C}$ the *function complex* $\hom(X, Y) \in \mathbf{S}$ which has as n-simplices $(n \geq 0)$ the maps $\Delta[n] \otimes X \to Y \in \mathrm{s}\mathbf{C}$ (with obvious faces and degeneracies [8, 6.4]).

One can also [9, Chapter II, Section 1] construct, for every object $Y \in \mathrm{s}\mathbf{C}$ and every finite simplicial set K, an object $Y^K \in \mathrm{s}\mathbf{C}$ and then define a simplicial structure on s**C** by assigning to every pair of objects $X, Y \in \mathrm{s}\mathbf{C}$ the simplicial set which has as n-simplices $(n \geq 0)$ the maps $X \to Y^{\Delta[n]} \in \mathrm{s}\mathbf{C}$. It turns out [9, Chapter II, Section 1] that *the functor* $(-)^K : \mathrm{s}\mathbf{C} \to \mathrm{s}\mathbf{C}$ *is right adjoint to the functor* $(K \otimes -) : \mathrm{s}\mathbf{C} \to \mathrm{s}\mathbf{C}$ and hence this simplicial structure coincides with the one defined above: in the terminology of [9, Chapter II, Section 2] *the above simplicial structure on the category* s**C** *satisfies axiom* SM0.

2.3. Latching objects and matching objects. If **C** is a category with *finite limits* *and colimits* one can also construct the following:

(i) *Latching objects.* Let $\mathbf{L}_n$ $(n \geq 0)$ be the category which has as objects the maps (2.1) $[j] \to [n] \in \Delta^{\mathrm{op}}$ with $j < n$ and which has as maps the obvious commutative triangles. Given an object $X \in \mathbf{sC}$, let, by a slight abuse of notation, $X|_{\mathbf{L}_n} : \mathbf{L}_n \to \mathbf{C}$ denote the composition of the forgetful functor $\mathbf{L}_n \to \Delta^{\mathrm{op}}$ with the functor $X : \Delta^{\mathrm{op}} \to \mathbf{C}$. The nth *latching object* L_nX of X then is defined by $L_nX = \varinjlim(X|_{\mathbf{L}_n})$. In particular, L_0X is the initial object of $\mathbf{C}$. Note that there is an obvious natural map $L_nX \to X_n$ $(n \geq 0)$.

(ii) *Matching objects.* In a similar way, let $\mathbf{M}_n$ $(n \geq 0)$ be the category which has as objects the maps $[n] \to [j] \in \Delta^{\mathrm{op}}$ with $j < n$ and which has as maps the obvious commutative triangles. Given an object $X \in \mathbf{sC}$, let, again by a slight abuse of notation, $X|_{\mathbf{M}_n} : \mathbf{M}_n \to \mathbf{C}$ be the composition of the forgetful functor $\mathbf{M}_n \to \Delta^{\mathrm{op}}$ with the functor $X : \Delta^{\mathrm{op}} \to \mathbf{C}$. The nth *matching object* M_nX of X then is defined by $M_nX = \varprojlim(X|_{\mathbf{M}_n})$. In particular, M_0X is the terminal object of $\mathbf{C}$. There is an obvious *natural map* $X_n \to M_nX \in \mathbf{C}$.

(iii) *Partial matching objects.* Later, in the proof of 3.5, we will need a generalization of matching objects which results from the observation that the category $\mathbf{M}_n$ has as objects the simplices of $\dot{\Delta}[n]$ (2.1) of dimension $< n$ and as maps the simplicial operators between them. Thus every subcomplex $A \subset \dot{\Delta}[n]$ gives rise to a subcategory $\mathbf{M}_A \subset \mathbf{M}_n$ and hence, for every object $X \in \mathbf{sC}$ to a functor $X|_{\mathbf{M}_A} : \mathbf{M}_A \to \mathbf{C}$ and a *partial matching object* $M_AX = \varprojlim(X|_{\mathbf{M}_A})$. In particular, $M_AX = M_nX$ if $A = \dot{\Delta}[n]$; we write M_n^kX instead of M_AX if $A = V[n,k]$ (2.1). The construction of M_AX is *natural* in A in the sense that every subcomplex $B \subset A$ gives rise to a natural map $M_AX \to M_BX \in \mathbf{C}$.

Now we are ready to formulate the following:

2.4. Reedy model category structures on sC. *Let* $\mathbf{C}$ *be a closed model category* (1.5). *Then* [12] $\mathbf{sC}$ *admits a closed model category structure in which*

(i) *a map* $X \to Y \in \mathbf{sC}$ *is a weak equivalence* (*called* Reedy weak equivalence) *whenever, for every* $n \geq 0$, *the restriction* $X_n \to Y_n \in \mathbf{C}$ *is a weak equivalence,*

(ii) *a map* $X \to Y \in \mathbf{sC}$ *is a* (*trivial*) *cofibration* (*called* (trivial) Reedy cofibration) *whenever, for every* $n \geq 0$, *the induced map* $(X_n \coprod_{L_nX} L_nY) \to Y_n \in \mathbf{C}$ *is a* (*trivial*) *cofibration, and*

(iii) *a map* $X \to Y \in \mathbf{sC}$ *is a* (*trivial*) *fibration* (*called* (trivial) Reedy fibration) *whenever, for every* $n \geq 0$, *the induced map* $X_n \to (Y_n \prod_{M_nY} M_nX) \in \mathbf{C}$ *is a* (*trivial*) *fibration.*

Of course there is a dual structure:

2.5. Reedy model category structure on cC. If $\mathbf{C}$ is a closed model category, then so is its opposite $\mathbf{C}^{\mathrm{op}}$ (1.5). Since *the category* $\mathbf{cC}$ *of cosimplicial objects over*

C *is the opposite of the category* s($\mathbf{C}^{op}$), the Reedy model category structure (2.4) on s($\mathbf{C}^{op}$) gives rise to a model category structure on c**C**, which we will refer to as its *Reedy model category structure.*

We end with some remarks.

2.6. Remarks on the simplicial structure. The Reedy model category structure on s**C** is *not* compatible with the simplicial structure of 2.2, in the sense that this simplicial structure does *not* turn s**C** into a closed *simplicial* model category [9, Chapter II, Section 2]. More precisely,

(i) the first of the two additional axioms, axiom SM0 (which does not involve weak equivalences, cofibrations or fibrations), *is indeed satisfied* (as we have already mentioned in 2.2), and

(ii) the first part of the second axiom SM7(b) also holds, i.e. *if* $X \to Y \in$ s**C** *is a (trivial) Reedy cofibration, then so is, for every* $n \geq 0$, *the induced map* (2.1) $(\Delta[n] \otimes X \coprod_{\dot\Delta[n]\otimes X} \dot\Delta[n] \otimes Y) \to \Delta[n] \otimes Y \in$ s**C**, but

(iii) the second part of axiom SM7(b) does not hold. One can verify readily that, for a Reedy cofibration $X \to Y \in$ s**C**, the maps $(\Delta[1] \otimes X \coprod_{\Delta[0]\otimes X} \Delta[0] \otimes Y) \to \Delta[1] \otimes Y \in$ s**C**, induced by the two maps $\Delta[0] \to \Delta[1] \in$ **S**, are Reedy cofibrations but *not* necessarily trivial ones.

Of course, if **C** itself is a closed simplicial model category, then s**C** admits another simplicial structure (the one "inherited" from **C**—see the paragraph preceeding Theorem B.6 in [2]) and this simplicial structure on s**C** is compatible with the Reedy model category structure.

3. The E^2 model category structure on s$\mathbf{T}_*$

We now formulate (in 3.1) our key result: the existence of the E^2 model category structure mentioned in 1.1(i) on the category s$\mathbf{T}_*$ of simplicial pointed topological spaces. In 3.6 we give a justification for the E^2 terminology.

3.1. The E^2 model category structure on s$\mathbf{T}_*$. *The category* s$\mathbf{T}_*$ *admits a closed simplicial model category structure* [9, Chapter II, Section 2] *in which the simplicial structure is as in* 2.2 *and in which the weak equivalences, the cofibrations and the fibrations are the* E^2 *weak equivalences, the* E^2 *cofibrations and the* E^2 *fibrations defined in* 3.2–3.4.

3.2. E^2 **weak equivalences.** Application of the ith homotopy group functor π_i ($i \geq 1$) to a simplicial space $X \in$ s$\mathbf{T}_*$ yields a simplicial group $\pi_i X$ [8, Section 17]. A map $X \to Y \in$ s$\mathbf{T}_*$ will be called an E^2 weak equivalence if, for every $i \geq 1$, the induced map $\pi_i X \to \pi_i Y$ is a weak equivalence of simplicial groups, i.e. [9, Chapter II, Section 3] if this induced map is a weak equivalence between the underlying simplicial sets or equivalently, if it induces isomorphisms $\pi_j\pi_i X \cong \pi_j\pi_i Y$ for all $j \geq 0$.

This definition clearly implies that *every Reedy weak equivalence* (2.4) *is an* E^2 *weak equivalence.* Also, if $X \in \mathbf{sT}_*$ and $X^{\mathrm{b}} \subset X$ consists of the basepoint components of the X_n $(n \geq 0)$, then *the inclusion* $X^{\mathrm{b}} \to X \in \mathbf{sT}_*$ *is an* E^2 *weak equivalence.* In other words, the non-basepoint components of the X_n do not contribute to the E^2 homotopy type of X.

3.3. E^2 cofibrations. A map in $\mathbf{sT}_*$ will be called an E^2 *cofibration* if it is a retract of an "S^1-free" map, where a map $X \to Y \in \mathbf{sT}_*$ is called S^1*-free* if, for every $n \geq 0$, there exist

(i) a CW-complex $Z_n \in \mathbf{T}_*$ which has the homotopy type of a wedge of spheres S^i $(i \geq 1)$, and

(ii) a map $Z_n \to Y_n \in \mathbf{T}_*$ such that the induced map

$$(X_n \textstyle\coprod_{L_n X} L_n Y)\coprod Z_n \to Y_n \quad \in \mathbf{T}_*$$

is a trivial cofibration.

Thus, *every* E^2 *cofibration is a Reedy cofibration* (2.6) *and every trivial Reedy cofibration is a trivial* E^2 *cofibration.*

3.4. E^2 fibrations. A map $X \to Y \in \mathbf{sT}_*$ will be called an E^2*-fibration* if it is a Reedy fibration (2.4) and if, for every $i \geq 1$, the induced map $\pi_i X \to \pi_i Y$ is a fibration of simplicial groups, i.e. [9, Chapter II, Section 3] if this induced map is a fibration of the underlying simplicial sets or equivalently, if the image of the simplicial group $\pi_i X$ in the simplicial group $\pi_i Y$ contains the identity component of $\pi_i Y$.

This definition clearly implies that *every Reedy fibrant object is* E^2 *fibrant* and that *every trivial Reedy fibration is a trivial* E^2 *fibration.*

Although this definition of E^2 fibration has the same flavor as the above definition of E^2 weak equivalence, it is also possible to characterize (trivial) E^2 fibrations in a manner which is more in line with the above definition of E^2 cofibrations. This goes as follows.

3.5. A characterization of (trivial) E^2 fibrations. *A Reedy fibration* $X \to Y \in \mathbf{sT}_*$ *is*

(i) *an* E^2 *fibration iff, for all* $i, n \geq 1$ *and* $0 \leq k \leq n$, *it has the right lifting property* (1.5) *with respect to the map* (2.1) $V[n,k]\otimes S^i \to \Delta[n]\otimes S^i \in \mathbf{sT}_*$ *induced by the inclusion* $V[n,k] \to \Delta[n]$, *and*

(ii) *a trivial* E^2 *fibration iff, for all* $i \geq 1$ *and* $n \geq 0$, *it has the right lifting property* (1.5) *with respect to the map* (2.1) $\dot{\Delta}[n]\otimes S^i \to \Delta[n]\otimes S^i \in \mathbf{sT}_*$ *induced by the inclusion* $\dot{\Delta}[n] \to \Delta[n]$.

Proof. We first remark that every commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array} \qquad \text{in } \mathbf{sT}_*$$

in which the vertical maps are Reedy fibrations and the horizontal maps are Reedy weak equivalences induces for each subcomplex $A \subset \dot{\Delta}[n]$ $(n \geq 1)$ an isomorphism (2.3)

$$\pi_i(Y_n \textstyle\prod_{M_A Y} M_A X) \cong \pi_i(Y'_n \prod_{M_A Y'} M_A X'), \quad i \geq 1.$$

This is obvious when $\dim(A) = -1$ (i.e. when A is empty) and the general case follows readily by induction from the observation that, for $\dim(A) = k \geq 0$ and $B \subset A$ its $(k-1)$-skeleton, one can obtain $Y_n \prod_{M_A Y} M_A X$ by pulling back a product of copies of the fibration $X_k \to Y_k \prod_{M_k Y} M_k X$ (one for each non-degenerate k-simplex of A) along a map from $Y_n \prod_{M_B Y} M_B X$. We will use this fact below to replace X and Y by Reedy fibrant objects X' and Y'.

To prove (i) note that a Reedy fibration $X \to Y$ is an E^2 fibration iff the induced maps (2.3(iii)) $\pi_i X_n \to (\pi_i Y_n \prod_{M_n^k \pi_i Y} M_n^k \pi_i X)$ are onto. The Reedy fibration $X \to Y$ has the right lifting property with respect to the maps $V[n,k] \otimes S^i \to \Delta[n] \otimes S^i$ iff the induced maps $\pi_i X_n \to \pi_i(Y_n \prod_{M_n^k Y} M_n^k X)$ are onto. It thus suffices to show that the canonical maps

$$\pi_i(Y_n \textstyle\prod_{M_n^k Y} M_n^k X) \to (\pi_i Y_n \prod_{M_n^k \pi_i Y} M_n^k \pi_i X)$$

are isomorphisms. Moreover, in view of the above remark, we may assume that X and Y are Reedy fibrant. The argument of [2, B.11] now readily yields that the canonical maps $\pi_i M_n^k X \to M_n^k \pi_i X$ and $\pi_i M_n^k Y \to M_n^k \pi_i Y$ are isomorphisms and the desired result then follows from the fact that the canonical maps $\pi_i Y_n \to M_n^k \pi_i Y$ are onto (because $\pi_i Y$ is a simplicial group [8, 17.1]).

The proof of (ii) is similar. Again one may assume that X and Y are Reedy fibrant. The key step in the proof then is the following inductive argument. Assume that, for some $n \geq 0$ and all $i \geq 1$, the canonical maps $\pi_i(Y_n \prod_{M_n Y} M_n X) \to (\pi_i Y_n \prod_{M_n \pi_i Y} M_n \pi_i X)$ are isomorphisms and that either the induced maps $\pi_i X_n \to (\pi_i Y_n \prod_{M_n \pi_i Y} M_n \pi_i X)$ are onto (because the map $X \to Y$ is a trivial E^2 fibration) or that the induced maps $\pi_i X_n \to \pi_i(Y_n \prod_{M_n Y} M_n X)$ are onto (because the map $X \to Y$ has the right lifting property with respect to the maps $\dot{\Delta}[n] \otimes S^i \to \Delta[n] \otimes S^i$). Then the canonical maps

$$\pi_i(Y_{n+1} \textstyle\prod_{M_{n+1} Y} M_{n+1} X) \to (\pi_i Y_{n+1} \prod_{M_{n+1} \pi_i X} M_{n+1} \pi_i X)$$

are also isomorphisms because $Y_{n+1} \prod_{M_{n+1} Y} M_{n+1} X$ can be obtained by pulling the map $X_n = (Y_n \prod_{Y_n} X_n) \to (Y_n \prod_{M_n Y} M_n X)$ back along the map $(Y_{n+1} \prod_{M_{n+1}^0 Y} M_{n+1}^0 X) \to (Y_n \prod_{M_n Y} M_n X)$. □

It remains to give a

3.6. Justification of the E^2 terminology. *Let $X \in \mathbf{sT}_*$ be Reedy fibrant. Then there exists a first quadrant spectral sequence $\{E^2_{p,q}\}$ with*

$$E^2_{p,q} = \pi_p \pi_{q+1} X \quad \Longrightarrow \quad \pi_{p+q} \hom(S^1, X)$$

where $\hom(S^1, X)$ *is as in* 2.2.

3.7. Remark. If X is also Reedy cofibrant, then the realization of the simplicial set $\hom(S^1, X)$ has the same homotopy type as the loop space on the realization [13] of the simplicial space $X^\flat$ (3.2).

Proof of 3.6. Let U be the cosimplicial pointed topological space consisting of the half smash products of S^1 with the topological p-simplices ($p \geq 0$) and let $\hom(U, X)$ denote the simplicial pointed simplicial set which in dimension $n \geq 0$ consists of the pointed simplicial set $\hom(U, X_n)$ which has as p-simplices ($p \geq 0$) the maps $U^p \to X_n \in \mathbf{sT}_*$. Then [5, Section 4] there are obvious isomorphisms of simplicial groups $\pi_i X \cong \pi_{i-1} \hom(U, X)$ ($i \geq 1$) and as (3.4) X is E^2 fibrant, 3.1 implies that the iterated codegeneracy maps $U^p \to U^0$ ($p \geq 0$) induce weak equivalences $\hom(U^0, X) \to \hom(U^p, X)$. The desired spectral sequence now is the Quillen–Bousfield–Friedlander spectral sequence of $\hom(U, X)$ [2, 2.5] which converges strongly to $\pi_* \operatorname{diag} \hom(U, X)$ and hence to $\pi_* \hom(S^1, X)$. □

4. Proof of 3.1

We start with some preliminaries.

First we note that [7] readily implies the following proposition:

4.1. Proposition. *If two maps $f, g : X \to Y \in \mathbf{sT}_*$ are simplicially homotopic* (in the weak sense that, considered as vertices of $\hom(X, Y)$ (2.2), they are in the same component), *and f is an E^2 weak equivalence, then so is g.* □

As, for every $n \geq 0$, *every map $\Delta[n] \to \Delta[n] \in \mathbf{S}$ which sends $\Delta[n]$ to one of its vertices is simplicially homotopic to the identity map,* this implies the following corollary:

4.2. Corollary. *For every object $X \in \mathbf{sT}_*$ and every map $\Delta[p] \to \Delta[q] \in \mathbf{S}$ ($p, q \geq 0$), the induced map $\Delta[p] \otimes X \to \Delta[q] \otimes X \in \mathbf{sT}_*$ is an E^2 weak equivalence.* □

The definition of E^2 fibration (3.4) suggests the following:

4.3. A pathlike construction. Let $D^{i+1} \in \mathbf{T}_*$ denote the reduced cone on the i-sphere $S^i \in \mathbf{T}_*$ $(i \geq 1)$ and, for every $n \geq 0$, consider the map $S^i = \Delta[0] \otimes S^i \to \Delta[n] \otimes S^i$ induced by the map $\Delta[0] \to \Delta[n] \in \mathbf{S}$ which sends (see 2.1) $\mathrm{id}_{[0]}$ to $d_0 \cdots d_0(\mathrm{id}_{[n]})$. Let $P(i,n) = D^{i+1} \coprod_{S^i} \Delta[n] \otimes S^i \in \mathrm{s}\mathbf{T}_*$. Given an object $Y \in \mathrm{s}\mathbf{T}_*$ denote by PY the coproduct of $P(i,n)$'s indexed by the 4-tuples (i,n,a,b) with $i,n \geq 1$ and $a : D^{i+1} \to Y_0$ and $b : S^i \to Y_n$ in $\mathbf{T}_*$ such that $a|_{S^i} = d_0 \cdots d_0 b$. Then Proposition 4.1, Corollary 4.2 and 3.5(i) imply that the map $* \to PY \in \mathrm{s}\mathbf{T}_*$ is a trivial E^2 cofibration which has the left lifting property (1.5) with respect to all E^2 fibrations and that the obvious map $PY \to Y \in \mathrm{s}\mathbf{T}_*$ induces, for every $i \geq 1$, a fibration of simplicial groups $\pi_i PY \to \pi_i Y$. Moreover, the same argument yields more generally the following lemma:

4.4. A factorization lemma. *Every map* $X \to Y \in \mathrm{s}\mathbf{T}_*$ *can be factored into a trivial* E^2 *cofibration* $X \to X \coprod PY \in \mathrm{s}\mathbf{T}_*$ *which has the left lifting property with respect to all* E^2 *fibrations, followed by a map* $X \coprod PY \to Y \in \mathrm{s}\mathbf{T}_*$ *which, for every* $i \geq 1$, *induces a fibration of simplicial groups* $\pi_i(X \coprod PY) \to \pi_i Y$. □

To obtain another factorization we need the following:

4.5. Resolution of a map. This is a generalization of the key construction of [14]. Given a map $A \to B \in \mathbf{T}_*$, let $WB = A \coprod VB \in \mathbf{T}_*$, where [14, Section 2] $VB \in \mathbf{T}_*$ is obtained by taking a wedge of spheres S^i, one for every $i \geq 1$ and map $S^i \to B \in \mathbf{T}_*$, and then attaching an $(i+1)$-cell for every $i \geq 1$ and map $D^{i+1} \to B \in \mathbf{T}_*$ (4.3). As WB comes with an obvious map $A \to WB \in \mathbf{T}_*$, one can repeat this construction and obtain [14, Section 2] an object $W_\bullet \in \mathrm{s}\mathbf{T}_*$ with $W_n B = W^{n+1}B$ for all $n \geq 0$ and a factorization of the map $A \to B$ into an E^2 cofibration $A \to W_\bullet \in \mathrm{s}\mathbf{T}_*$, followed by a map $W_\bullet B \to B \in \mathrm{s}\mathbf{T}_*$ which (in view of the argument of [14, Section 2]) is an E^2 weak equivalence. A diagonal argument now yields more generally the following lemma:

4.6. Another factorization lemma. *Every map* $X \to Y \in \mathrm{s}\mathbf{T}_*$ *can be factored into an* E^2 *cofibration* $X \to \operatorname{diag} W_\bullet Y \in \mathrm{s}\mathbf{T}_*$, *followed by a map* $\operatorname{diag} W_\bullet Y \to Y \in \mathrm{s}\mathbf{T}_*$ *which, for every* $i \geq 1$, *induces a trivial fibration of simplicial groups* $\pi_i(\operatorname{diag} W_\bullet Y) \to \pi_i Y$. □

Now we are finally ready to give a

Proof of 3.1. First we verify the closed model category axioms CM1–CM5 (1.5). The first three axioms are easy. To verify CM5 one factors the given map as in Lemma 4.4 or Lemma 4.6 and then factors the second map so obtained into a trivial Reedy cofibration followed by a Reedy fibration. Axiom

CM4(i) follows immediately from 3.5(ii) and to obtain CM4(ii) one factors a given trivial E^2 cofibration f, as done just above in verifying CM5(i), into a trivial E^2 cofibration q (4.4), followed by a trivial Reedy cofibration q' and an E^2 fibration which is trivial in view of CM2. One then notes that (in view of CM4(i)) f is a retract of $q'q$ and that q and q' both have the left lifting property with respect to all E^2 fibrations.

It remains to verify the simplicial axioms SM0 and SM7(b). The first of these was dealt with in 2.2, while SM7(b) is not difficult to verify using 2.5, Proposition 4.1 and Corollary 4.2 and the fact that 3.3 and Lemma 4.4, together with the above arguments, readily imply the following proposition. □

4.8. Proposition. *Every E^2 cofibration is a retract of a composition hg in which h is a trivial Reedy cofibration and g is "strongly S^1-free", i.e., if $g : X \to Y \in \mathbf{sT}_*$, then there are, for every $n \geq 0$, a CW-complex $Z_n \in \mathbf{T}_*$ which has the homotopy type of a wedge of spheres S^i $(i \geq 1)$ and a map $Z_n \to Y_n \in \mathbf{T}_*$ such that the induced map $(X_n \coprod_{L_nX} L_nY) \coprod Z_n \to Y_n \in \mathbf{T}_*$ is a homeomorphism.* □

4.9. Proposition. *Every trivial E^2 cofibration is a retract of a composition hgf in which h and f are trivial Reedy cofibrations and g is a homotopy equivalence, i.e., if $g : X \to Y \in \mathbf{sT}_*$, then there is a map $g' : Y \to X \in \mathbf{sT}_*$ such that $g'g$ and gg' are simplicially homotopic* (4.1) *to the identity maps of X and Y respectively.* □

5. Other E^2 model category structures

To obtain the E^2 model category structure on $\mathbf{sT}_*$ we really only used

(i) the fact that the category $\mathbf{sT}_*$ is a *pointed* (i.e. initial object $= * =$ terminal object) *closed model category* with *arbitrary* (not just finite) *colimits*, in which *every object is fibrant*, and

(ii) the choice of a *cofibrant "co-grouplike" object* in $\mathbf{sT}_*$ (the 1-sphere S^1).

Thus, given any closed model category $\mathbf{C}_*$ with these properties and a cofibrant co-grouplike object in $\mathbf{C}_*$, there is an associated E^2 model category structure on $\mathbf{sC}_*$. Before explicitly describing this model category structure, we first briefly recall the notion of co-grouplike object and the notion of suspension object.

5.1. Co-grouplike objects. Let $\mathbf{C}_*$ be a pointed closed model category in which every object is fibrant and, for every two objects $X, Y \in \mathbf{C}_*$ with X cofibrant, let $[X, Y]$ denote the pointed set of homotopy classes of maps $X \to Y \in \mathbf{C}_*$ [9, Chapter I, Section 1]. A *cofibrant co-grouplike object* then will be a cofibrant object $M \in \mathbf{C}_*$, together with a homotopy class of maps $M \to M \coprod M$ which (see [15, p. 122]) induces, for every object $Y \in \mathbf{C}_*$, a group structure on

$[M, Y]$, natural in Y. Important examples of co-grouplike objects are provided by the following:

5.2. Suspension objects. With $\mathbf{C}_*$ as in 5.1, one can, for every cofibrant object $L \in \mathbf{C}_*$, construct *iterated suspension objects* $\Sigma^n L$ of L $(n \geq 0)$ by [9, Chapter I, Section 2] putting $\Sigma^0 L = L$ and requiring that each $\Sigma^{n+1} L$ $(n \geq 0)$ be the colimit of a diagram $C\Sigma^n L \leftarrow \Sigma^n L \rightarrow C\Sigma^n L$ in which the map $c : \Sigma^n L \rightarrow C\Sigma^n L$ is obtained by factoring the map $\Sigma^n L \rightarrow * \in \mathbf{C}_*$ into a cofibration $c : \Sigma^n L \rightarrow C\Sigma^n L \in \mathbf{C}_*$, followed by a weak equivalence $C\Sigma^n L \rightarrow *$. Clearly each $\Sigma^n L$ $(n \geq 0)$ is cofibrant and comes with a homotopy class of maps $\Sigma^n L \rightarrow \Sigma^n L \coprod \Sigma^n L$ which turns it into a co-group object. Moreover, *for every object* $Y \in \mathbf{C}_*$, *the resulting group* $[\Sigma^n L, Y]$ *does not depend on the choices made in constructing* $\Sigma^n L$.

Now we can formulate

5.3. The E^2 model category structure. *Given a pointed closed model category* $\mathbf{C}_*$ *with arbitrary colimits in which every object is fibrant, and given a cofibrant co-grouplike object* $M \in \mathbf{C}_*$, *there is an associated closed simplicial model category structure on* $s\mathbf{C}_*$, *in which the simplicial structure is as in* 2.2 *and in which the weak equivalences, the cofibrations, and the fibrations are the* E^2 *weak equivalences, the* E^2 *cofibrations, and the* E^2 *fibrations defined in* 5.4–5.6.

5.4. E^2 weak equivalences. A map $X \rightarrow Y \in s\mathbf{C}_*$ will be called an E^2 *weak equivalence* (with respect to M) if, for each $j \geq 0$, the induced map of simplicial groups $[\Sigma^j M, X] \rightarrow [\Sigma^j M, Y]$ is a weak equivalence of simplicial groups (see 3.2). This definition clearly implies that *every Reedy weak equivalence* (2.4) *is an* E^2 *weak equivalence.*

5.5. E^2 cofibrations. A map in $s\mathbf{C}_*$ will be called an E^2 *cofibration* (with respect to M) if it is a retract of an "M-free" map, where a map $X \rightarrow Y \in s\mathbf{C}_*$ is called *M-free* if, for each $n \geq 0$, there is a cofibrant object $Z_n \in \mathbf{C}_*$ which is weakly equivalent to a coproduct of copies of the $\Sigma^j M$ $(j \geq 0)$ and a map $Z_n \rightarrow Y_n \in \mathbf{C}_*$ such that the induced map (1.3) $(X_n \coprod_{L_n X} L_n Y) \rightarrow Y_n \in \mathbf{C}_*$ is a trivial cofibration. This implies that *every* E^2 *cofibration is a Reedy cofibration* and that *every trivial Reedy cofibration is a trivial* E^2 *cofibration.*

5.6. E^2 fibrations. A map $X \rightarrow Y \in s\mathbf{C}_*$ will be called an E^2 *fibration* (with respect to M) if it is a Reedy fibration (2.4) and an "E^2 fibration up to homotopy", i.e. for every $j \geq 0$, the induced map of simplicial groups $[\Sigma^j M, X] \rightarrow [\Sigma^j M, Y]$ is a fibration (see 3.4). Thus *every Reedy fibrant object is* E^2 *fibrant* and *every trivial Reedy fibration is a trivial* E^2 *fibration.*

Following 3.5, *E^2 fibrations and trivial E^2 fibrations can be characterized by having the right lifting property with respect to the maps* $V[n,k] \otimes \Sigma^j M \to \Delta[n] \otimes \Sigma^j M \in \mathbf{sC}_*$ *($j \geq 0, n \geq 1, 0 \leq k \leq n$) and* $\dot{\Delta}[n] \otimes \Sigma^j M \to \Delta[n] \otimes \Sigma^j M \in \mathbf{sC}_*$ *($j, n \geq 0$), respectively.* In the first part of the proof of this statement one needs the dual of the corollary of Theorem B in [12, Section 2]. The rest of the proof proceeds as before. (The corollary of Theorem B in [12, Section 2] states that if

$$\begin{array}{ccccc} A_2 & \xleftarrow{i_1} & A_1 & \xrightarrow{i_2} & A_3 \\ \downarrow & & \downarrow & & \downarrow \\ B_2 & \xleftarrow{j_1} & B_1 & \xrightarrow{j_2} & B_3 \end{array}$$

is a commutative diagram in a closed model category such that all of the objects in the diagram are cofibrant, the vertical arrows are weak equivalences and the maps i_1 and j_1 are cofibrations, then the natural induced map from the colimit of the top row to the colimit of the bottom row is a weak equivalence.)

The proof of 5.3 is essentially the same as that of 3.1, and as in 3.6 one can again justify the E^2 terminology by the existence of

5.7. A spectral sequence. *Let $X \in \mathbf{sC}_*$ be Reedy fibrant. Then there exists a first quadrant spectral sequence $\{E^r_{p,q}\}$ with*

$$E^2_{p,q} = \pi_p[\Sigma^q M, X] \quad \Longrightarrow \quad \pi_{p+q}\,\hom(M,X)$$

where $\hom(M,X)$ *is as in* 2.2.

To prove this one takes a cosimplicial resolution of M in the sense of [5, 4.3] and then proceeds as in the proof of 3.6.

Examples. Obvious examples are obtained by taking $\mathbf{sC}_* = \mathbf{sT}_*$ and choosing for M a *Moore space which is a suspension* or by taking $\mathbf{sC}_*$ to be the category of simplicial groups and choosing for M the *constant simplicial group* $\mathbb{Z}$ (the additive group of integers). Note that the constant simplicial group $\mathbb{Z}$ is co-grouplike, although it is not a suspension.

Less obvious, but potentially useful, examples result from the observation that the *opposite* $\mathbf{S}_*^{\mathrm{op}}$ of the category $\mathbf{S}$ of *pointed simplicial sets* has the required properties. Thus, given a "fibrant grouplike object" $M \in \mathbf{S}$, there is (2.5) an associated E^2 model category structure on the category $\mathbf{cS}_* = (\mathbf{sS}_*^{\mathrm{op}})^{\mathrm{op}}$ of cosimplicial pointed simplicial sets. In particular, if p is a prime and one takes for M the product of the $K(\mathbb{Z}/p,n)$'s ($n \geq 0$) [8, Section 23], then one gets the model category structure mentioned in 1.1 (ii).

We end with some remarks.

5.9. Concluding remarks. One might wonder to what extent the restrictions imposed in 5.3 on $\mathbf{C}_*$ and M are indeed necessary.

The requirement that M be a *cofibrant co-grouplike object* causes the simplicial set $[M, X]$ to be a simplicial group for every $X \in s\mathbf{C}_*$. This makes it possible to obtain factorizations CM5(i) and CM5(ii) without the small object argument [9, Chapter II, Section 3]. The small object argument would have worked for $\mathbf{C}_* = \mathbf{T}_*$ but would not work in general.

The assumption that *arbitrary colimits exist* is rather harmless since this assumption is in any case almost always satisfied, and one would expect the requirement that *every object be fibrant* to be superfluous. However, it seems difficult to prove the general version of Lemma 4.6 without this restriction. Nevertheless, although some simplicial sets are not fibrant, the E^2 model category structures on $s\mathbf{T}_*$ give rise (as they should) to

5.10. E^2 model category structures on $s\mathbf{S}_*$. *Let $M \in \mathbf{T}_*$ be a cofibrant co-grouplike object. Then there exists a closed simplicial model category structure on the category $s\mathbf{S}_*$ of simplicial pointed simplicial sets, in which the simplicial structure is as in* 2.2 *and in which*

(i) *a map is a weak equivalence whenever its realization* [8, Section 14] *is an E^2 weak equivalence with respect to M,*

(ii) *a map is a cofibration whenever it is a Reedy cofibration and its realization is an E^2 cofibration with respect to M, and*

(iii) *a map is a fibration whenever it is an Reedy fibration and its realization is an* (5.6) *E^2 fibration up to homotopy, with respect to M.*

This is not difficult to prove using the facts that the realization and singular functors satisfy the *conditions of Quillen* [9, Chapter I, Section 4, Theorem 3] and that the usual closed model category structure on $s\mathbf{S}_*$ is a *proper* one in the sense of [2, 1.2].

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