

Conjectural calculations of general linear group homology

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ABSTRACT. We use étale homotopy theory together with the work of Artin and Verdier to translate some of the K-theory conjectures of Quillen and Lichtenbaum into explicit conjectures about general linear group homology.

§1. INTRODUCTION. Let F be an algebraic number field (a finite extension of the rational numbers \mathbb{Q}), let \mathcal{O} be the ring of algebraic integers in F , and let ℓ be an ordinary prime number. Quillen [12] and Lichtenbaum [8] have offered some remarkable conjectures relating the (ℓ -adic) algebraic K-theory of \mathcal{O} to more classical invariants of the field F . The most dramatic statement posits a connection between orders of various K-groups which are known to be finite and values of the zeta-function of F :

CONJECTURE 1.1. [8, 2.4] If F is totally real and m is an odd positive integer, then, up to powers of two,

$$\#K_{2m}(\mathcal{O})/\#K_{2m+1}(\mathcal{O}) = |\zeta(F, -m)|.$$

Another formulation, which is tied to 1.1 by known and suspected connections between values of the zeta-function and orders of étale cohomology groups, gives an explicit description of ℓ -adic K-theory in terms of étale cohomology:

CONJECTURE 1.2. [12, §9] If ℓ is odd or F is totally imaginary, there are isomorphisms

$$K_n(\mathcal{O}[1/\ell]) \otimes \mathbb{Z}_\ell \approx H_{\text{cont}}^j(\text{Spec } \mathcal{O}[1/\ell]_{\text{et}}, \mathbb{Z}_\ell(i))$$

where $n = 2i - j$; $j = 1, 2$; and $n \geq 1$.

As in [6] (generalized according to §3 below if $\ell = 2$ and F has a real embedding) let $\hat{K}_*^{\text{et}}(\mathcal{O}[1/\ell])$ denote the ℓ -adic étale K-theory of $\mathcal{O}[1/\ell]$, which

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is defined in terms of the etale homotopy theory of $\text{Spec } O[1/\ell]$. There is a natural map $\phi_* : K_*(O[1/\ell]) \otimes \mathbb{Z}_\ell \rightarrow \hat{K}_*^{\text{et}}(O[1/\ell])$. The following conjecture includes 1.2 [6, 8.8]:

CONJECTURE 1.3. ([6, §8] and §3 below) The map

$$\phi_* : K_n(O[1/\ell]) \otimes \mathbb{Z}_\ell \rightarrow \hat{K}_n^{\text{et}}(O[1/\ell])$$

is an isomorphism if $n \geq 1$.

The main technical attraction of 1.3 is that the map ϕ_* is geometric, that is, it can be identified with the homotopy map $\pi_* U \rightarrow \pi_* V$ induced by a certain map $\phi : U \rightarrow V$ between connected pointed spaces (see §3). In [6] we exploited this geometric quality to show that the map ϕ_* is often surjective. In this paper we observe that ϕ_* is an isomorphism iff ϕ induces a homology isomorphism $H_*(U, \mathbb{Z}/\ell) \rightarrow H_*(V, \mathbb{Z}/\ell)$. By construction $H_*(U, \mathbb{Z}/\ell)$ is exactly the Eilenberg-MacLane group homology of the infinite general linear group $GL(O[1/\ell])$, and in some situations we can explicitly compute $H_*(V, \mathbb{Z}/\ell)$. This leads to some specific conjectures about general linear group homology (4.5, 4.6). These conjectures would be implied by the truth of 1.3.

Our main technique in computing $H_*(V, \mathbb{Z}/\ell)$ is to combine the etale homotopy theoretic methods of [6] with the etale cohomology calculations of Artin and Verdier [3] (cf. [9], [14]). The expert will appreciate that we can work only in those number-theoretic contexts in which class group considerations (e.g. numerators of Bernoulli numbers) do not come into play.

NOTATION. Throughout the paper, ℓ denotes a fixed prime, R the ring $\mathbb{Z}[1/\ell]$, F an algebraic number field, and O the ring of algebraic integers in F .

§2. ETALE HOMOLOGY OF NUMBER RINGS.

The Artin-Verdier duality theorem for the etale cohomology of rings of integers in number fields [3] enables us to compute the etale homology of O . This computation is just a re-interpretation of the duality theorem and related results as presented by Mazur [9] and Zink [14].

If A is a noetherian ring, the i 'th etale homology group of A with trivial \mathbb{Z}/k coefficients is the pro-abelian-group $H_i(\text{Spec } A_{\text{et}}, \mathbb{Z}/k)$ described in [7, p. 68].

DEFINITION. The i 'th etale homology group of A with trivial (pro-finite) integral coefficients, denoted $H_i(\text{Spec } A_{\text{et}})$, is the composite pro-abelian group $\{H_i(\text{Spec } A_{\text{et}}, \mathbb{Z}/k)\}$ in which the additional structure maps are induced by the natural surjections $\mathbb{Z}/k \rightarrow \mathbb{Z}/k'$ for $k' \mid k$.

The pro-groups $H_i(\text{Spec } \mathcal{O}_{\text{et}})$ are the analogues, for the étale topology on $\text{Spec } \mathcal{O}$, of the ordinary homology groups of a space with trivial (i.e. untwisted) integral coefficients. Describing these pro-groups for a ring of algebraic integers \mathcal{O} requires some notation. Let r_1 denote the number of real embeddings of \mathcal{O} . A non-zero element of \mathcal{O} is said to be totally positive if its image under any real embedding of \mathcal{O} is a positive real number. Let $\mathcal{O}_{\text{pos}}^*$ denote the multiplicative group of totally positive units of \mathcal{O} , and $\text{Cl}_{\text{pos}}(\mathcal{O})$ the ray class group of \mathcal{O} (i.e. the group of fractional ideals modulo principal fractional ideals with totally positive generators). Let $\widehat{\quad}$ denote profinite completion.

THEOREM 2.1. ([3], [9], [14]) As above, let \mathcal{O} be the ring of algebraic integers in a number field. Then there are natural isomorphisms

$$\begin{array}{ll}
 H_i(\text{Spec } \mathcal{O}_{\text{et}}) \simeq & \mathbb{Z}^{\wedge} & i = 0 \\
 & \text{Cl}_{\text{pos}}(\mathcal{O}) & i = 1 \\
 & (\mathcal{O}_{\text{pos}}^*)^{\wedge} & i = 2 \\
 & (\mathbb{Z}/2)^{r_1} & i \text{ odd, } i \geq 3 \\
 & 0 & i \text{ even, } i \geq 3
 \end{array}$$

SKETCH OF PROOF. If \mathcal{O} is totally imaginary ($r_1 = 0$), Theorem 3.1 of [3] gives isomorphisms

$$H_q(\text{Spec } \mathcal{O}_{\text{et}}, \mathbb{Z}/k) \rightarrow \text{Ext}^{3-q}(\text{Spec } \mathcal{O}_{\text{et}}; \mathbb{Z}/k, G_m)$$

(see also [9, 2.4] and [14, 3.2.1]). The Ext-groups on the right-hand-side can be calculated easily using [3, Cor. 1.5] (see also [9, p. 539]) and the theorem follows by passing to the limit in k . If \mathcal{O} has a real embedding ($r_1 > 0$) the theorem is proved by combining the more delicate duality theorem of [14, 3.2.1] with several long-exact sequence arguments.

From the calculation it is easy to derive information more directly relevant to K-theory about $\text{Spec } \mathcal{O}[1/\ell]_{\text{et}}$. For each prime α of \mathcal{O} above ℓ , let \mathcal{O}_{α} denote the completion of \mathcal{O} at α and F_{α} the completion of the quotient field F of \mathcal{O} at α . According to the decomposition lemma [9] [1] there is a homotopy pushout diagram

$$\begin{array}{ccc}
 \prod_{\alpha|\ell} (\text{Spec } F_{\alpha})_{\text{et}} & \rightarrow & (\text{Spec } O[1/\ell])_{\text{et}} \\
 \downarrow & & \downarrow \\
 \prod_{\alpha|\ell} (\text{Spec } O_{\alpha})_{\text{et}} & \rightarrow & (\text{Spec } O)_{\text{et}}
 \end{array}$$

of etale homotopy types. According to local class field theory [9], [13] the relative homology group $H_i((\text{Spec } O_{\alpha})_{\text{et}}, (\text{Spec } F_{\alpha})_{\text{et}})$ is $(O^*)^{\wedge}$ if $i = 2$ and zero otherwise. Calculating with the long exact homology sequence of the right-hand vertical map above then gives the following theorem. To simplify notation, let $A_{\text{pos}}^{\ell}(O)$ and $B_{\text{pos}}^{\ell}(O)$ denote the kernel and cokernel respectively of the natural map

$$(O^*_{\text{pos}})^{\wedge} \rightarrow \left(\prod_{\alpha|\ell} O^*_{\alpha} \right)^{\wedge}$$

THEOREM 2.2. There are natural isomorphisms

$$\begin{array}{ll}
 H_i(\text{Spec } O[1/\ell]_{\text{et}}) \simeq & \mathbb{Z}^{\wedge} & i = 0 \\
 & A_{\text{pos}}^{\ell}(O) & i = 2 \\
 & (\mathbb{Z}/2)^{\Gamma_1} & i \text{ odd, } i \geq 3 \\
 & 0 & i \text{ even, } i \geq 3
 \end{array}$$

as well as a short exact sequence

$$0 \rightarrow B_{\text{pos}}^{\ell}(O) \rightarrow H_1(\text{Spec } O[1/\ell]_{\text{et}}) \rightarrow \text{Cl}_{\text{pos}}(O) \rightarrow 0$$

REMARK. By global class field theory it is possible to identify $H_1(\text{Spec } O[1/\ell]_{\text{et}})$ as a more or less evident quotient of the group of idèles of F .

53. REFORMULATION OF THE QUILLEN-LICHTENBAUM CONJECTURE.

The purpose of this section is to reformulate the Quillen-Lichtenbaum conjecture into an assertion that the mod ℓ cohomology of the infinite general linear group over O can be calculated in terms of the mod ℓ cohomology of the reduced etale K-theory space. This observation, implicit in [6], becomes more apparent if one employs the invariance property of Corollary 3.3.

As usual, R is the ring $\mathbb{Z}[1/\ell]$. For each $n \geq 0$ let GL_n denote the rank n general linear group scheme over R . According to [6, 2.5], for any noetherian R -algebra A there exists a natural map

$$\phi_n : BGL_n(A) = \text{Hom}(A, BGL_n)_{R} \rightarrow \text{Hom}_{\ell}(A, BGL_n)_{R} .$$

The maps ϕ_n pass to a limit map

$$\phi : BGL(A) \rightarrow \lim_{\rightarrow n} \text{Hom}(A, BGL_n)_{R} .$$

If A has finite mod- ℓ etale cohomological dimension, we will let $\tilde{K}^{\text{et}}(A)$ denote the component of $\lim_{\rightarrow n} \text{Hom}_{\ell}(A, BGL_n)_{R}$ which contains the image of ϕ .

PROPOSITION 3.1. Suppose that O is the ring of algebraic integers in a number field, and that ℓ is odd or that $\ell = 2$ and $\sqrt{-1} \in O$. Then the ℓ -adic Quillen-Lichtenbaum conjecture for O is true iff the above map

$$\phi : BGL(O[1/\ell]) \rightarrow \tilde{K}^{\text{et}}(O[1/\ell])$$

induces an isomorphism on mod ℓ cohomology.

PROOF. This follows immediately from [6; 4.5, 8.8] and the appropriate Whitehead theorem.

REMARK. If ℓ is odd or O has no real embeddings (i.e., if O has finite mod ℓ etale cohomological dimension) we will refer to the conjecture that the map ϕ of 3.1 induces an isomorphism on mod ℓ cohomology as the ℓ -adic Quillen-Lichtenbaum conjecture for O , even in cases to which 3.1 does not apply.

Let $f : X \rightarrow \text{Spec } R_{\text{et}}$ be a map of pro-spaces. If X has finite mod ℓ cohomological dimension, let $\tilde{K}^{\text{top}}(X)$ denote the identity component of the direct limit

$$\lim_{\rightarrow n} \text{Hom}_{\ell}(X, (BGL_n)_{\text{et}})_{R_{\text{et}}}$$

(See [6, 2.3]. The identity component of $\text{Hom}_{\ell}(X, (BGL_n)_{\text{et}})_{R_{\text{et}}}$ is the component which contains the composite map $X \rightarrow \text{Spec } R_{\text{et}} \rightarrow (BGL_n)_{\text{et}}$ induced by the section $\text{Spec } R \rightarrow BGL_n$ corresponding to the identity matrix in $GL_n(R)$).

Observe that if A is a noetherian R -algebra of finite mod ℓ etale cohomological dimension, X a pro-space of finite mod ℓ cohomological dimension, and $X \rightarrow \text{Spec } A_{\text{et}}$ a map, the commutative diagram

$$\begin{array}{ccc} X & \rightarrow & (\text{Spec } A)_{\text{et}} \\ & \searrow & \swarrow \\ & & (\text{Spec } R)_{\text{et}} \end{array}$$

induces a map $\tilde{K}^{\text{et}}(A) \rightarrow \tilde{K}^{\text{top}}(X)$.

Let ζ_ℓ denote a primitive ℓ 'th root of unity. If $X \rightarrow (\text{Spec } R)_{\text{et}}$ is a map of pro-spaces, let \tilde{X} denote the pullback to X of the $(\ell - 1)$ -fold Galois covering $\text{Spec } R[\zeta_\ell]_{\text{et}} \rightarrow \text{Spec } R_{\text{et}}$.

PROPOSITION 3.2. Suppose that A is a noetherian R -algebra of finite mod ℓ etale cohomological dimension, X a pro-space of finite mod ℓ cohomological dimension, and $f : X \rightarrow (\text{Spec } A)_{\text{et}}$ a map. Then if $\tilde{f} : \tilde{X} \rightarrow (\text{Spec } A)_{\text{et}}^{\sim}$

induces an isomorphism on mod ℓ cohomology, the induced map $\tilde{K}^{\text{et}}(A) \rightarrow \tilde{K}^{\text{top}}(X)$ is a homotopy equivalence.

PROOF. Let $\mathbb{Z}_\ell(i)$ ($i \geq 0$) denote the coefficient system of [6, §5]. It follows from the hypotheses, together with a limit argument [6, 2.9], that f induces isomorphisms

$$H_{\text{cont}}^i(\text{Spec } A_{\text{et}}, \mathbb{Z}_\ell(i)) \rightarrow H_{\text{cont}}^i(X, \mathbb{Z}_\ell(i))$$

for any $i \geq 0$. The result now follows from [6, Proof of 5.1] and the obstruction theory spectral sequences [6, Proof of 2.11].

Assume for the moment that $\ell = 2$. In this case none of the above discussion applies to $O[1/\ell]$ if the number ring O has a real embedding, since then $O[1/\ell]$ has infinite mod ℓ etale cohomological dimension. However, there does exist a finite etale cover of $\text{Spec } O[1/\ell]$, namely $\text{Spec } O[1/\ell, \sqrt{-1}]$, which does have finite mod ℓ etale cohomological dimension. With this in mind we will extend the above machinery as follows.

A noetherian R -algebra A is said to have virtually finite mod ℓ etale cohomological dimension if there is a finite etale Galois covering $p : \text{Spec } A' \rightarrow \text{Spec } A$ with the property that $\text{Spec } A'$ has finite mod ℓ etale cohomological dimension. For such an A , define $\tilde{K}^{\text{et}}(A)$ to be the "identity component" of the function space

$$\text{Hom}^{\Gamma}(\text{E}\Gamma, \tilde{\mathbb{K}}^{\text{et}}(A'))$$

where Γ is the Galois group of A' over A , $\text{E}\Gamma$ is a contractible space on which Γ acts freely, $\tilde{\mathbb{K}}^{\text{et}}(A')$ is as defined above, and the action of Γ on $\tilde{\mathbb{K}}^{\text{et}}(A')$ is through maps induced by the action of Γ on A' . It is easy to construct a map $\phi : \text{BGL}(A) \rightarrow \tilde{\mathbb{K}}^{\text{et}}(A)$ and to check, using [6, 7.1], that neither $\tilde{\mathbb{K}}^{\text{et}}(A)$ nor the map ϕ depend up to homotopy on the choice of A' ; in particular, if A already has finite mod ℓ etale cohomological dimension, this more general definition of $\tilde{\mathbb{K}}^{\text{et}}(A)$ agrees with the earlier one. For any number ring O , we will now call the conjecture that the map

$\phi : \text{BGL}(O[1/\ell]) \rightarrow \tilde{\mathbb{K}}^{\text{et}}(O[1/\ell])$ induces an isomorphism on mod ℓ cohomology the ℓ -adic Quillen-Lichtenbaum conjecture for O . It is easy to define what it means for a pro-space X to have virtually finite mod ℓ cohomological dimension, and to formulate and prove a generalization of Proposition 3.2. We will content ourselves with stating the following corollary of this generalization.

COROLLARY 3.3. Let A be a noetherian R -algebra of virtually finite mod ℓ etale cohomological dimension, X a pro-space of virtually finite mod ℓ cohomological dimension, and $f : X \rightarrow \text{Spec } A_{\text{et}}$ a map which induces an isomorphism on mod ℓ cohomology. Then if A contains a primitive ℓ 'th root of unity the natural map $\tilde{\mathbb{K}}^{\text{et}}(A) \rightarrow \tilde{\mathbb{K}}^{\text{top}}(X)$ induced by f is a homotopy equivalence.

PROOF. This follows easily from the fact that the structure map $\text{Spec } A \rightarrow \text{Spec } R$ factors through $\text{Spec } R[\zeta_{\ell}]$, where ζ_{ℓ} is a primitive ℓ 'th root of unity.

§4. SPECIFIC CALCULATIONS

In this section we will use the results of §2 - §3 to calculate the space $\tilde{\mathbb{K}}^{\text{et}}(A)$ explicitly for particular R -algebras A .

We will use BU to denote the topological classifying space of the infinite unitary group and BO the topological classifying space of the infinite orthogonal group. For any number k which is relatively prime to ℓ ,

$\text{F}\psi^k$ will stand for the space of [10], i.e., the homotopy fibre of the map $\psi^k - 1 : \text{BU} \rightarrow \text{BU}$. The symbol " $\underset{\ell}{\sim}$ " will stand for ℓ -adic homotopy equivalence.

PROPOSITION 4.1. There is an equivalence

$$\tilde{\mathbb{K}}^{\text{et}}(\mathbb{R}) \underset{\ell}{\sim} \text{BO}.$$

If \mathbb{F}_q is a field with q elements, $(q, \ell) = 1$, then there is an equivalence

$$\tilde{K}^{\text{et}}(\mathbb{F}_q) \underset{\ell}{\simeq} \mathbb{F}\psi^q$$

PROOF. By definition (§3) $\tilde{K}^{\text{et}}(\mathbb{R})$ is $\text{Hom}^{\mathbb{Z}/2}(\mathbb{E}\mathbb{Z}/2, \tilde{K}^{\text{et}}(\mathbb{C}))$. Since $\text{Spec } C_{\text{et}} \simeq *$, it follows from the remark in [6, proof of 5.1] and from the comparison theorem [2] that $\tilde{K}^{\text{et}}(\mathbb{C})$ is ℓ -equivalent, as a $\mathbb{Z}/2$ -space, to BU with the standard action of complex conjugation. The first part of the proposition is therefore a consequence of [4]. The second part is a consequence of [6, 8.6] together with Quillen's calculations [10].

Now we are ready to work out a more substantial example. Set $\ell = 2$ and observe that the image of the prime 3, say, in the quotient group $\mathbb{Z}_2^*/\{\pm 1\}$ is a topological generator. It follows easily from Theorem 2.2 and a cohomology ring argument that the natural map

$$(\text{Spec } \mathbb{R}_{\text{et}}) \vee (\text{Spec } \mathbb{F}_3)_{\text{et}} \rightarrow \text{Spec } \mathbb{Z}[1/2]_{\text{et}}$$

is a mod 2 (co-)homology equivalence. By Corollary 3.3 and the fact that a fibre square of mapping spaces arises from a wedge decomposition of the domain, there is a homotopy fibre square

$$\begin{array}{ccc} \tilde{K}^{\text{et}}(\mathbb{Z}[1/2]) & \rightarrow & \tilde{K}^{\text{et}}(\mathbb{R}) \\ \downarrow & & \downarrow \\ \tilde{K}^{\text{et}}(\mathbb{F}_3) & \rightarrow & \tilde{K}^{\text{top}}(*) \end{array}$$

In light of Proposition 4.1, this gives the following result.

PROPOSITION 4.2. There is a 2-adic homotopy fibre square

$$\begin{array}{ccc} \tilde{K}^{\text{et}}(\mathbb{Z}[1/2]) & \rightarrow & \text{BO} \\ \downarrow & & \downarrow \\ \mathbb{F}\psi^3 & \rightarrow & \text{BU} \end{array}$$

COROLLARY 4.3. Suppose that the 2-adic Quillen-Lichtenbaum conjecture is true for the ring \mathbb{Z} . Then

(i) there is a natural isomorphism

$$H^*(GL^{\text{top}}(\mathbb{R})/GL(\mathbb{Z}), \mathbb{Z}/2) \simeq H^*(SU, \mathbb{Z}/2)$$

and

(ii) there is a natural filtration of $H^*(BGL(\mathbb{Z}), \mathbb{Z}/2)$ together with an isomorphism

$$\text{Gr}H^*(BGL(\mathbb{Z}), \mathbb{Z}/2) \simeq H^*(BO, \mathbb{Z}/2) \oplus H^*(SU, \mathbb{Z}/2)$$

REMARK. Here $GL^{\text{top}}(\mathbb{R})/GL(\mathbb{Z})$ denotes the direct limit over n of the homogeneous spaces $GL_n^{\text{top}}(\mathbb{R})/GL_n(\mathbb{Z})$, and SU denotes the infinite special unitary group.

REMARK. It appears from Proposition 4.2 that $\tilde{K}^{\text{et}}(\mathbb{Z}[1/2])$ is essentially identical to the space $JK(\mathbb{Z})$ which appears in the remarkable work of Ekedahl [5].

PROOF OF 4.3. Under the stated hypothesis, $BGL(\mathbb{Z}[1/2])^+$ is equivalent at 2 to $\tilde{K}^{\text{et}}(\mathbb{Z}[1/2])$. Since the composite of the map (§3) $BGL(\mathbb{Z}[1/2])^+ \rightarrow \tilde{K}^{\text{et}}(\mathbb{Z}[1/2])$ with the map (4.2) $\tilde{K}^{\text{et}}(\mathbb{Z}[1/2]) \rightarrow BO$ is induced by the natural map $GL(\mathbb{Z}[1/2]) \rightarrow GL^{\text{top}}(\mathbb{R})$, it follows from Proposition 4.2 that the homotopy fibre of $BGL(\mathbb{Z}[1/2])^+ \rightarrow BGL^{\text{top}}(\mathbb{R})$ is equivalent, at the prime 2, to the homotopy fibre of $F\psi^3 \rightarrow BU$, i.e., to U . By the localization theorem [11] and Quillen's calculations [10], the space $BGL(\mathbb{Z}[1/2])^+$ is equivalent, at 2, to the product $BGL(\mathbb{Z})^+ \times S^1$. One concludes that the homotopy fibre of $BGL(\mathbb{Z})^+ \rightarrow BGL^{\text{top}}(\mathbb{R})$, which can be identified with $[GL^{\text{top}}(\mathbb{R})/GL(\mathbb{Z})]^+$, is equivalent at 2 to SU . This gives (i). The map $BGL(\mathbb{Z})^+ \rightarrow BGL^{\text{top}}(\mathbb{R})$ is a map of infinite loop spaces; moreover, this map induces an injection on mod 2 cohomology, since $H^*(BGL^{\text{top}}(\mathbb{R}), \mathbb{Z}/2)$ can be detected on the group of $\mathbb{Z}/2$ -characters in $GL(\mathbb{Z})$. It follows that the Serre spectral sequence of this map collapses. This proves (ii).

Using the above ideas, it is possible to make a whole family of calculations at the prime 2 with little additional effort. Let $\mathbb{R}P^\infty(\sim B\mathbb{Z}/2)$ denote infinite real projective space and S^1 the circle. Observe that there is a map $\mathbb{R}P^\infty \vee S^1 \rightarrow (\text{Spec } \mathbb{R})_{\text{et}} \vee (\text{Spec } \mathbb{F}_3)_{\text{et}}$ which induces an isomorphism on $\mathbb{Z}/2$ -cohomology. Let A be any ring finite, étale, and Galois over $\mathbb{Z}[1/2]$, and assume that the Galois group of A over $\mathbb{Z}[1/2]$ is a 2-group. Then if

X is defined by the homotopy pullback diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\quad\quad\quad} & \text{Spec } A_{\text{et}} \\
 \downarrow & & \downarrow \\
 \mathbb{R}P^\infty \vee S^1 \vee (\text{Spec } \mathbb{R})_{\text{et}} \vee (\text{Spec } \mathbb{F}_3)_{\text{et}} & \rightarrow & \text{Spec } \mathbb{Z}[1/2]_{\text{et}}
 \end{array}$$

the map $X \rightarrow \text{Spec } A_{\text{et}}$ induces an isomorphism on mod 2 (co-) homology, and the consequent equivalence $\tilde{K}^{\text{et}}(A) \rightarrow \tilde{K}^{\text{top}}(X)$ allows as above an explicit calculation of $\tilde{K}^{\text{et}}(A)$. Here are some examples of the spaces X that can arise

(ζ_{2^n} denotes a primitive 2^n 'th root of unity, and $\bar{\zeta}_{2^n}$ its complex conjugate):

<u>Ring A</u>	<u>Space X</u>
$\mathbb{Z}[1/2, \zeta_{2^n}], n \geq 2$	$S^1 \vee \dots \vee S^1, 2^{n-2} + 1$ summands
$\mathbb{Z}[1/2, \zeta_{2^n} + \bar{\zeta}_{2^n}], n \geq 2$	$S^1 \vee \mathbb{R}P^\infty \vee \dots \vee \mathbb{R}P^\infty, 2^{n-2} \mathbb{R}P^\infty$ summands
$\mathbb{Z}[1/2, \zeta_{2^n} - \bar{\zeta}_{2^n}], n \geq 3$	$S^1 \vee \dots \vee S^1, 2^{n-3} + 1$ summands

In fact, rings A of the specified type are in bijective correspondence, via X, with suitable finite covering spaces of $\mathbb{R}P^\infty \vee S^1$.

Suppose now that ℓ is an odd prime and let $O = \mathbb{Z}[\zeta_\ell]$, where ζ_ℓ is a primitive ℓ 'th root of unity. Assume in addition that O is regular in the sense of number theory, i.e., that the ideal class group of O contains no ℓ -torsion.

LEMMA 4.4. There are isomorphisms

$$H_i(\text{Spec } O[1/\ell]_{\text{et}}, \mathbb{Z}/\ell) \approx \begin{cases} \mathbb{Z}/\ell & i = 0 \\ (\mathbb{Z}/\ell)^{(\ell+1)/2} & i = 1 \\ 0 & i \geq 2 \end{cases}$$

PROOF. Let α be the unique prime of O above ℓ , O_α the completion of O at α , and $f : (O^*)^\wedge \rightarrow O_\alpha^*$ the natural map. By Theorem 2.2 and a counting argument, it is enough to show that

- (i) $\ker f$ is uniquely ℓ -divisible, and
- (ii) $\text{coker } f$ has no ℓ -torsion.

By a diagram chase (i) and (ii) together are implied by

- (i)' f induces an isomorphism on ℓ -torsion subgroups, and
- (ii)' $f \otimes \mathbb{Z}/\ell$ is injective

Condition (i)' is obvious, since the ℓ -torsion subgroups of $(O^*)^\wedge$ and O_α^* are each generated by a primitive ℓ 'th root of unity. For (ii)', let u be a unit of O which represents a non-zero element of $\ker(f \otimes \mathbb{Z}/\ell)$, so that u is not an ℓ 'th power in O but becomes an ℓ 'th power in O_α . Adjoining an ℓ 'th root of u to the quotient field F of O then produces a degree ℓ abelian extension of F which is clearly unramified away from α and is split and therefore unramified at α itself. This contradicts the assumption that the class group of O contains no ℓ -torsion.

By Lemma 4.4, it is possible to choose a map g from the wedge of $(\ell + 1)/2$ circles to $\text{Spec } O[1/\ell]_{\text{et}}$ which induces an isomorphism on mod ℓ (co-) homology. Let $\theta : \pi_1 \text{Spec } \mathbb{Z}[1/\ell]_{\text{et}} \rightarrow (\mathbb{Z}^*)^\wedge$ be the homomorphism given by the action of the Galois group of the maximal unramified extension of $\mathbb{Z}[1/\ell]$ on the ℓ -primary roots of unity. It is clear that the map g can be chosen so that under the composite

$$\begin{aligned} \pi_1 \left(\bigvee_{\frac{\ell+1}{2}} S^1 \right) &\xrightarrow{g} \pi_1 \text{Spec } O[1/\ell]_{\text{et}} \\ &\rightarrow \pi_1 \text{Spec } \mathbb{Z}[1/\ell]_{\text{et}} \xrightarrow{\theta} (\mathbb{Z}_\ell^*)^\wedge \end{aligned}$$

all of the circle generators except the first, say, map trivially and the generator of the fundamental group of the first circle maps to a topological generator of

$$\ker(\mathbb{Z}_\ell^* \rightarrow (\mathbb{Z}/(\ell - 1))^*)^\wedge \simeq (\mathbb{Z}_\ell)^\wedge$$

Find a rational prime q whose image in \mathbb{Z}_ℓ^* is a topological generator of $\ker(\mathbb{Z}_\ell^* \rightarrow (\mathbb{Z}/\ell - 1)^*)$, choose a map $\text{Spec } \mathbb{F}_q \rightarrow \text{Spec } O[1/\ell]$, and replace the first circle summand in the domain of g above by $(\text{Spec } \mathbb{F}_q)_{\text{et}}$; this gives a mod ℓ (co-) homology isomorphism

$$(\text{Spec } \mathbb{F}_q)_{\text{et}} \vee \bigvee_{\frac{\ell-1}{2}} S^1 \xrightarrow{g'} \text{Spec } O[1/\ell]_{\text{et}}$$

One can calculate using the machinery of [6, §5] that if $h : S^1 \rightarrow \text{Spec } \mathbb{Z}[1/\ell]_{\text{et}}$ is a map with the property that the composite $\theta h_{\#} : \pi_1 S^1 \rightarrow (\mathbb{Z}_{\ell}^*)^{\wedge}$ is trivial, then $\tilde{K}^{\text{top}}(S^1)$, computed with respect to the map h , is ℓ -equivalent to the unpointed function space BU^{S^1} . Along the lines of Proposition 4.2 this leads to the following conclusion.

PROPOSITION 4.5. Let ℓ be an odd regular prime, $O = \mathbb{Z}[\zeta_{\ell}]$, and q a rational prime whose image in \mathbb{Z}_{ℓ}^* is a topological generator of $\ker(\mathbb{Z}_{\ell}^* \rightarrow (\mathbb{Z}/\ell - 1)^*)$. Then there is an ℓ -adic homotopy fibre square

$$\begin{array}{ccc} \tilde{K}^{\text{et}}(O[1/\ell]) & \rightarrow & BU^W \\ \downarrow & & \downarrow \\ \mathbb{F}\psi^q & \rightarrow & BU \end{array}$$

where W is the wedge of $(\ell - 1)/2$ circles, BU^W denotes the function space of (unpointed) maps from W to BU , and the right-hand vertical map is evaluation at the basepoint.

Since BU is an infinite loop space, the function space BU^W , W as in Proposition 4.5, is equivalent to a product $BU \times (\prod_{\frac{\ell-1}{2}} U)$

COROLLARY 4.6. In the setting of Proposition 4.5 there is an ℓ -adic equivalence

$$\tilde{K}^{\text{et}}(O[1/\ell]) \underset{\ell}{\sim} \mathbb{F}\psi^q \times \left(\prod_{\frac{\ell-1}{2}} U \right)$$

Consequently, if the ℓ -adic Quillen-Lichtenbaum conjecture is true for O there is a cohomology isomorphism

$$H^*(\text{BGL}(O), \mathbb{Z}/\ell) \simeq H^*(\mathbb{F}\psi^q \times SU \times \left(\prod_{\frac{\ell-3}{2}} U \right), \mathbb{Z}/\ell)$$

REMARK. It would be interesting to obtain similar results without the assumption that ℓ is regular. There are strong reasons to believe that this would be difficult.

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