CLASSIFYING SPACES AND HOMOLOGY DECOMPOSITIONS

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ABSTRACT. Suppose that G is a finite group. We look at the problem of expressing the classifying space $\mathrm{B}G$, up to mod p cohomology, as a homotopy colimit of classifying spaces of smaller groups. A number of interesting tools come into play, such as simplicial sets and spaces, nerves of categories, equivariant homotopy theory, and the transfer.

CONTENTS

1. INTRODUCTION

In these notes we discuss a particular technique for trying to understand the classifying space $\mathrm{B}G$ of a finite group G. The technique is especially useful for studying the cohomology of $\mathrm{B}G$, but it can also serve other purposes. The approach is pick a prime number p and

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construct BG, up to mod p cohomology, by gluing together classifying spaces of proper subgroups of G. A construction like this is called a *homology decomposition* of BG; in principle it gives an inductive way to obtain information about BG from information about classifying spaces of smaller groups. These homology decompositions are certainly interesting on their own, but another reason to work with them is that it illustrates how to use some everyday topological machinery. We try to make the machinery easier to understand by explaining some things that are usually taken for granted.

The outline of the paper is this. Section 2 introduces classifying spaces and shows how to construct them. Section 3 faces the issue that for our purposes it is much easier to work with combinatorial models for topological spaces than with topological spaces themselves. There is an extended attempt to motivate the particular combinatorial models we will use, called *simplicial sets*; the section ends with a description of the simplicial sets that correspond to classifying spaces. Section 4 gives a systematic account of a large class of gluing constructions called *homotopy colimits*. The easiest way to describe homotopy colimits is with the help of *simplicial spaces* (although from the point of view of §3 simplicial spaces look a little peculiar!). Simplicial spaces lead to an appealing spectral sequence for the homology of a homotopy colimit (4.16). Section 5 focuses on the *nerve* of a category; this is the homotopy colimit of the constant functor with value a one-point space. It turns out that there is a way using the "Grothendieck Construction" to represent other more complex homotopy colimits as nerves. The advantage of this is that it is easy (by using natural transformations between functors) to construct homotopies between maps of nerves, while constructing combinatorial homotopies between maps of arbitrary simplicial sets is a tedious and error-prone business. Section 6 introduces another specialized homotopy colimit (the *homotopy orbit space*) associated to a group action.

In §7, at last, the main characters come on stage: there is a definition of *homology decomposition* in terms of homotopy colimits, and an explanation (using the results of §5) of how any collection of subgroups of G which satisfies a certain "ampleness" property (7.7) gives rise to three distinct homology decompositions for BG. Section 8 points out that some homology decompositions, called *sharp* decompositions, are better than others, in that a sharp decomposition gives a formula for the homology of BG instead of just a spectral sequence. In this section there are also some examples of homology decompositions, although only a few trivial examples are analyzed in full. The rest of the paper is devoted to working out some nontrivial cases. Section 9 shows that the spectral sequence associated to a homology decomposition can be interpreted as an "isotropy spectral sequence" associated to the action of G on a space X. The following sections (§10, §11) give a homological interpretation of the E^2-page of the isotropy spectral sequence, and develop methods for proving that this E^2-page collapses in such a way that $\mathrm{H}_* \mathrm{B}G$ appears along the vertical axis and all other groups are zero. If such a collapse takes place, the corresponding homology decomposition in sharp. Sections 12 and 13 apply this homological machinery to study two specific collections of subgroups of G, the collection of all nontrivial p-subgroups, and the collection of all nontrivial elementary abelian p-subgroups. It turns out that both collections are ample (as long as p divides the order of G), and that many of the six homology decompositions associated to these collections are sharp. The appendix contains a result on G-spaces which is referred to in §10.

2. Classifying spaces

Let G be a discrete group. Later on, we will assume that G is finite.

2.1. **Definition.** A *classifying space* for G is a pointed connected CW–complex B such that $\pi_1 B$ is isomorphic to G and $\pi_i B$ is trivial for $i > 1$.

2.2. *Remark.* We will usually assume that a classifying space B for G comes with a chosen isomorphism $\iota_B : \pi_1 B \approx G$.

2.3. *Example.* The circle S^1 is a classifying space for the infinite cyclic group $\mathbb{Z}$; real projective space $\mathbb{R}P^\infty$ is a classifying space for $\mathbb{Z}/2$. Let $\mathbb{Z}/p^k$ act on the unit sphere S^{2n-1} in $\mathbb{C}^n$ in such a way that the generator acts by multiplying each coordinate by $e^{2\pi i/k}$. The infinite lens space $X = \cup_n S^{2n-1}/(\mathbb{Z}/p^k)$ is a classifying space for $\mathbb{Z}/p^k$. If X is a classifying space for G and Y is a classifying space for K, then $X \times Y$ is a classifying space for $G \times K$.

2.4. **Theorem.** *Any discrete group G has a classifying space.*

Sketch of proof. Start with a presentation of G by generators $\{x_\alpha\}$ and relations $\{r_\beta\}$. Let B_0 be a one-point space, and B_1 a space obtained from B_0 by attaching a 1-cell e_α for each x_α, so that B_1 is a bouquet of circles and $\pi_1 B_1$ is isomorphic to the free group $F(x_\alpha)$ on the symbols $\{x_\alpha\}$. Construct B_2 from B_1 by attaching a 2-cell e_β for each relation r_β; the attaching map for e_β should be the word in $F(x_\alpha)$ given by r_β. By the van Kampen theorem, $\pi_1 B_2$ is isomorphic in a natural way to G. Now by induction build B_n from B_{n-1} ($n > 2$) by attaching n-cells to B_{n-1} via maps $\{f_\gamma : S^{n-1} \to B_{n-1}\}$ which run through a collection of generators for $\pi_{n-1}B_{n-1}$. It is not hard to see that $\pi_1 B_n \approx G$ and $\pi_i B_n \approx 0$ for $2 \le i \le n-1$. Let $B = \cup_n B_n$. Since the sphere S^k is compact, any map $f : S^k \to B$ has image contained in B_n for some n, and so f is null homotopic for $k \ge 2$. It follows from this that B is a classifying space for G. □

With minor adjustments, the above method can be used to make more general constructions. Let X be a pointed space, W a pointed finite CW–complex, and $\Sigma^k W = S^k \wedge W$ the k-fold suspension of W. Let $X_1 = X$ and by induction build X_n ($n \ge 2$) from X_{n-1} by attaching a copy of a cone on $\Sigma^k W$ for each map $\Sigma^k W \to X$ ($k \ge 0$). Let $X_\infty = \cup X_k$. Then any map $f : \Sigma^k W \to X_\infty$ has image contained in X_n for some n, and so f is null homotopic; in fact X_∞ is in a certain sense universal with respect to this mapping property [4]. The space X_∞ is denoted $P_W(X)$. One can remove the requirement that W be a finite complex by iterating the process transfinitely. One further generalization (replacing W by a map $f : U \to V$) yields a functor L_f; essentially all known homotopy theoretic localization functors are of the form L_f for suitable f [5, 10].

For the rest of this section we will let $\mathrm{B}G$ denote some chosen classifying space for G.

2.5. **Proposition.** *If X is any classifying space for G, then up to homotopy there is a unique basepoint-preserving map $f : \mathrm{B}G \to X$ such that $\iota_{\mathrm{BG}} = \iota_X \cdot f_\#$ (cf. 2.2). The map f is a homotopy equivalence.*

Sketch of proof. Let f send the basepoint of $\mathrm{B}G$ to the basepoint of X and each 1-cell e_α of $\mathrm{B}G$ to a loop in X representing $(\iota_X)^{-1}\iota_{\mathrm{BG}}(\langle e_\alpha \rangle)$. Now extend over the 2-skeleton of $\mathrm{B}G$ by using the fact that any relation among the homotopy classes $\{\langle e_\alpha \rangle\}$ in $\pi_1\,\mathrm{B}G$ also holds among their images in $\pi_1 X$, and extend over higher skeleta of $\mathrm{B}G$ by using the fact that the higher homotopy groups of X are trivial. The same idea gives a basepoint-preserving homotopy between f and any other map f' of the same type. The map f is obviously a weak equivalence (i.e. induces isomorphisms $\pi_i\,\mathrm{B}G \approx \pi_i X$); since the domain and range of f are CW–complexes, f is a homotopy equivalence. □

A similar argument shows that if Y is any pointed connected CW–complex, then the space of pointed maps $Y \to \mathrm{B}G$ is weakly equivalent to the discrete set of homomorphisms $\pi_1 Y \to G$. What is the homotopy type of the space of *all* maps $Y \to \mathrm{B}G$?

Suppose that X is a pointed, connected CW–complex, and that $\tilde{X}$ is its universal cover. The group $G = \pi_1(X)$ acts freely on $\tilde{X}$ (say on the left) by covering transformations. Suppose that M is a right G-module and that N is a left G-module. For any space Y, let $S_*(Y)$ denote the singular chain complex of Y. Observe that $S_*(\tilde{X})$ is a chain complex of *free* modules over $\mathbb{Z}[G]$; this follows from the fact that if σ is a singular simplex of X, the singular simplices of $\tilde{X}$ which lift σ are permuted freely and transitively by G.

2.6**. Definition.** The *homology of X with coefficients in M*, denoted $\mathrm{H}_*(X;M)$, is the homology of the chain complex $M \otimes_{\mathbb{Z}[G]} S_*(\tilde{X})$. The *cohomology of X with coefficients in N*, denoted $\mathrm{H}^*(X;N)$ is the cohomology of the cochain complex $\mathrm{Hom}_{\mathbb{Z}[G]}(S_*(\tilde{X}),N)$.

These are sometimes called *twisted* or *local coefficient* (co-)homology groups of X. If $\mathbb{Z}$ is a trivial G-module, there is a natural isomorphism

$$\mathbb{Z} \otimes_{\mathbb{Z}[G]} S_*(\tilde{X}) \approx S_*(X) \ .$$

If M is an arbitrary trivial G-module, the resulting isomorphisms

$$M \otimes_{\mathbb{Z}[G]} S_*(\tilde{X}) \approx M \otimes_{\mathbb{Z}} \mathbb{Z} \otimes_{\mathbb{Z}[G]} S_*(\tilde{X}) \approx M \otimes_{\mathbb{Z}} S_*(X)$$

show that $\mathrm{H}_*(X;M)$ as defined above agrees with the usual singular homology of X with coefficients in M. A similar result holds for cohomology.

The antiautomorphism of G sending g to g^{-1} gives a way of passing from right G-modules to left G-modules, and vice versa. For this reason the distinction between right G-modules and left G-modules is not too important, and from now on we will not always pay attention to it.

If $X = \mathrm{B}G$, then $\tilde{X}$ is contractible, so $S_*(\tilde{X})$ is a free resolution of the trivial G-module $\mathbb{Z}$ over $\mathbb{Z}[G]$ [23, V.11]. In this case elementary homological algebra gives formulas for the homology and cohomology groups of X.

2.7**. Proposition.** *Suppose that M is a right G-module and that N is a left G-module. Then there are natural isomorphisms*

$$\mathrm{H}_*(\mathrm{B}G;M) \approx \mathrm{Tor}_*^{\mathbb{Z}[G]}(M,\mathbb{Z})$$
$$\mathrm{H}^*(\mathrm{B}G;N) \approx \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z},N)$$

For a general connected CW–complex X with fundamental group G there is a first-quadrant spectral sequence converging to $\mathrm{H}_*(X;M)$ whose E_2-page depends in an algebraic way on the group G and the G-modules $\mathrm{H}_*(\tilde{X};\mathbb{Z})$ and M. Can you construct this spectral sequence and identify its E_2-page?

2.8. *Remark.* The twisted homology $\mathrm{H}_*(\mathrm{B}G;M)$ (resp. the twisted cohomology $\mathrm{H}^*(\mathrm{B}G;N)$) is sometimes denoted $\mathrm{H}_*(G;M)$ (resp. $\mathrm{H}^*(G;N)$) and called the *group homology* of G with coefficients in M (resp. the *group cohomology* of G with coefficients in N). There is a small risk of confusing $\mathrm{H}_*(G;M)$ and $\mathrm{H}^*(G;M)$ with the homology or cohomology of the discrete space G, but usually this is not a problem. The groups

$\mathrm{H}_*(G; M)$ have an algebraic interpretation; they are (2.7) the left derived functors of the construction which assigns to any G-module M the abelian group $M' = M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, i.e., the largest quotient M' of M which is a trivial G-module. Similarly, $\mathrm{H}^*(G; N)$ is the collection of right derived functors of the construction $N \mapsto N^G = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, N)$.

Let G be a finitely generated abelian group. Starting from 2.3, compute $\mathrm{H}^*(\mathrm{B}G; \mathbb{F}_p)$ and $\mathrm{H}^*(\mathrm{B}G; \mathbb{Q})$ as rings. A more ambitious project is to compute the homology and cohomology of $\mathrm{B}\mathbb{Z}_{p^\infty}$, where $\mathbb{Z}_{p^\infty} = \cup_n \mathbb{Z}/p^n$. The integral homology of this space is trivial, but the integral cohomology is torsion-free. What is the rational cohomology?

2.9. *Remark.* Fix a prime number p, and let $\mathbb{F}_p$ denote the field with p elements. Let G be a finite group. In these notes we will construct various maps $f : A \to \mathrm{B}G$ which approximate $\mathrm{B}G$ up to (co-)homology, in the sense that f induces isomorphisms on $\mathrm{H}_*(-; \mathbb{F}_p)$ or $\mathrm{H}^*(-; \mathbb{F}_p)$.

One could also look for maps $f : A \to \mathrm{B}G$ which induce isomorphisms on $\mathrm{H}_*(-; M)$ or on $\mathrm{H}^*(-; N)$ for suitable $\mathbb{F}_p[G]$-modules M and N. Let G be a finite group, X a CW–complex with a map $h : \pi_1(X) \to G$, and X' the regular covering space of X corresponding to $\ker(h)$. Show that $\mathrm{H}_*(X; \mathbb{F}_p[G])$ is isomorphic to $\mathrm{H}_*(X'; \mathbb{F}_p)^{\oplus n}$, where n is the index of $\mathrm{im}(h)$ in G. Conclude that $f : A \to \mathrm{B}G$ induces an isomorphism $\mathrm{H}_*(A; M) \to \mathrm{H}_*(\mathrm{B}G; M)$ for all $\mathbb{F}_p[G]$-modules M if and only if $\pi_1(f)$ is surjective and the homotopy fibre of f has the $\mathbb{F}_p$-homology of a point. Is it necessary to assume that G is finite? What happens if f induces an isomorphism $\mathrm{H}^*(\mathrm{B}G; N) \to \mathrm{H}^*(A; N)$ for all $\mathbb{F}_p[G]$-modules N? What happens if $\mathrm{B}G$ is replaced by another space Y?

2.10. *Notation.* From now on, all homology and cohomology will have coefficients in $\mathbb{F}_p$, unless some other coefficients are specified. If there is a fundamental group G involved, we will always assume that $\mathbb{F}_p$ has a trivial G-action.

3. Simplicial complexes and simplicial sets

In order to make progress on building homology approximations to $\mathrm{B}G$ (2.9), we need to develop a method of describing spaces combinatorially. CW–complexes have nice topological properties, but without additional machinery it is not easy to give explicit cellular recipes for complicated spaces.

Abstract simplicial complexes. An *abstract simplicial complex* K is a pair (V_K, S_K), where V_K is a set and S_K is a collection of nonempty finite subsets of V_K. The elements of V_K are the *vertices* of K and the elements of S_K are the *simplices* of K. The collection S_K is required to be closed under passage to subsets: if $\sigma \in S_K$ and $\sigma' \subset \sigma$, then $\sigma' \in S_K$. To avoid carrying around extra baggage, we also insist that if $v \in V_K$ the singleton subset $\{v\}$ should belong to S_K. The subsets of a simplex σ are called its *faces*.

Associated to an abstract simplicial complex K is a space $|K|$, called the *geometric realization* of K, obtained as the space of formal linear combinations

$$\sum_{v \in V_K} t_v v \,, \tag{3.1}$$

where $0 \leq t_v \leq 1$, $\text{supp}\{t_v\} \in S_K$, and $\sum t_v = 1$. (Here $\text{supp}\{t_v\}$, called the support of $\{t_v\}$, is the set of all vertices v such that $t_v \neq 0$). The topology on $|K|$ is given as follows [26, §1.1–3]: for each simplex $\sigma \in S_K$ the subspace $\sum_{v\in\sigma} t_v v$ is topologized as a subspace of $\mathbb{R}^{\text{card}(\sigma)}$, and then $|K|$ is given the finest topology with respect to which the inclusions of these subspaces are continuous. A simplex $\sigma \in S_K$ with $\text{card}(\sigma) = n + 1$ is called an n-simplex of K, because the subspace of $|K|$ corresponding to σ is a topological n-simplex.

Abstract simplicial complexes form the objects of a category **ASC**, in which a map $K \to L$ is a map of sets $f : V_K \to V_L$ such that for each simplex σ of K, $f(\sigma)$ is a simplex of L. Such a map induces a continuous map $f : |K| \to |L|$ according to the formula

$$f(\sum t_v v) = \sum t_v f(v) \,.$$

We do not require f to be a monomorphism, so it might be necessary to rearrange terms on the right hand side of the above equality and add up the coefficients of any given vertex of L in order to obtain a sum in standard form (3.1).

3.2. *Examples.* The *abstract n-simplex* $D[n]$ is the abstract simplicial complex whose vertex set is $\mathbf{n} = \{0, 1, \ldots, n\}$ and whose collection of simplices is the set of all subsets of $\mathbf{n}$. The geometrical realization $|D[n]|$ is a standard topological n-simplex. The automorphism group of $D[n]$ is the symmetric group Σ_{n+1}, which acts by permuting vertices. There is a unique map $D[n] \to D[0]$, which on taking geometrical realizations gives the map from a topological n-simplex to a one-point space.

Describing spaces in terms of abstract simplicial complexes has a direct geometric appeal [32, §3] [26, §1], but for a couple of reasons it is difficult to use the category of abstract simplicial complexes as a foundation for homotopy theory. One problem is that the categorical product in **ASC** is not well-behaved, in the sense that it does not commute with geometric realization.

If X and Y are objects of some category $\mathbf{C}$, an object Z of $\mathbf{C}$ is called a *(categorical) product* of X and Y if Z is provided with morphisms (called projection maps) $\mathrm{pr}_1 : Z \to X$ and $\mathrm{pr}_2 : Z \to Y$ such that for any object W of $\mathbf{C}$, composition with these maps induces a bijection

$$\mathrm{Hom}_{\mathbf{C}}(W, Z) \to \mathrm{Hom}_{\mathbf{C}}(W, X) \times \mathrm{Hom}_{\mathbf{C}}(W, Y) \ .$$

It is easy to check that if Z and Z' are two products for X and Y, then there is a unique isomorphism $Z \to Z'$ which is compatible with the projection maps, and so it is common to speak of "the" product of X and Y. See [22, III.4] for an extended discussion of products and, more generally, limits.

Suppose that K and L are abstract simplicial complexes. Let $V_{K\otimes L}$ denote the cartesian product $V_K \times V_L$, let $\mathrm{pr}_1 : V_{K\otimes L} \to V_K$ and $\mathrm{pr}_2 : V_{K\otimes L} \to V_L$ be the (set theoretic) projection maps, and let $S_{K\otimes L}$ denote the collection of all subsets σ of $V_{K\otimes L}$ such that $\mathrm{pr}_1(\sigma) \in V_K$ and $\mathrm{pr}_2(\sigma) \in V_L$.

3.3. **Lemma.** *In the above situation, the abstract simplicial complex* $K \otimes L = (V_{K\otimes L}, S_{K\otimes L})$ *is the categorical product of* K *and* L *in* **ASC**. *In general, the natural map* $|K \otimes L| \to |K| \times |L|$ *is* not *a homeomorphism.*

Proof. The first statement in the lemma is obvious. For the second, it's only necessary to notice that $D[1] \otimes D[1]$ contains the 3-simplex $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$. □

The example $D[1] \otimes D[1]$ suggests that the categorical product $K\otimes L$ of two abstract simplicial complexes is too big because it is forced to be a target of too many maps. One way to deal with this problem is to reduce the number of maps in the category by introducing some extra structure on the objects. It turns out that a good way to do this is to introduce a total ordering on the vertex set of each simplex.

Ordered simplicial complexes. An *ordered simplicial complex* K is an abstract simplicial complex (V_K, S_K) with the property that V_K is furnished with a partial ordering [32, 1.1] which restricts to a total ordering on each simplex of K. The geometric realization $|K|$ of an ordered simplicial complex is defined to be the geometric realization of the underlying simplicial complex. Ordered simplicial complexes form the objects of a category **OSC**, in which the morphisms are the maps $K \to L$ of simplicial complexes which respect the partial orderings on the vertex sets.

3.4. *Remark.* A set with a partial ordering is called a partially ordered set, or *poset* for short.

3.5. *Examples.* Any abstract simplicial complex can be converted to an ordered simplicial complex by choosing a total ordering on its set of vertices. For instance, the *ordered n-simplex* $\Delta[n]$ is the ordered simplicial complex whose vertex set is the set $\mathbf{n}$ (3.2) with the usual numerical ordering and whose collection of simplices is the set of all nonempty subsets of $\mathbf{n}$. The geometrical realization $|\Delta[n]|$ is a standard topological n-simplex. The automorphism group of $\Delta[n]$ is trivial, since there are no nonidentity order-preserving bijections $\mathbf{n} \to \mathbf{n}$. As in 3.2, there is a unique map $\Delta[n] \to \Delta[0]$ which corresponds under geometric realization to the map from a topological n-simplex to a one-point space.

3.6. *Example.* Suppose that S is a poset. Associated to S is a "largest possible" ordered simplicial complex $\mathrm{Cx}(S)$ with vertex poset S; this is the complex whose simplices consist of *all* of the totally ordered subsets of S. Given any simplicial complex K, ordered or not, let $S = S_K$ be the set of simplices of K, ordered under inclusion. The ordered simplicial complex $\mathrm{Cx}(S_K)$ is called the *barycentric subdivision* [32, p. 123–124] of K, and denoted $\mathrm{sd}\, K$. The geometric realization of $\mathrm{sd}\, K$ is homeomorphic to the geometric realization of K.

Suppose that K and L are ordered simplicial complexes. Let $V_{K\times L}$ denote the cartesian product $V_K \times V_L$ with the product partial ordering (i.e., $(v, w) \le (v', w')$ if $v \le v'$ and $w \le w'$). Let $\mathrm{pr}_1 : V_{K\times L} \to V_K$ and $\mathrm{pr}_2 : V_{K\times L} \to V_L$ be the projection maps, and let $S_{K\times L}$ denote the collection of all totally ordered subsets σ of $V_{K\times L}$ such that $\mathrm{pr}_1(\sigma) \in S_K$ and $\mathrm{pr}_2(\sigma) \in S_L$.

The next proposition shows that replacing the category **ASC** by **OSC** solves the problem of products.

3.7. **Proposition.** *In the above situation, the ordered simplicial complex $K \times L = (V_{K\times L}, S_{K\times L})$ is the product of K and L in the category of ordered simplicial complexes. The natural map $|K \times L| \to |K| \times |L|$ is a homeomorphism.*

Proof. The first statement in the lemma is obvious. The indicated natural map takes a point $\sum c_k(v_k, w_k)$ of $|K \times L|$ to the point

$$(\sum c_k v_k, \sum c_k w_k)$$

of $|K| \times |L|$. The inverse homeomorphism $|K| \times |L| \to |K \times L|$ takes $(\sum s_i v_i, \sum t_j w_j)$ to the point $\sum c_k p_k$ determined as follows. Relabel so that the vertices of K and L which appear with a nonzero coefficient are

$$v_0 < v_1 < \cdots < v_m \quad \text{and} \quad w_0 < w_1 < \cdots < w_n \ .$$

Let $S_i = \sum_{i' \le i} s_{i'}$ and $T_j = \sum_{j' \le j} t_{j'}$. Note that $S_m = T_n = 1$. Let

$$C_0 \le C_1 \le \cdots \le C_{m+n}$$

be the sequence that results from arranging the $(n+m+2)$ numbers T_i and S_j in order and deleting one of the repeated 1's at the high end. Then $p_k = (v_{a(k)}, w_{b(k)})$, where $a(k)$ is the cardinality of the set $\{i \mid S_i \le C_k\}$ and $b(k)$ is the cardinality of $\{j \mid T_j \le C_k\}$. The coefficient c_k is $C_k - C_{k-1}$. See [15, p. 68]. □

3.8. *Remark.* The best way to understand the above proposition is to look at the triangulations it gives for products of low-dimensional simplices. Start with $|\Delta[1]| \times |\Delta[1]|$ and go on to $|\Delta[1]| \times |\Delta[2]|$. The process is less complicated than it seems to be at first sight. The top-dimensional simplices of $\Delta[p] \times \Delta[q]$ are indexed by (p,q)-*shuffles* [24, p. 17].

3.9. *Remark.* The careful reader will worry about whether the "inverse homeomorphism" described in the proof of 3.7 is continuous. In fact it is *not* continuous unless K and L satisfy some finiteness conditions or (which is usually the case in homotopy theory) the product $|K| \times |L|$ is tacitly given the compactly generated topology [33].

Now that we have handled products, we have to deal with a second deficiency of the category **ASC**, a deficiency which is inherited by **OSC**; this is the poor behavior of pushouts (or more generally, of colimits).

Given a diagram $Z \xleftarrow{g} X \xrightarrow{f} Y$ in some category **C**, a *pushout* for the diagram is an object P of **C** together with a commutative square (on the left)

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ Z & \xrightarrow{v} & P \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u'} \\ Z & \xrightarrow{v'} & P' \end{array}$$

such that for any commutative square on the right there is a unique morphism $t : P \to P'$ with $tu = u'$ and $tv = v'$. See [22, p. 66]; any two pushouts for the same diagram are isomorphic in a unique way respecting the pushout structure maps. The intuitive content of this notion is that the pushout P is obtained by gluing Y and Z together along X. See [22, III.1] for a discussion of the more general notion of colimit.

If $L \leftarrow K \to L'$ is a diagram of ordered simplicial complexes, one would hope to be able to form its pushout W in such a way that $|W|$ would be homeomorphic to the pushout of $|L| \leftarrow |K| \to |L'|$. This is impossible. Let $\partial\Delta[n]$ be the subcomplex of $\Delta[n]$ containing all simplices except $\{0, 1, \ldots n\}$. The pushout of the diagram

$$\Delta[0] \leftarrow \partial\Delta[n] \to \Delta[n] \ .$$

in **OSC** is isomorphic to $\Delta[0]$: just calculate with vertices, and observe that any ordered simplicial complex with one vertex is isomorphic to

$\Delta[0]$. The pushout of the corresponding geometric diagram

$$\begin{array}{ccccc} |\Delta[0]| & \longleftarrow & |\partial\Delta[n]| & \longrightarrow & |\Delta[n]| \\ \approx\downarrow & & \approx\downarrow & & \approx\downarrow \\ * & \longleftarrow & S^{n-1} & \longrightarrow & D^n \end{array} \tag{3.10}$$

is the n-sphere. The problem here is that a simplex in an ordered simplicial complex is by definition determined by its vertices, so that identifying vertices with one another automatically collapses any simplices which they form. In the topological category, however, it is possible to identify all of the vertices of a simplex to a point, or even to collapse the whole boundary of a simplex to a point, without collapsing the simplex itself.

The solution to this problem is to stop thinking of a simplex as determined by its vertices, and instead to think of it as an indivisible object which has vertices (and faces of other dimensions) associated to it in some explicit way. Of course these vertices (and other faces) are again indivisible objects of the same general type. This leads to the notion of a *simplicial set*.

Simplicial sets. Let K be an ordered simplicial complex. We think of a map $\sigma : \Delta[n] \to K$ as a kind of generalized simplex of K. Note that the maps $\Delta[n] \to K$ which give monomorphisms $\mathbf{n} \to V_K$ correspond bijectively to ordinary simplices of K, but it is a bad idea to restrict attention to these special kinds of maps, because they don't behave functorially.

What does "they don't behave functorially" mean?

A generalized simplex σ of K corresponds to a face of σ' if $\sigma = \sigma' \cdot i$ for some monomorphism $i : \Delta[n] \to \Delta[m]$. Similarly, σ is obtained from σ' by repeating vertices if $\sigma = \sigma \cdot j$ for some epimorphism $j : \Delta[n] \to \Delta[m]$. This suggests considering the following structure.

3.11. **Definition.** Let $\mathbf{\Delta}$ denote the full subcategory of **OSC** with objects $\Delta[n]$ $(n \geq 0)$. A *simplicial set* X is a functor

$$X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Sets} .$$

The category $\mathbf{\Delta}^{\mathrm{op}}$ is the *opposite category* of $\mathbf{\Delta}$ [22, p.33]. A functor with domain $\mathbf{\Delta}^{\mathrm{op}}$ is the same thing as a contravariant functor with domain $\mathbf{\Delta}$.

Simplicial sets form a category **Sp** in which a morphism is a natural transformation between functors. If X is a simplicial set, we write X_n for the set $X(\Delta[n])$ and call this the set of n-*simplices* of X. The above considerations provide a "singular" functor Sing : **OSC** $\to$ **Sp** given by

$$\mathrm{Sing}(K)_n = \mathrm{Hom}_{\mathbf{OSC}}(\Delta[n], K)\ .$$

It is easy to see that the category $\mathbf{\Delta}$ can be identified with the category whose objects are the ordered sets $\mathbf{n}$ ($n \geq 0$) and whose maps are the set maps $f : \mathbf{n} \to \mathbf{m}$ which are weakly order preserving (in the sense that $f(i) \leq f(j)$ if $i \leq j$); this amounts to noticing that a map $\Delta[n] \to \Delta[m]$ is completely determined by its effect on vertices. One can then check that the morphisms in $\mathbf{\Delta}$ are generated by injective maps $d^i : \mathbf{n-1} \to \mathbf{n}$ and surjective maps $s^i : \mathbf{n+1} \to \mathbf{n}$ ($0 \leq i \leq n$) given by the following formulas:

$$(3.12) \qquad d^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases} \qquad\qquad s^i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

These morphisms satisfy a few obvious relations

$$(3.13) \qquad \begin{aligned} d^j d^i &= d^i d^{j-1} \text{ if } i < j \\ s^j s^i &= s^i s^{j+1} \text{ if } i \leq j \\ s^j d^i &= d^i s^{j-1} \text{ if } i < j \\ s^j d^j &= \text{ identity } = s^j d^{j+1} \\ s^j d^i &= d^{i-1} s^j \text{ if } i > j+1 \end{aligned} \quad .$$

and it is possible to verify that all relations between composites of the d^i's and s^j's are consequences of these specific ones [24, p. 4]. This leads to another way to look at a simplicial set X. Recall that X is a contravariant functor on $\mathbf{\Delta}$; let $d_j = X(d^j)$ and $s_j = X(s^j)$. Then X is effectively a collection of sets X_n ($n \geq 0$) together with maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ ($0 \leq i \leq n$) satisfying a list of identities [24, p. 1] which are the opposites of the ones in (3.13). For completeness, we will write these identities out:

$$(3.14) \qquad \begin{aligned} d_i d_j &= d_{j-1} d_i \text{ if } i < i \\ s_i s_j &= s_{j+1} s_i \text{ if } i \leq j \\ d_i s_j &= s_{j-1} d_i \text{ if } i < j \\ d_j s_j &= \text{ identity } = d_{j+1} s_j \\ d_i s_j &= s_j d_{i-1} \text{ if } i > j+1\ . \end{aligned}$$

The d_i's are called *face operators* and the s_i's *degeneracy operators*. If x is an n-simplex of X, then $d_i x$ is an $(n-1)$-simplex of X which is the i'th face of x.

3.15. *Remark.* More generally, if **C** is some category, a *simplicial object* in **C** is by definition just a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$ (see 3.11); these simplicial objects form a category, in which the morphisms are the natural transformations of functors. A simplicial set is a simplicial object in **Sets**. More explicitly, a simplicial object X in **C** is a collection $\{X_n\}_{n\geq 0}$ of objects of **C** together with morphisms $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ $(0 \leq i \leq n)$ satisfying the identities 3.14.

If **C** is the category of abelian groups (or more generally the category of modules over a ring R), then the category of simplicial objects over **C** is equivalent (via a normalization functor, see 3.21 or [24, 22.4]) to the category of nonnegatively graded chain complexes over **C**. The study of these chain complexes is homological algebra. The idea of "chain complex" does not extend very well to more general categories **C**, but the idea of "simplicial object" does, and replacing chain complexes by simplicial objects gives a way to generalize homological algebra to essentially arbitrary algebraic settings. Following Quillen [27], this generalization is called *homotopical algebra*. Since simplicial sets are closely connected to topological spaces, ordinary homotopy theory can be thought of as the homotopical algebra of the category of sets! Another instance comes up in [29]; here Quillen studies the homotopical algebra of the category of commutative rings.

The geometric realization construction for ordered simplicial complexes extends to a similar construction for simplicial sets, although the description of this extended construction is a little more complicated. Let $\Delta_n = |\Delta[n]|$ be the topological n-simplex, and for each morphism $d^i : \Delta[n] \to \Delta[n+1]$ or $s^i : \Delta[n] \to \Delta[n-1]$ (3.12) denote by the same symbols the induced continuous maps $\Delta_n \to \Delta_{n+1}$ and $\Delta_n \to \Delta_{n-1}$. (The map $d^i : \Delta_n \to \Delta_{n+1}$ is the i'th face inclusion, and $s^i : \Delta_n \to \Delta_{n-1}$ is a linear collapse obtained by pinching together the i'th and $(i+1)$'st vertices.) If X is a simplicial set, the geometric realization $|X|$ is the space obtained from the disjoint union

$$\bigcup_n X_n \times \Delta_n$$

(here X_n is treated as a space with the discrete topology) by making the identifications

$$(3.16) \qquad \begin{aligned} (d_i x, p) &\sim (x, d^i p) \text{ for } (x,p) \in X_n \times \Delta_{n-1} \\ (s_i x, p) &\sim (x, s^i p) \text{ for } (x,p) \in X_{n-1} \times \Delta_n \end{aligned}$$

See [24, III]. A simplex $x \in X_n$ is said to be *degenerate* if $x = s_i x'$ for some i and some $x' \in X_{n-1}$. The space $|X|$ is a CW–complex in which the n-cells correspond to the nondegenerate n-simplices of X [24, 14.1]. If $X = \mathrm{Sing}(K)$ for an ordered simplicial complex K, then the nondegenerate simplices of X are essentially the ordinary simplices of K, and $|X|$ is homeomorphic to $|K|$.

From the categorical point of view, the geometric realization of a simplicial set X is a curious construction; it is built by gluing, but it certainly is not given as a colimit over $\mathbf{\Delta}$. The best way to understand it is as a *coend* [22, p. 223], or equivalently as a kind of tensor product over $\mathbf{\Delta}$ of the covariant functor $\mathbf{n} \mapsto \Delta_n$ with the contravariant functor $\mathbf{n} \mapsto X_n$.

All small limits and colimits exist in the category **Sp** and can be constructed dimensionwise; see [22, p. 111] for limits, and the dual (i.e. opposite-category) version for colimits. For example, the categorical product $X \times Y$ of two simplicial sets is given by

$$(X \times Y)_n = X_n \times Y_n$$

with simplicial operators $d_i = d_i^X \times d_i^Y$ and $s_i = s_i^X \times s_i^Y$. If $Z \xleftarrow{g} X \xrightarrow{g} Y$ is a diagram of simplicial sets then the pushout P of the diagram exists, and P_n can be calculated as the pushout of the diagram $Z_n \xleftarrow{g_n} X_n \xrightarrow{f_n} Y_n$ in the category of sets.

The category of simplicial sets solves the two problems we encountered above. First of all, for any simplicial sets X and Y the natural map

$$|X \times Y| \to |X| \times |Y|$$

is a homeomorphism [24, 14.3], at least if $|X| \times |Y|$ is given a suitable topology (3.9). This is an extension of the result above for ordered simplicial complexes, since if K and L are objects of **OSC**, there is an isomorphism $\mathrm{Sing}(K) \times \mathrm{Sing}(L) \approx \mathrm{Sing}(K \times L)$. Secondly, the realization functor $\mathbf{Sp} \to \mathbf{Top}$ commutes with pushouts, and, even better, with all colimits [24, 16.1] [22, p. 115]. Note how (3.10) works in **Sp**. The pushout of the diagram

$$\mathrm{Sing}(\Delta[0]) \leftarrow \mathrm{Sing}(\partial\Delta[n]) \to \mathrm{Sing}\,(\Delta[n])$$

in **Sp** is a simplicial set X which has only two nondegenerate simplices, one of dimension 0 and one of dimension n; X is an object of **Sp** which is genuinely new: it does not come from **OSC**. The geometrical realization $|X|$ is obtained from the topological n-simplex Δ_n by collapsing its boundary to a point, and is homeomorphic to the n-sphere.

3.17. *Remark.* From now on we will identify the ordered simplicial complex $\Delta[n]$ with the corresponding simplicial set $\mathrm{Sing}(\Delta[n])$ and call it the *standard n-simplex.* The set $\Delta[n]_k$ of its k-simplices is the set of nondecreasing sequences $\langle i_0, \ldots, i_k\rangle$ of elements of $\mathbf{n}$.

3.18. **Homotopy and homology of simplicial sets.** For the rest of this paper we will refer to simplicial sets as *spaces*, and sets with topologies as *topological spaces.* This is partly for convenience, and partly to emphasize the fact that in almost everything we do here, point-set topology plays no significant role; what is important are the combinatorial patterns in which simplices are matched with one another. The category **Sp** of simplicial sets is a convenient one for making homotopy theoretic manipulations (see for instance [6]).

3.19. *Remark.* We will think of the category of sets as embedded in **Sp** by the functor which assigns to a set S the constant simplicial set (i.e. constant functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Sets}$) with value S. These simplicial sets are called *discrete* spaces. Equivalently, this functor assigns to S a coproduct of copies of $\Delta[0]$, one copy for each element of S. The geometric realization of this simplicial set is homeomorphic to the set S itself (endowed with the discrete topology).

The homotopical relationship between **Sp** and **Top** is very close; besides a realization functor $\mathbf{Sp} \to \mathbf{Top}$, there is a singular functor $\mathrm{Sing} : \mathbf{Top} \to \mathbf{Sp}$ given by

$$\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X) \ .$$

Almost by definition, the cellular homology of the CW–complex $|\mathrm{Sing}(X)|$ is the singular homology of X, so it is not surprising that the more or less obvious natural map of spaces

$$|\mathrm{Sing}(X)| \to X$$

is an isomorphism on homology [24, 16.2] or even a weak homotopy equivalence. The realization functor is *left adjoint* [22, p .81] to the singular functor, and this gives a simple explanation for the fact that the realization functor commutes with colimits [22, p. 114–115]. See [17] for an extended discussion of simplicial theory.

3.20**. Definition.** A map f of spaces is said to be an *equivalence* or *weak equivalence* if the geometric realization of f is a weak equivalence of topological spaces.

The *homotopy groups* $\pi_* X$ of a space X can be defined either in terms of the ordinary homotopy groups of $|X|$, or by a direct combinatorial formula [24, §3] [24, 26.7]. Needless to say the combinatorial formula is pretty complicated, since so far it has not led to a calculation of $\pi_*(\Delta[n]/\partial\Delta[n])$! A map of spaces is a weak equivalence in the above sense if and only if it is bijective on components and for every choice of basepoint induces an isomorphism on homotopy groups. Try to find a combinatorial description of the set of components of a space X. How about a construction of the fundamental group of X (based at a vertex)?

Before considering the homology of a space, we have to make an algebraic definition.

3.21**. Definition.** Suppose that A is a simplicial abelian group (3.15). The *normalization* of A, denoted $N(A)$, is the chain complex obtained by defining

$$N(A)_n = A_n/(s_0 A_{n-1} + \cdots + s_{n-1} A_{n-1})$$

and letting $\partial : N(A)_n \to N(A)_{n-1}$ be the quotient map [24, 22.2] induced by the alternating sum of face operators

$$\sum_{i=0}^{n} d_i : A_n \to A_{n-1} \ . \qquad (3.22)$$

3.23. *Remark.* The homology groups of $N(A)$ are denoted $\mathrm{H}_* N(A)$ or sometimes, for brevity, just $\mathrm{H}_*(A)$. It is possible to form another chain complex $C(A)$ by setting $C(A)_n = A_n$ and letting $\partial : C(A)_n \to C(A)_{n-1}$ be given by formula (3.22) (without passing to any quotient). This change does not make much of a difference: the obvious surjection $C(A) \to N(A)$ is a chain map which induces an isomorphism on homology groups [24, 22.2].

3.24. **Definition.** If X is a space, let $\mathbb{Z}[X]$ denote the simplicial abelian group obtained by applying to X (dimensionwise) the free abelian group functor from the category of sets to the category of abelian groups. If M is an abelian group, let $M[X]$ denote the simplicial abelian group $M \otimes_{\mathbb{Z}} (\mathbb{Z}[X])$. The *homology of* X *with coefficients in* M, denoted $\mathrm{H}_*(X; M)$ is defined to be the homology of the chain complex $N(M[X])$.

How would you define $\mathrm{H}^*(X; M)$? There is a latent ambiguity in the above notation: if A is a simplicial abelian group, then $\mathrm{H}_*(A)$ could mean either the homology of the chain complex $N(A)$ (3.23) or the homology of the chain complex $N(\mathbb{Z}[A])$ (3.24). We promise not to use the second interpretation. Read a little further, and try to sort out how this second interpretation would involve the homology of Eilenberg–MacLane spaces.

3.25. *Remark.* If Y is a subspace of X, the relative homology $\mathrm{H}_*(X, Y; M)$ is defined to be the homology of the chain complex

$$N(M[X]/M[Y]) \approx N(M[X])/N(M[Y]) \ .$$

It is not hard to see that $\mathrm{H}_*(X, Y; M)$ is exactly the cellular homology, with coefficients in M, of the pair $(|X|, |Y|)$ of CW–complexes.

If A is a simplicial abelian group, there is a natural isomorphism [24, 22.1]

$$\pi_*(A) \approx \mathrm{H}_*(N(A)) \ .$$

In particular, if X is a space, $|\mathbb{Z}[X]|$ is space whose homotopy groups are the ordinary (singular or cellular) integral homology groups of $|X|$. Compare this to the Dold-Thom theorem [9], which expresses the homology of a topological space Y as the homotopy of the infinite symmetric product $\mathrm{SP}^\infty(Y)$. There is quite a similarity here: $\mathbb{Z}[X]$ is the simplicial free abelian group on X, while $\mathrm{SP}^\infty(Y)$ is the topological free abelian monoid on Y.

3.26. **Classifying spaces.** An action of a (discrete) group G on a space X amounts to actions of G on the sets X_n ($n \geq 0$) which commute with all face and degeneracy operators. An action of G on X is said to be *free* if the induced action of G on each X_n is free. A space X is said to be *weakly contractible* if the unique map $X \to * = \Delta[0]$ is a weak equivalence, or, in other words, if $|X|$ is a contractible topological space. Suppose that E is a weakly contractible space with a free action of G. It is not hard to see that the natural map

$$|E| \to |E/G|$$

is a principal covering map with group G, and so $|E/G|$ is a topological classifying space (2.1) for G. In the combinatorial context we have chosen to work in, we will call E/G itself a classifying space for G. Later on (5.9) we will describe a functorial construction for classifying spaces.

4. Simplicial spaces and homotopy colimits

The construction of the homotopy colimit is motivated by the fact that ordinary colimits are not well-behaved with respect to weak equivalences. For instance, consider the following commutative diagram of topological spaces (where D^n is the n-disk and S^{n-1} its boundary sphere).

$$\begin{array}{ccccc} D^n & \longleftarrow & S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \downarrow{\scriptstyle =} & & \downarrow \\ * & \longleftarrow & S^{n-1} & \longrightarrow & * \end{array} \tag{4.1}$$

All three vertical arrows are weak equivalences (even homotopy equivalences) but the colimit of the top row is homeomorphic to S^n, the colimit of the bottom row is a one-point space $*$, and the map $S^n \to *$ induced by the diagram is not a weak equivalence. The same sort of thing can happen with spaces (i.e., simplicial sets (3.18)); consider the diagram

$$\begin{array}{ccccc} \Delta[n] & \longleftarrow & \partial\Delta[n] & \longrightarrow & \Delta[n] \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & \partial\Delta[n] & \longrightarrow & * \end{array} \tag{4.2}$$

where in this case $* = \Delta[0]$. The lesson from elementary homotopy theory is that the colimit of the top row in (4.1) is the "correct" homotopy pushout: in a homotopical context, before taking the pushout of a diagram of topological spaces you should first replace the maps by equivalent cofibrations. There is a parallel principle in **Sp**; before taking the pushout of a diagram of simplicial sets, you should first replace the maps by weakly equivalent injections. What is gained by such replacement is homotopy invariance (cf. 4.14). The general homotopy colimit construction we will describe below makes these procedures systematic, and gives a way to generalize them to colimits more complicated than pushouts. The result is that we will be able to use a single language to discuss a large family of homotopy invariant ways to glue spaces together.

See [13, §10] for a conceptual approach to homotopy pushouts. From the point of view presented there (which can be traced back to [6] and [27]), the homotopy pushout is a derived functor of the pushout [13, 10.7], in a sense closely related to the way Tor is a derived functor of $\otimes$ [13, 9.6]. There is a similar relationship between general colimits and the corresponding homotopy colimits.

The homotopy colimit is defined in terms of a construction which is also useful for other purposes. This is the construction of the *realization of a simplicial space.*

Simplicial spaces and their realizations. Suppose that X is a simplicial space (3.15), i.e., a simplicial object in the category of spaces.

4.3. **Definition.** The *realization* of X is defined to be the space constructed by taking the disjoint union

$$\coprod_n X_n \times \Delta[n] \tag{4.4}$$

and making the analogues of identifications (3.16).

4.5. *Remark.* There is another very different way to construct the realization. The object X is a sequence $\{X_n\}$ of spaces, with "horizontal" maps $d_i^h : X_n \to X_{n-1}$, $s_i^h : X_n \to X_{n-1}$. Each X_n is itself a sequence of sets $X_{m,n}$ with "vertical" maps $d_i^v : X_{m,n} \to X_{m-1,n}$, $s_i^v : X_{m,n} \to X_{m-1,n}$. Combining the two directions allows X to be considered as a rectangular array $\{X_{m,n}\}$ of sets with both horizontal and vertical simplicial operators. The following proposition may be hard to believe, but it is easy to check.

4.6. **Proposition.** *If X is a simplicial space, then the realization of X is naturally isomorphic to the space* $\operatorname{diag}(X)$ *with* $\operatorname{diag}(X)_n = X_{n,n}$. *The face operators d_i and degeneracy operators s_i of* $\operatorname{diag}(X)$ *are given in terms of those of X by the formulas $d_i = d_i^h \cdot d_i^v$, $s_i = s_i^h \cdot s_i^v$.*

This proposition states that if the simplicial space X is interpreted as a functor

$$X : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Sets}\,,$$

then the realization of X is isomorphic to the functor $\boldsymbol{\Delta}^{\mathrm{op}} \to \mathbf{Sets}$ obtained by composing X with the diagonal functor $\boldsymbol{\Delta}^{\mathrm{op}} \to \boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}}$. This is the reason for the notation "diag(X)". In fact, because of 4.6, the realization of a simplicial space X is often called the *diagonal* of X.

4.7. *A homology spectral sequence.* For a simplicial space X there is an increasing filtration

$$F_0 \operatorname{diag}(X) \subset \cdots \subset F_n \operatorname{diag}(X) \subset \cdots$$

of $\operatorname{diag}(X)$ given by letting $F_n \operatorname{diag}(X)$ be the image (4.4) of the space $\cup_{i\le n} X_i \times \Delta[n]$. This gives increasing filtrations of the simplicial abelian group $\mathbb{Z}[X]$ (3.18) and of the chain complex $N(\mathbb{Z}[X])$. Associated to this filtration is a spectral sequence for $\mathrm{H}_*(\operatorname{diag}(X); \mathbb{Z})$. It turns out to be a first quadrant spectral sequence of homological type (e.g. the differential d^r has bidegree $(-r, r-1)$). We will denote this spectral sequence $E^r_{i,j}(X; \mathbb{Z})$; if there is a coefficient module M the corresponding spectral sequence is written $E^r_{i,j}(X; M)$. The E^2-page is described by the following proposition. Let $\mathrm{H}_n(X; M)$ denote the simplicial abelian group obtained by applying the functor $\mathrm{H}_n(-; M)$ (cf. 3.24) dimensionwise to X (so that $\mathrm{H}_n(X; M)_j = \mathrm{H}_n(X_j; M)$) and let $\mathrm{H}_m \mathrm{H}_n(X; M)$ denote the m'th homology group of $\mathrm{H}_n(X; M)$ (cf. 3.23).

4.8**. Proposition.** [6, XII 5.7] *If X is a simplicial space, there are natural isomorphisms*

$$E^2_{i,j}(X; M) \approx \mathrm{H}_i \mathrm{H}_j(X; M) \ .$$

The diagonal construction for a simplicial space has the following basic homotopy invariance property. It's not hard to give a direct proof by examining how $\operatorname{diag}(X)$ is glued together from the spaces $X_n \times \Delta[n]$.

4.9**. Proposition.** [6, XII 4.2, 4.3] *If $f : X \to Y$ is a map of simplicial spaces which induces weak equivalences (3.20) $X_i \to Y_i$ ($i \ge 0$), then f induces a weak equivalence* $\operatorname{diag}(X) \to \operatorname{diag}(Y)$.

4.10**. Homotopy colimits.** Suppose that $\mathbf{D}$ is a small category. Consider the poset $\mathbf{n}$ as a category with one morphism $i \to j$ if $i \le j$, and no other morphisms. The *singular complex* or *nerve* $\mathrm{N}(\mathbf{D})$ of $\mathbf{D}$ is the space given by

$$\mathrm{N}(\mathbf{D})_n = \mathrm{Hom}_{\mathbf{Cat}}(\mathbf{n}, \mathbf{D}) \ .$$

Here **Cat** is the category in which the objects are small categories and the morphisms are functors. More concretely, an n-simplex σ of $\mathrm{N}(\mathbf{D})$ is just a length n sequence

$$\sigma(0) \xrightarrow{\alpha_1} \sigma(1) \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} \sigma(n) \qquad (4.11)$$

of composable arrows in $\mathbf{D}$. The face and degeneracy operators of $\mathrm{N}(\mathbf{D})$ are given by composition with the functors determined by the formulas (3.12). In other words, $d_i\sigma$ is obtained from σ by leaving out $\sigma(0)$ if $i = 0$, by composing α_{i+1} with α_i if $0 < i < n$, and by leaving out $\sigma(n)$ if $i = n$; $s_i\sigma$ is obtained by inserting the identity map $\sigma(i) \to \sigma(i)$ between α_i and α_{i+1}.

4.12. *Example.* Any poset S can be treated as a category in which the objects are the elements of S and in which there is exactly one morphism $x \to y$ if $x \le y$ (there are no other morphisms). The nerve $\mathrm{N}(S)$ is then the same (3.19) as the ordered simplicial complex associated to S in 3.6.

Suppose now that $F : \mathbf{D} \to \mathbf{Sp}$ is a functor. (Such an F is sometimes called a *diagram of spaces with the shape of* $\mathbf{D}$.) The *simplicial replacement* of F is the simplicial space $\coprod_* F$ which in dimension n consists of the coproduct (i.e. disjoint union)

$$(\textstyle\coprod_* F)_n = \coprod_{\sigma \in \mathrm{N}(\mathbf{D})_n} F(\sigma(0)) \ .$$

The horizontal degeneracy operator $s_i : (\coprod_* F)_n \to (\coprod_* F)_{n+1}$ maps the space $F(\sigma(0))$ indexed by σ to $F((s_i\sigma)(0))$ by the identity map; the horizontal face operator d_i maps $F(\sigma(0))$ to $F((d_i\sigma)(0))$ by the identity map if $i > 0$ and by the map α_0 if $i = 0$.

The simplicial space defined here as $\coprod_* F$ differs from the one described in [6, XII.5.1] by an automorphism of the category $\mathbf{\Delta}^{\mathrm{op}}$. This is a technical point which does not affect any of its properties.

4.13. **Definition.** Let $\mathbf{D}$ be a small category and $F : \mathbf{D} \to \mathbf{Sp}$ a functor. The *homotopy colimit* of F is the space $\mathrm{hocolim}(F)$ given by $\mathrm{diag}(\coprod_* F)$.

4.14. *Remark.* The homotopy colimit construction is functorial, in the sense that a natural transformation $\tau : F \to F'$ induces a map

$$\mathrm{hocolim}\,\tau : \mathrm{hocolim}\,F \to \mathrm{hocolim}\,F' \ .$$

It follows from 4.9 that homotopy colimits have a strong homotopy invariance property: if τ induces a weak equivalence $\tau_d : F(d) \to F'(d)$ for each object d of $\mathbf{D}$, then $\mathrm{hocolim}(\tau)$ is a weak equivalence.

The homotopy colimit construction is also functorial in $\mathbf{D}$, in the sense that if $j : \mathbf{D}' \to \mathbf{D}$ is a functor and $j^*(F)$ denotes the composite $F \cdot j$, then there is a natural map

$$\mathrm{hocolim}\, j^*(F) \to \mathrm{hocolim}\,F \ .$$

4.15. *Remark.* More explicitly, the space $\mathrm{hocolim}(F)$ can be constructed [6, XII, §2] by taking

- for each object d_0 of $\mathbf{D}$ a copy of $F(d_0)$,
- for each arrow $\alpha_0 : d_0 \to d_1$ of $\mathbf{D}$ a copy of $F(d_0) \times \Delta[1]$, and in general,
- for each chain $d_0 \to \cdots \to d_n$ of composable arrows in $\mathbf{D}$ a copy of $F(d_0) \times \Delta[n]$,

and making appropriate identifications. The identifications amount to

- collapsing $F(d_0) \times \Delta[n]$ if it arises from a chain $d_0 \to \cdots \to d_n$ containing an identity map, and
- identifying the subspace $F(d_0) \times \partial\Delta[n]$ of $F(d_0) \times \Delta[n]$ with an appropriate subspace of $\operatorname{hocolim}(F)$ arising from chains of smaller length.

There is a more or less obvious natural map

$$\operatorname{hocolim} F \to \operatorname{colim} F \ .$$

This map is not usually a weak equivalence [6, XII.2.5].

4.16. *Remark.* It follows from 4.7 and the construction of $\operatorname{hocolim}(F)$ that for any abelian group M there is a natural spectral sequence $E^r_{i,j}(F; M)$ converging to $\mathrm{H}_*(\operatorname{hocolim} F; M)$. This is called the *Bousfield-Kan homology spectral sequence* of the homotopy colimit. Let $\mathbf{Ab}$ be the category of abelian groups and $\mathbf{Ab}^{\mathbf{D}}$ the category of functors $\mathbf{D} \to \mathbf{Ab}$ (with morphisms being natural transformations). According to [6, XII.5.7] the E^2-page $E^2_{i,j}(F; M)$ can be described by the formula

$$E^2_{i,j}(F; M) = \operatorname{colim}_i \mathrm{H}_j(F; M) \ .$$

where

- colim is the colimit functor $\mathbf{Ab}^{\mathbf{D}} \to \mathbf{Ab}$,
- colim_i is the i'th left derived functor of colim, and
- $\mathrm{H}_j(F; M)$ is the functor $\mathbf{D} \to \mathbf{Ab}$ obtained by composing F with the homology functor $\mathrm{H}_j(–; M)$.

What are the projective objects of $\mathbf{Ab}^{\mathbf{D}}$? Show that there are enough of them to construct left derived functors. See [16, p. 154]. There is a parallel cohomology spectral sequence

$$E_2^{i,j}(F; M) = \lim^i \mathrm{H}^j(F; M) \Rightarrow \mathrm{H}^{i+j}(\operatorname{hocolim} F; M) \ ,$$

where

- lim is the limit functor $\mathbf{Ab}^{\mathbf{D}^{\mathrm{op}}} \to \mathbf{Ab}$,
- $\lim^i$ is the i'th right derived functor of lim, and
- $\mathrm{H}^j(F; M)$ is the functor $\mathbf{D}^{\mathrm{op}} \to \mathbf{Ab}$ obtained by composing F with the cohomology functor $\mathrm{H}^j(–; M)$.

This cohomological version appears more frequently in the literature than the homological one. If $M = \mathbb{F}_p$, for instance, this cohomology spectral sequence is the $\mathbb{F}_p$-dual of $E^r_{i,j}(F; M)$.

The left derived functors of colimit are sometimes easier to calculate than you might think. Suppose that **D** is the pushout category of 4.18 below. Show that if

$$F = (A \overset{u}{\leftarrow} B \overset{v}{\rightarrow} C)$$

is a functor from **D** to abelian groups, then

$$\operatorname{colim}_i F = \begin{cases} \operatorname{coker} B \xrightarrow{(u,-v)} A \oplus C & i = 0 \\ \ker B \xrightarrow{(u,-v)} A \oplus C & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Conclude that the homology spectral sequence for homotopy pushouts amounts to the usual Mayer-Vietoris sequence. If **G** is the category of a group G (5.9) and M is a G-module, treated as a functor from **G** to abelian groups, observe that $\operatorname{colim} M = \operatorname{colim}_0 M = \mathrm{H}_0(G; M)$ (group homology, as in 2.8) and conclude that $\operatorname{colim}_i M$ is isomorphic to $\mathrm{H}_i(G; M)$.

Examples of homotopy colimits. How complicated a particular homotopy colimit construction turns out to be depends mostly on the shape of the indexing category **D**. In the following examples, we assume that F is a functor from **D** to **Sp**.

4.17. *Homotopy coproducts.* If **D** is a *trivial category* with a collection $\{d_\alpha\}$ of objects and no nonidentity morphisms, then $\operatorname{hocolim}(F)$ is $\coprod_\alpha F(d_\alpha)$ and the map $\operatorname{hocolim}(F) \to \operatorname{colim}(F)$ is an isomorphism. For example, let **D** be an arbitrary category, let d be an object of **D**, and let $i_d : \{d\} \to \mathbf{D}$ be the inclusion functor whose domain is the trivial category with the single object d. By naturality (4.14) there is an induced map

$$j_d : F(d) = \operatorname{hocolim} i_d^*(F) \to \operatorname{hocolim} F \ .$$

This shows that each space which appears as a value of the functor F has a natural map to $\operatorname{hocolim} F$.

4.18. *Homotopy pushouts.* Let 0 and 1 denote the two copies of $\Delta[0]$ inside the simplicial interval $\Delta[1]$. If **D** is the category

$$a \overset{f}{\leftarrow} b \overset{g}{\rightarrow} c$$

then $\operatorname{hocolim}(F)$ is isomorphic to the space obtained from the coproduct

$$F(a) \ \coprod \ (F(b)_f \times \Delta[1]) \ \coprod \ F(b) \ \coprod \ (F(b)_g \times \Delta[1]) \ \coprod \ F(c)$$

by gluing

- $F(b)_f \times 1$ to $F(a)$ by $F(f)$,
- $F(b)_f \times 0$ to $F(b)$ by the identity map of $F(b)$,
- $F(b)_g \times 0$ to $F(b)$ by the identity map of $F(b)$, and
- $F(b)_g \times 1$ to $F(c)$ by $F(g)$.

This is a slightly modified form of the usual double mapping cone construction of the homotopy pushout. The map from hocolim(F) to colim(F) is not usually a weak equivalence.

Sequential colimits. If $\mathbf{D}$ is the category given by the poset of nonnegative integers

$$1 \to 2 \to \cdots \to n \to \cdots$$

then, although hocolim(F) on the face of it looks large and complicated, the map hocolim(F) $\to$ colim(F) is always a weak equivalence. The same result holds if $\mathbf{D}$ is replaced by a more general "right filtering" category [6, XII.3.5].

4.19. *Nerves.* If F is the constant functor $*$ with value the one-point space, then hocolim(F) is isomorphic to the nerve $\mathrm{N}(\mathbf{D})$. More generally, if F is the constant functor with value X, then $\operatorname{hocolim} F$ is isomorphic to $X \times \mathrm{N}(\mathbf{D})$. For any F, the unique natural transformation $F \to *$ induces a map $\operatorname{hocolim} F \to \mathrm{N}(\mathbf{D})$.

In general, the map $q : \operatorname{hocolim} F \to \mathrm{N}(\mathbf{D})$ is a kind of "singular fibration" in which, after geometric realization, the fibre over each point of the interior of a nondegenerate simplex σ is homeomorphic to $|F(\sigma(0))|$. One form of Quillen's Theorem B [31, p. 97] is the statement that if F carries each morphism of $\mathbf{D}$ to a weak equivalence, then $|q|$ is up to homotopy a fibration in which the fibre over the point corresponding to an object d of $\mathbf{D}$ is weakly equivalent to $|F(d)|$.

Simplicial objects. [6, XII.3.4] If $\mathbf{D} = \boldsymbol{\Delta}^{\mathrm{op}}$, so that F is a simplicial space, then hocolim(F) is weakly equivalent to diag(F). In particular, any space X is weakly equivalent to hocolim(F), where $F : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathbf{Sets}$ has $F(\mathbf{n}) = X_n$ (see 3.19 for how to treat a set as a space).

We will see other examples of homotopy colimits later on.

5. Nerves of categories and the Grothendieck construction

The nerve of a category (4.10) is a particularly simple homotopy colimit (4.19). In this section we will describe some special properties of nerves, and then give a construction (due to Thomason [35]) which allows many homotopy colimits to be interpreted as the nerves of auxiliary categories.

Properties of the nerve. Suppose that X and Y are spaces (3.18). Two maps $f, g : X \to Y$ are said to be *simplicially homotopic* if there is a map

$$H : X \times \Delta[1] \to Y$$

such that the restriction of H to $X \times 0$ (4.18) is f and the restriction of H to $X \times 1$ is g (see [24, 5.1] for a daunting combinatorial formulation). The notion of simplicial homotopy is unsatisfactory in some ways; in particular, it does *not* give an equivalence relation on the set of maps from X to Y. Nevertheless, since the geometric realization functor preserves products and the realization of $\Delta[1]$ is a unit interval, the following proposition is clear.

5.1. **Proposition.** *Suppose that f and g are maps of spaces. If f is simplicially homotopic to g, then $|f|$ is homotopic (in the topological sense) to $|g|$. In particular, f is a weak equivalence (3.20) if and only if g is.*

In practice, explicit simplicial homotopies arise most frequently by applying the nerve construction to natural transformations between functors.

5.2. **Proposition.** *Suppose that F and G are functors between small categories. If F and G are related by a natural transformation, then* $\mathrm{N}(F)$ *is simplicially homotopic* $\mathrm{N}(G)$*. In particular,* $\mathrm{N}(F)$ *is a weak equivalence if and only if* $\mathrm{N}(G)$ *is.*

Proof. The fact that F and G are related by a natural transformation is equivalent to the statement that F and G extend to a functor

$$\tau : \mathbf{D} \times \mathbf{1} \to \mathbf{D}' .$$

(Here $\mathbf{1}$ is the category of the poset $\mathbf{1}$ (4.10)). By construction the functor $\mathrm{N}(\text{–}) : \mathbf{Cat} \to \mathbf{Sp}$ preserves products, and $\mathrm{N}(\mathbf{1})$ is exactly $\Delta[1]$, so $\mathrm{N}(\tau)$ gives the required simplicial homotopy. □

Proposition 5.2 has some immediate consequences.

5.3. **Proposition.** *If $F : \mathbf{D} \to \mathbf{D}'$ is an equivalence of categories, then* $\mathrm{N}(F) : \mathrm{N}(\mathbf{D}) \to \mathrm{N}(\mathbf{D})'$ *is a weak equivalence of spaces.*

Proof. If $F' : \mathbf{D}' \to \mathbf{D}$ is an inverse equivalence, then the composites FF' and $F'F$ are naturally equivalent to the respective identity functors. □

The conclusion of 5.3 continues to hold under the weaker assumption that there is a functor $F' : \mathbf{D}' \to \mathbf{D}$ which is either left adjoint [22, p .81] or right adjoint to F. In either case, the composites FF' and $F'F$ are connected to the respective identity functors by natural transformations. These natural transformations are not necessarily natural equivalences, but they still provide simplicial homotopies. (5.2).

An object d of a category $\mathbf{D}$ is said to be an *initial object* (resp. a *terminal object*) if for any object d' of $\mathbf{D}$ there is exactly one morphism

$d \to d'$ (resp. $d' \to d$). For instance, the empty topological space is an initial object of **Top**, and any one-point space is a terminal object.

5.4. **Proposition.** *If* $\mathbf{D}$ *has either an initial object or a terminal object, then* $\mathrm{N}(\mathbf{D})$ *is weakly contractible.*

Proof. Let $*$ denote the trivial category with one object and only the identity morphism. There is a unique functor $F : \mathbf{D} \to *$, as well as a unique functor $F' : * \to \mathbf{D}$ sending the object of $*$ to d. The composite FF' is the identity functor of $*$. If d is either an initial or a terminal object of $\mathbf{D}$, there is an obvious natural transformation connecting the identity functor of $\mathbf{D}$ to the composite $F'F$. By 5.2, $\mathrm{N}(\mathbf{D})$ is weakly equivalent to $\mathrm{N}(*) = \Delta[0]$. □

5.5. *Example.* If S is a poset with a maximal or minimal element, then the nerve of S (4.12) is weakly contractible.

An argument similar to the previous one gives the following.

5.6. **Proposition.** *If the identity functor of* $\mathbf{D}$ *is connected to some constant functor* $\mathbf{D} \to \mathbf{D}$ *by zigzag of natural transformations, then* $\mathrm{N}(\mathbf{D})$ *is weakly contractible.*

A *constant functor* is one which takes the same value d_0 on each object of $\mathbf{D}$ and sends all morphisms of $\mathbf{D}$ to the identity map of d_0. The hypothesis of the proposition means that there exists some chain of natural transformations

$$F_0 \leftarrow F_1 \to F_2 \leftarrow \cdots \to F_n$$

in which F_0 is the identity functor and F_n is a constant functor.

The following result is a bit trickier to prove with category theoretic arguments.

5.7. **Proposition.** [31, p. 86] *The nerve of* $\mathbf{D}$ *is weakly equivalent in a natural way to the nerve of* $\mathbf{D}^{op}$

5.8. *Remark.* It is pretty clear that there is in general no functor $\mathbf{D} \to \mathbf{D}^{\mathrm{op}}$ inducing a weak equivalence of nerves. Quillen's proof of 5.7 proceeds by introducing a third category $S(\mathbf{D})$ and two functors $s : S(\mathbf{D}) \to \mathbf{D}^{\mathrm{op}}$, $t : S(\mathbf{D}) \to \mathbf{D}$, both of which do induce weak equivalences when the nerve construction is applied. All of this data depends functorially on $\mathbf{D}$; this is the meaning of the phrase "in a natural way" in 5.7. Another proof of 5.7 is to notice that the geometric realization of $\mathrm{N}(\mathbf{D})$ is naturally homeomorphic to the geometric realization of $\mathrm{N}(\mathbf{D})^{\mathrm{op}}$.

We can now give a few more examples of nerves.

5.9. *Classifying spaces as nerves.* Let G be a discrete group. The *category of* G is the category $\mathbf{G}$ with one object $*$, and with the monoid of maps $* \to *$ isomorphic to G. A functor $F : \mathbf{G} \to \mathbf{Sp}$ is essentially a space $X = F(*)$ with an action (3.26) of G. We claim that the nerve $\mathrm{N}(\mathbf{G})$ is a classifying space $\mathrm{B}G$. To see this, let $\mathbf{EG}$ denote the category whose objects are the elements x of G, and in which there is exactly one morphism between any two objects, and let $\mathrm{E}G = \mathrm{N}(\mathbf{EG})$. The space $\mathrm{E}G$ is weakly contractible (5.4), because every object of $\mathbf{EG}$ is both initial and terminal. The group G acts on $\mathbf{EG}$ (via functors) by the rule $g \cdot x = gx$. The induced action of G on $\mathrm{E}G$ is free, and it is easy to see that the quotient $(\mathrm{E}G)/G$ is isomorphic to $\mathrm{N}(\mathbf{G})$. Therefore (3.26), $\mathrm{N}(\mathbf{G})$ is a classifying space for G. From now on we will let $\mathrm{B}G$ denote this specific classifying space. Note that $\mathrm{B}G$ is functorial in G, in the sense that a homomorphism $G \to H$ induces a functor $\mathbf{G} \to \mathbf{H}$ and hence a map $\mathrm{B}G \to \mathrm{B}H$.

5.10. *Groupoids.* A *groupoid* is a small category in which every morphism is invertible. Suppose that $\mathbf{D}$ is a groupoid. For any object x of $\mathbf{D}$ the *vertex group* G_x is the group $\mathrm{Hom}_{\mathbf{D}}(x, x)$. There is an obvious functor $\mathbf{G}_x \to \mathbf{D}$ taking the unique object of $\mathbf{G}_x$ to x. Choose representatives $\{x_\alpha\}$ of isomorphism classes of objects from $\mathbf{D}$, and consider the functor

$$\coprod_\alpha \mathbf{G}_{x_\alpha} \to \mathbf{D} \ .$$

It is not hard to see that this functor is an equivalence of categories [22, p. 91] and so induces a weak equivalence on nerves. This gives a weak equivalence

$$\coprod_\alpha \mathrm{B}G_{x_\alpha} \to \mathrm{N}(\mathbf{D}) \ .$$

In other words, the nerve of a groupoid is a disjoint union of classifying spaces of groups; there is one component for each isomorphism class of objects in the groupoid, and this component has the homotopy type of the classifying space of the group of automorphisms of any object in the isomorphism class.

In calculating with nerves we will use the following observation.

5.11. **Proposition.** *Suppose that* G *is a group acting on a category* $\mathbf{D}$*, and that* $\mathbf{D}^G$ *is the fixed subcategory (that is, the subcategory of* $\mathbf{D}$ *containing those objects and morphisms fixed by the action). Then the natural map*

$$\mathrm{N}(\mathbf{D}^G) \to (\mathrm{N}(\mathbf{D}))^G$$

is an isomorphism.

The Grothendieck construction. Suppose that $F : \mathbf{D} \to \mathbf{Sets}$ is a functor. The *transport category* $\mathrm{Tr}(F)$ of F is the category whose objects consist of pairs (d, x), where d is an object of $\mathbf{D}$ and $x \in F(d)$. A map $(d, x) \to (d', x')$ is a morphism $f : d \to d'$ in $\mathbf{D}$ such that $F(f)(x) = x'$. These morphisms compose according to the composition of morphisms in $\mathbf{D}$. The next proposition points out that this construction gives a categorical model for $\operatorname{hocolim} F$.

5.12. **Proposition.** *For any functor* $F : \mathbf{D} \to \mathbf{Sets}$ *there is a natural isomorphism*

$$\mathrm{N}(\mathrm{Tr}(F)) \approx \operatorname{hocolim} F\ .$$

Proof. Check that an n-simplex of $\operatorname{hocolim} F$ amounts to a pair (σ, x), where σ is an n-simplex (4.11) of the nerve of $\mathbf{D}$ and $x \in F(\sigma(0))$. These are exactly the n-simplices of $\mathrm{N}(\mathrm{Tr}(F))$. □

5.13. *Example.* Let X be a space, considered as a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Sets}$. The transport category $\mathrm{Tr}(X)$ can be thought of as a category whose objects are the simplices x of X; a morphism $x \to x'$ is a suitable morphism Φ of $\mathbf{\Delta}^{\mathrm{op}}$ (i.e. a composite of face and degeneracy operators (3.14)) such that $\Phi(x) = x'$. The nerve of $\mathrm{Tr}(X)$ is isomorphic to $\operatorname{hocolim} X$, and so is weakly equivalent to X (4). This shows that every space can be constructed up to weak equivalence as the nerve of a category.

For related results see [25] (every connected space is weakly equivalent to the nerve of a category with a single object, i.e. to the classifying space of a monoid) and [21] (every connected space is homology equivalent to the classifying space of a discrete group).

5.14. *Example.* Let G be a group and S a G-set, considered as a functor $S : \mathbf{G} \to \mathbf{Sets}$ (5.9). The transport category of S is the groupoid (5.10) whose objects are the elements of S and in which a morphism $s \to s'$ is an element of $g \in G$ such $gs = s'$. The isomorphism classes of objects in this category correspond to the orbits of the action of G on X, and the vertex group of an object $x \in S$ is the isotropy subgroup G_x. By 5.10, $\mathrm{Tr}(S)$ (or $\operatorname{hocolim} S$) is equivalent to a disjoint union $\coprod_\alpha \mathrm{B}G_{x_\alpha}$, where $\{x_\alpha\}$ runs through a set of orbit representatives for the action of G on S.

Proposition 5.12 has a wonderful generalization, due to Thomason. Suppose that $\mathbf{D}$ is a small category, and that $F : \mathbf{D} \to \mathbf{Cat}$ is a functor. The *Grothendieck Construction on* F, denoted $\mathrm{Gr}(F)$, is the category whose objects are the pairs (d, x) where d is an object of $\mathbf{D}$ and x is an object of $F(d)$. An arrow $(d, x) \to (d', x')$ in

$\mathrm{Gr}(F)$ is a pair (f, g), where $f : d \to d'$ is a morphism in $\mathbf{D}$ and $g : (F(f))(x) \to x'$ is a morphism in $F(d')$. Arrows compose according to the rule $(f, g) \cdot (f', g') = (f'', g'')$, where f'' is the composite $f \cdot f'$ and g'' is the composite of g with the image of g' under the functor $F(f)$.

5.15. **Theorem.** [35, 1.2] *Suppose that* $\mathbf{D}$ *is a small category and* $F : \mathbf{D} \to \mathbf{Cat}$ *is a functor. Let* $\mathrm{Gr}(F)$ *be the Grothendieck Construction on* F*. Then there is a natural weak equivalence*

$$\mathrm{N}(\mathrm{Gr}(F)) \sim \operatorname{hocolim} \mathrm{N}(F) \ .$$

Variations. Suppose that $F : \mathbf{D} \to \mathbf{Cat}$ is a functor. Let F^{op} denote the composite of F with the "opposite" construction $\mathbf{Cat} \to \mathbf{Cat}$; note that F^{op} is again a functor $\mathbf{D} \to \mathbf{Cat}$. It follows from 5.15, 5.7, and the homotopy invariance of homotopy colimits (4.14) that the four categories $\mathrm{Gr}(F)$, $\mathrm{Gr}(F)^{\mathrm{op}}$, $\mathrm{Gr}(F^{\mathrm{op}})$ and $\mathrm{Gr}(F^{\mathrm{op}})^{\mathrm{op}}$ *all* have nerves which are weakly equivalent in a natural way to $\operatorname{hocolim} \mathrm{N}(F)$.

6. Homotopy orbit spaces

In this section we give a description of the *homotopy orbit space*, which is a particular kind of homotopy colimit that plays a key role in the description of homology decompositions.

Suppose that X is a G-space (i.e. a space with an action of the group G), and denote by the same letter X the corresponding (5.9) functor $\mathbf{G} \to \mathbf{Sp}$.

6.1. **Definition.** The *homotopy orbit space* of the action of G on X, denoted $X_{\mathrm{h}G}$, is the space

$$X_{\mathrm{h}G} = \operatorname{hocolim}_{\mathbf{G}} X \ .$$

6.2. *Remark.* It is easy to check that the space $X_{\mathrm{h}G}$ is isomorphic to the quotient space

$$(X \times \mathrm{E}G)/G \,,$$

where $\mathrm{E}G$ is the free contractible G-space of 5.9 and G acts diagonally on the product $X \times \mathrm{E}G$. In particular there is (essentially) a fibration sequence

$$X \to X_{\mathrm{h}G} \to (\mathrm{E}G)/G = \mathrm{B}G \ . \qquad (6.3)$$

(See [24, II.7] and [28] for information about fibrations of simplicial sets). The map $X_{\mathrm{h}G} \to (*)_{\mathrm{h}G} = \mathrm{B}G$ is induced by the unique G-map $X \to *$. The space $X_{\mathrm{h}G}$ is sometimes called *the fibration over* $\mathrm{B}G$ *associated to the action of* G *on* X or the *Borel construction* of the action of G on X. Note that $\operatorname{colim}_{\mathbf{G}} X$ is the orbit space of this action, so that there is a natural map (4.15)

$$X_{\mathrm{h}G} \to X/G$$

from the homotopy orbit space to the usual orbit space. The property which distinguishes the homotopy orbit space from the usual orbit space is that (like the homotopy pushout) the homotopy orbit space is homotopy invariant (4.14): if $f : X \to Y$ is a map of G-spaces which is an ordinary weak equivalence of spaces, then $(f)_{\mathrm{h}G}$ is a weak equivalence.

The homology spectral sequence (4.16) of $X_{\mathrm{h}G}$ can be identified with the Serre spectral sequence of 6.3; it has the form

$$E^2_{i,j} = \mathrm{H}_i(\mathrm{B}G; \mathrm{H}_j(X; M)) \Rightarrow \mathrm{H}_{i+j}(X_{\mathrm{h}G}; M)\ .$$

6.4. *Example.* If S is a transitive G-set, then by 5.14 the space $S_{\mathrm{h}G}$ is weakly equivalent to $\mathrm{B}G_s$, where G_s is the isotropy subgroup of some element $s \in S$.

Let **GSp** denote the category of G-spaces. If **D** is a small category and $F : \mathbf{D} \to \mathbf{GSp}$ is a functor, then, either by inspection or by general naturality principles, $\operatorname{hocolim}(F)$ is in a natural way a G-space. The following proposition could be rephrased as a statement that homotopy limits commute.

6.5**. Proposition.** *Suppose that G is a group,* **D** *is a small category, and $F : \mathbf{D} \to \mathbf{GSp}$ is a functor. Then there is a natural isomorphism of spaces*

$$(\operatorname{hocolim} F)_{hG} \approx \operatorname{hocolim}(F_{hG})\ .$$

Proof. Check using 4.6 that the n-simplices of both spaces correspond to triples (x, σ, τ) where τ is an n-simplex of $\mathrm{N}(\mathbf{G})$ (5.9), σ is an n-simplex of $\mathrm{N}(\mathbf{D})$, and x is an n-simplex of $\sigma(0)$. □

7. Homology decompositions

Suppose that G is a discrete group. Recall (2.9) that p is a fixed prime number.

7.1**. Definition.** A *homology decomposition* for $\mathrm{B}G$ consists of a mod p homology isomorphism

$$\operatorname{hocolim} F \xrightarrow[p]{\sim} \mathrm{B}G$$

where **D** is a small category, $F : \mathbf{D} \to \mathbf{Sp}$ is a functor, and, for each object d of **D**, $F(d)$ is weakly equivalent to $\mathrm{B}H_d$ for some subgroup H_d of G.

7.2. *Remark.* In any reasonable homology decomposition, the composite map

$$\mathrm{B}H_d \simeq F(d) \to \operatorname{hocolim} F \ \to \mathrm{B}G$$

(see 4.17) agrees in some homotopical sense with the map $\mathrm{B}H_d \to \mathrm{B}G$ induced (5.9) by the inclusion of H_d in G as a subgroup.

7.3. *Remark.* Finding a good homology decomposition is a process of striking a balance. On one extreme, if $\mathbf{D}$ is the trivial category with one object and F is the constant functor with value $\mathrm{B}G$, there is a weak equivalence (even an isomorphism)

$$\operatorname{hocolim} F \approx \mathrm{B}G \ .$$

This is a homology decomposition in which all of the complexity is concentrated in the functor F. On the other extreme, if $\mathbf{D}$ is category of the group G (or the transport category (5.13) of $\mathrm{B}G$) and F is the constant functor with value the one-point space ($= \mathrm{B}\{e\}$), there is also a weak equivalence $\operatorname{hocolim} F \simeq \mathrm{B}G$. In this case F is trivial, and the complexity of the decomposition is concentrated in the shape of the category $\mathbf{D}$. Neither of these decompositions is very interesting. In the most useful homology decompositions the category $\mathbf{D}$ is simple, the subgroups H_d are small, and the focus of the structure is p-primary, in the sense that the map $\operatorname{hocolim} F \to \mathrm{B}G$ is a mod p homology isomorphism but *not* a weak equivalence.

We will get all of our homology decompositions by the method described in the following proposition.

7.4. **Proposition.** *Suppose that* $\mathbf{D}$ *is a small category and that* F *is a functor from* $\mathbf{D}$ *to the category of transitive* G*-sets. Suppose that the natural map (6.2)*

$$(\operatorname{hocolim} F)_{hG} \to \mathrm{B}G \tag{7.5}$$

is a mod p *homology isomorphism. Then there is a homology decomposition*

$$\operatorname{hocolim}(F_{hG}) \xrightarrow[p]{\sim} \mathrm{B}G \ .$$

Proof. It is easy to see that the natural maps $F(d)_{\mathrm{h}G} \to \mathrm{B}G$ induce a map $\operatorname{hocolim}(F_{\mathrm{h}G}) \to \mathrm{B}G$ which under the equivalence of 6.5 corresponds to the mod p homology isomorphism (7.5). In order to finish the proof, then, all we have to do is check that each space $F(d)_{\mathrm{h}G}$ is weakly equivalent to $\mathrm{B}H_d$ for some subgroup H_d of G. This follows from 6.4. □

Any homology decomposition can be constructed as in 7.4. Can you prove this?

Proposition 7.4 suggests that to construct a homology decomposition for BG, the first thing to search for is a G-space X with the property that the natural map $X_{\mathrm{h}G} \to \mathrm{B}G$ is a mod p homology equivalence. We will find such spaces by looking at complexes associated to *collections* of subgroups of G.

7.6. **Definition.** A *collection* $\mathcal{C}$ of subgroups of G is a set of subgroups of G which is closed under conjugation, in the sense that if $H \in \mathcal{C}$ and $g \in G$, then $gHg^{-1} \in \mathcal{C}$.

A collection $\mathcal{C}$ of subgroups is a poset with respect to inclusion of one subgroup in another, i.e., with respect to the convention that $H \le H'$ if $H \subset H'$. Let $K_{\mathcal{C}}$ be the associated ordered simplicial complex (3.6), or equivalently the nerve of the category (4.12) associated to $\mathcal{C}$. We will let $\mathbf{K}_{\mathcal{C}}$ denote this category, so that $K_{\mathcal{C}} = \mathrm{N}(\mathbf{K}_{\mathcal{C}})$. The n-simplices of $K_{\mathcal{C}}$ are just chains

$$H_0 \subset H_1 \subset \cdots \subset H_n$$

of elements of $\mathcal{C}$; in particular, if G is finite then $K_{\mathcal{C}}$ is a finite simplicial complex. The group G acts on $\mathcal{C}$ by conjugation (if $g \in G$ and $H \in \mathcal{C}$ then $g \cdot H = gHg^{-1}$). This action preserves the inclusion relation and passes to an action of G on $K_{\mathcal{C}}$.

7.7. **Definition.** A collection $\mathcal{C}$ of subgroups of G is said to be *ample* if the natural map

$$(K_{\mathcal{C}})_{\mathrm{h}G} \to \mathrm{B}G \tag{7.8}$$

is a mod p homology isomorphism.

7.9. *Example.* If $K_{\mathcal{C}}$ is contractible the collection $\mathcal{C}$ is ample, since then the map 7.8 is a weak equivalence (4.14). The homology decomposition maps (7.1) associated below to such a collection are weak equivalences, not just mod p homology isomorphisms.

At this point it is probably not clear that there exist *any* ample collections of subgroups, Before looking for some, we want to motivate the search. Given a collection $\mathcal{C}$, we will associate to it three functors

$$\begin{array}{rccl} \tilde{\alpha}_{\mathcal{C}}: & (\mathbf{A}_{\mathcal{C}})^{\mathrm{op}} & \to & \mathbf{GSets} \\ \tilde{\beta}_{\mathcal{C}}: & \mathbf{O}_{\mathcal{C}} & \to & \mathbf{GSets} \\ \tilde{\delta}_{\mathcal{C}}: & \bar{\mathrm{s}}\mathrm{d}\mathbf{K}_{\mathcal{C}} & \to & \mathbf{GSets} \end{array} \tag{7.10}$$

Of course, one issue will be to define the domain categories of these functors. We will then prove the following proposition. (The proof involves combining 5.12 with 5.3. The most interesting feature of it is the way in which working with functors and natural transformations makes it possible to construct what amount to explicit simplicial homotopies (§5) with only a small amount of work.)

7.11. **Proposition.** *If $\mathcal{C}$ is an ample collection of subgroups of G, then all three functors of* (7.10) *satisfy the hypotheses of 7.4, and the corresponding functors*

$$\alpha_{\mathcal{C}} = (\tilde{\alpha}_{\mathcal{C}})_{hG}$$
$$\beta_{\mathcal{C}} = (\tilde{\beta}_{\mathcal{C}})_{hG}$$
$$\delta_{\mathcal{C}} = (\tilde{\delta}_{\mathcal{C}})_{hG}$$

give homology decomposition for $\mathrm{B}G$*. Conversely, if any one of these functors gives a homology decomposition for* $\mathrm{B}G$*, then $\mathcal{C}$ is ample.*

The three decompositions of 7.11 are called the *centralizer decomposition*, the *subgroup decomposition*, and the *normalizer decomposition* associated to $\mathcal{C}$. The names come from the fact that

- the values of $\alpha_{\mathcal{C}}$ have the homotopy type of $\mathrm{B}C$, where C is the centralizer in G of some element of $\mathcal{C}$,
- the values of $\beta_{\mathcal{C}}$ have the homotopy type of $\mathrm{B}H$, where H is an element of $\mathcal{C}$, and
- the values of $\delta_{\mathcal{C}}$ have the homotopy type of $\mathrm{B}N$, where N is an intersection in G of normalizers of elements of $\mathcal{C}$.

The centralizer decomposition. The *$\mathcal{C}$-conjugacy category* $\mathbf{A}_{\mathcal{C}}$ is the category in which the objects are pairs (H, Σ), where H is a group and Σ is a conjugacy class of monomorphisms $i : H \to G$ with $i(H) \in \mathcal{C}$. A morphism $(H, \Sigma) \to (H', \Sigma')$ is a group homomorphism $j : H \to H'$ which under composition carries Σ' into Σ.

We are going to want to take homotopy colimits over $\mathbf{A}_{\mathcal{C}}$ (actually over its opposite category) and, given the way we have defined $\mathbf{A}_{\mathcal{C}}$, this is not possible: the category $\mathbf{A}_{\mathcal{C}}$ is not small. This difficulty is not serious, since $\mathbf{A}_{\mathcal{C}}$ is equivalent to a small category. What details have to be worked out to make this precise? Can you find an explicit small subcategory of $\mathbf{A}_{\mathcal{C}}$ which is equivalent to $\mathbf{A}_{\mathcal{C}}$?

There is a functor

$$\tilde{\alpha}_{\mathcal{C}} : (\mathbf{A}_{\mathcal{C}})^{\mathrm{op}} \to \mathbf{GSets}$$

which assigns to each object (H, Σ) the set Σ itself, on which G acts transitively by conjugation. Here is the first third of 7.11.

7.12. **Proposition.** *The collection $\mathcal{C}$ is ample if and only if the natural map*

$$(\mathrm{hocolim}\, \tilde{\alpha}_{\mathcal{C}})_{hG} \to \mathrm{B}G$$

is a mod p homology isomorphism.

Proof. According to 5.12, the space $X^\alpha_{\mathcal{C}} = \operatorname{hocolim} \tilde{\alpha}_{\mathcal{C}}$ is the nerve of a category $\mathbf{X}^\alpha_{\mathcal{C}}$ whose objects consist of pairs (H, i), where H is a group and $i : H \to G$ is a monomorphism with $i(H) \in \mathcal{C}$. A map $(H, i) \to (H', i')$ is a homomorphism $j : H \to H'$ such that $i'j = i$. The action of G on $\operatorname{hocolim} \tilde{\alpha}_{\mathcal{C}}$ corresponds to the action of G on $\mathbf{X}^\alpha_{\mathcal{C}}$ obtained by letting $g \cdot (H, i) = (H, gig^{-1})$. There is a functor

$$u : \mathbf{X}^\alpha_{\mathcal{C}} \to \mathbf{K}_{\mathcal{C}}$$

which sends (H, i) to $i(H)$, a functor which by inspection is G-equivariant. To finish the proof it is enough to show that u induces a weak equivalence from the nerve of $\mathbf{X}^\alpha_{\mathcal{C}}$ to $K_{\mathcal{C}}$. In fact, if this is true, then by 6.2 there is a commutative diagram

$$\begin{array}{ccc} (\operatorname{hocolim} \tilde{\alpha}_{\mathcal{C}})_{\mathrm{h}G} = (X^\alpha_{\mathcal{C}})_{\mathrm{h}G} & \xrightarrow{\simeq} & (K_{\mathcal{C}})_{\mathrm{h}G} \\ \downarrow & & \downarrow \\ \mathrm{B}G & \xrightarrow{=} & \mathrm{B}G \end{array}$$

in which by 4.14 the top arrow is a weak equivalence. This implies that the left hand arrow is an isomorphism on mod p homology if and only if the right hand one is.

To see that u has the desired property, use 5.3 and note that u is an equivalence of categories. In fact, there is a functor $u' : \mathbf{K}_{\mathcal{C}} \to \mathbf{X}^\alpha_{\mathcal{C}}$ which sends an object $H \in \mathcal{C}$ to the pair (H, i) where i is the inclusion map $H \to G$. The composite uu' is the identity map of $\mathbf{K}_{\mathcal{C}}$, while the composite $u'u$ is connected to the identity map of $\mathbf{X}^\alpha_{\mathcal{C}}$ by an obvious natural equivalence. □

7.13. *Remark.* If $H \subset G$ is a subgroup, let $C_G(H)$ denote the centralizer of H in G. By 6.4, the centralizer decomposition functor $\alpha_{\mathcal{C}} = (\tilde{\alpha}_{\mathcal{C}})_{\mathrm{h}G}$ assigns to any object (H, Σ) of $\mathbf{A}_{\mathcal{C}}$ a space which, for any $i \in \Sigma$, is weakly equivalent to $\mathrm{B}C_G(i(H))$.

The subgroup decomposition. The $\mathcal{C}$ *orbit category* $\mathbf{O}_{\mathcal{C}}$ is the category whose objects are the G-sets G/H, $H \in \mathcal{C}$, and whose morphisms are G-maps. There is a functor $\tilde{\beta}_{\mathcal{C}} : \mathbf{O}_{\mathcal{C}} \to \mathbf{GSets}$ which assigns to G/H the G-set G/H itself. Here is the second third of 7.11.

7.14. **Proposition.** *The collection $\mathcal{C}$ is ample if and only if the natural map*

$$(\operatorname{hocolim} \tilde{\beta}_{\mathcal{C}})_{hG} \to \mathrm{B}G$$

is a mod p homology isomorphism.

Proof. By 5.12, the space $X^\beta_{\mathcal{C}} = \operatorname{hocolim} \tilde{\beta}_{\mathcal{C}}$ is the nerve of a category $\mathbf{X}^\beta_{\mathcal{C}}$ whose objects consist of pairs $(G/H, x)$, where G/H is a coset space of G with $H \in \mathcal{C}$, and x is an element of G/H. A morphism $(G/H, x) \to (G/H', x')$ is a G-map $f : G/H \to G/H'$ such that $f(x) = x'$. The action of G on $X^\beta_\mathcal{C}$ corresponds to the action of G on $\mathbf{X}^\beta_\mathcal{C}$ obtained by letting $g \cdot (G/H, x) = (G/H, gx)$. There is a functor

$$v : \mathbf{X}^\beta_\mathcal{C} \to \mathbf{K}_\mathcal{C}$$

which sends $(G/H, x)$ to the isotropy subgroup G_x, and by inspection this functor is G-equivariant. As in the proof of 7.12, it is enough to show that v is an equivalence of categories. This is easy; there is a functor $v' : \mathbf{K}_\mathcal{C} \to \mathbf{X}^\beta_\mathcal{C}$ which sends $H \in \mathcal{C}$ to the pair $(G/H, eH)$, where $e \in G$ is the identity element. The composite vv' is the identity functor of $\mathbf{K}_\mathcal{C}$, and the composite $v'v$ is connected to the identity functor of $\mathbf{X}^\beta_\mathcal{C}$ by an obvious natural equivalence. □

7.15. *Remark.* The subgroup decomposition functor $\beta_\mathcal{C} = (\tilde\beta_\mathcal{C})_{\mathrm{h}G}$ assigns to the object G/H of $\mathbf{O}_\mathcal{C}$ the space $(G/H)_{\mathrm{h}G}$, which by 6.4 is weakly equivalent to $\mathrm{B}H$.

The normalizer decomposition. This one is a little trickier. Let $\bar{\mathrm{s}}\mathrm{d}\mathbf{K}_\mathcal{C}$ be the category of "orbit simplices" for the action of G on the simplicial complex $K_\mathcal{C}$. The objects of $\bar{\mathrm{s}}\mathrm{d}\mathbf{K}_\mathcal{C}$ are the orbits $\bar\sigma$ of the action of G on the simplices of $K_\mathcal{C}$, and there is one morphism $\bar\sigma \to \bar\sigma'$ if for some simplices $\sigma \in \bar\sigma$ and $\sigma' \in \bar\sigma'$, σ' is a face of σ. (This might look backwards, but it's necessary to define morphisms this way in order to get the functor $\tilde\delta_\mathcal{C}$ below.) Note that here we are definitely thinking of $K_\mathcal{C}$ as a simplicial complex, and not as a simplicial set; degenerate simplices play no role. An object of $\bar{\mathrm{s}}\mathrm{d}\mathbf{K}_\mathcal{C}$ is an equivalence class, under the action of G by conjugation, of chains

$$H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_n \tag{7.16}$$

such that each H_i belongs to $\mathcal{C}$.

There is a functor

$$\tilde\delta_\mathcal{C} : \bar{\mathrm{s}}\mathrm{d}\mathbf{K}_\mathcal{C} \to \mathbf{GSets}$$

which assigns to an orbit $\bar\sigma$ the transitive G-set provided by $\bar\sigma$ itself. Here is the last part of 7.11.

7.17. **Proposition.** *The collection $\mathcal{C}$ is ample if and only if the natural map*

$$(\mathrm{hocolim}\,\tilde\delta_\mathcal{C})_{hG} \to \mathrm{B}G$$

is a mod p homology isomorphism.

Proof. By 5.12, the space $X^\delta_\mathcal{C} = \mathrm{hocolim}\,\tilde\delta_\mathcal{C}$ is the nerve of a category $\mathbf{X}^\delta_\mathcal{C}$ whose objects are the simplices σ of $K_\mathcal{C}$. There is one morphism

$\sigma \to \sigma'$ if σ' is a face of σ and no other morphisms. The category $\mathbf{X}^\delta_\mathcal{C}$ is the opposite of the category whose nerve is the barycentric subdivision (3.6) of $K_\mathcal{C}$. There are G-equivariant homeomorphisms (5.8, 3.6)

$$|X^\delta_\mathcal{C}| \cong |\,\mathrm{N}((\mathbf{X}^\delta_\mathcal{C})^{\mathrm{op}})| \cong |K_\mathcal{C}|\ .$$

However, these homeomorphisms are not realized by functors or simplicial maps, so there is some more work to be done. We leave this to the reader: the problem is to devise a way of relating $X^\delta_\mathcal{C}$ to $K_\mathcal{C}$ inside the category **Sp**, by maps which are G-equivariant and are weak equivalences. □

7.18. *Remark.* If H is a subgroup of G, let $N_G(H)$ denote the normalizer of H in G. By 6.4, the normalizer decomposition functor $\delta_\mathcal{C} = (\tilde{\delta}_\mathcal{C})_{\mathrm{h}G}$ assigns to the orbit of the simplex (7.16) of $K_\mathcal{C}$ a space which has the homotopy type of $\mathrm{B}(\cap_i N_G(H_i))$

8. Sharp homology decompositions. Examples

Associated to any homology decomposition for $\mathrm{B}G$ (7.1) is a first quadrant mod p homology spectral sequence (4.16)

$$E^2_{i,j}(F) = \mathrm{colim}_i\,\mathrm{H}_j(F) \Rightarrow \mathrm{H}_{i+j}(\mathrm{B}G)\ .$$

8.1. **Definition.** A homology decomposition for $\mathrm{B}G$ is *sharp* if its homology spectral sequence collapses onto the vertical axis, in the sense that $E^2_{i,j} = 0$ for $i > 0$.

The usefulness of a sharp homology decomposition functor F is that it gives an isomorphism

$$\mathrm{colim}\,\mathrm{H}_*(F) = \mathrm{colim}_0\,\mathrm{H}_*(F) \xrightarrow{\approx} \mathrm{H}_*\,\mathrm{B}G\ . \tag{8.2}$$

This is essentially a formula for $\mathrm{H}_*\,\mathrm{B}G$ in terms of the homology groups of subgroups of G.

It turns out that sharp homology decompositions are sometimes easier to recognize than arbitrary ones. One way in which we will show that a collection $\mathcal{C}$ of subgroups is ample (7.7) is to show that one of the three functors $\alpha_\mathcal{C}$, $\beta_\mathcal{C}$, or $\delta_\mathcal{C}$ associated to $\mathcal{C}$ gives a sharp homology decomposition for $\mathrm{B}G$. The other two functors then give homology decompositions too (although these may or may not be sharp).

Examples of homology decompositions. From now on we will assume that G is a *finite* group.

8.3. **Definition.** An ample collection $\mathcal{C}$ of subgroups of G is said to be *centralizer-sharp* (resp. *subgroup-sharp*, *normalizer-sharp*) if the centralizer decomposition (resp. subgroup decomposition, normalizer decomposition) associated to $\mathcal{C}$ is sharp.

Here are some examples of homology decompositions.

8.4. *Example:* $\mathcal{C} = \{\{e\}\}$. Suppose that $\mathcal{C}$ contains only the trivial subgroup of G. Then $K_{\mathcal{C}}$ is a one-point space, so $\mathcal{C}$ is ample. Moreover, $\mathcal{C}$ is both centralizer-sharp and normalizer-sharp, since both the centralizer and normalizer decomposition diagrams reduce to the trivial diagram with $\mathrm{B}G$ as its only constituent. The subgroup decomposition category $\mathbf{O}_{\mathcal{C}}$ has only one object, and the group G is the space of self-maps of this object. The functor $\beta_{\mathcal{C}}$ assigns to this object the trivial one-point space. By 6.2, the homology spectral sequence of $\beta_{\mathcal{C}}$ has

$$E^2_{i,j} = \begin{cases} \mathrm{H}_i(G) & j = 0 \\ 0 & j > 0 \end{cases} .$$

This spectral sequence collapses, but onto the wrong axis! The collection $\mathcal{C}$ is subgroup-sharp if and only if the reduced mod p homology of $\mathrm{B}G$ is trivial.

If G is a finite group, the reduced mod p homology of $\mathrm{B}G$ is trivial if and only if the order of G is prime to p. Can you prove this? (Hint: the hard part is to show that if the mod p (co-)homology is trivial, then the order of G is prime to p. Suppose that the order of G is not prime to p. Take a faithful complex representation ρ of G, and try computing the Chern classes of the restriction of ρ to a cyclic subgroup of G of order p.)

8.5. *Example:* $\mathcal{C} = \{G\}$. Suppose that $\mathcal{C}$ contains only G itself. Again $K_{\mathcal{C}}$ is has only one point, so $\mathcal{C}$ is ample. The collection $\mathcal{C}$ is both subgroup-sharp and normalizer-sharp, since both the subgroup and normalizer decomposition diagrams reduce to the trivial diagram with $\mathrm{B}G$ as its only constituent. Let Z be the center of G. The category $\mathbf{A}_{\mathcal{C}}$ has one object whose group of self-maps is G/Z and the functor $\alpha_{\mathcal{C}}$ assigns to this object the space $\mathrm{B}Z$. By 6.2, the homology spectral sequence of $\alpha_{\mathcal{C}}$ is the Lyndon-Hochschild-Serre spectral sequence of the group extension $Z \to G \to G/Z$. From this it is possible to prove that $\mathcal{C}$ is centralizer-sharp if and only if Z contains a Sylow p-subgroup of G; this is the case if and only if G is the product of an abelian p-group and a group of order prime to p.

Can you prove these last statements?

8.6. *Other trivial examples.* If $\mathcal{C}$ contains either $\{e\}$ or G, then $K_{\mathcal{C}}$ is contractible, because the category associated to $\mathcal{C}$ (4.12) has either $\{e\}$ as an initial object or G as a terminal object (see 5.4). In particular, $\mathcal{C}$ is ample. The corresponding homology decompositions are in some sense circular, since (as suggested in the above two examples) G itself is somehow encoded in each of the three homotopy colimits associated to $\mathcal{C}$.

Suppose that $\mathcal{C}$ is the set of conjugates of H, where H is some subgroup of G different from $\{e\}$ and from G itself. Can you think of examples of subgroups H for which $\mathcal{C}$ is ample? What are the corresponding three decomposition functors? What sharpness properties does $\mathcal{C}$ have? Think first about the special case in which H is normal, so that $\mathcal{C} = \{H\}$.

8.7. *Non-identity p-subgroups.* Suppose that p divides the order of G, and let $\mathcal{C}$ be the collection of all non-identity p-subgroups of G. In one form or another it has been known for a long time that $\mathcal{C}$ is ample. We will prove this in §12, and show in addition that $\mathcal{C}$ is centralizer-sharp, subgroup-sharp, and normalizer-sharp; Most of the technical results that go into proving sharpness are due originally to Webb [36] [37]. The colimit formula (8.2) for $\mathrm{H}_* \mathrm{B}G$ that arises from the associated subgroup decomposition (§7) is essentially the $\mathbb{F}_p$-dual of the formula of Cartan and Eilenberg [8, p. 259] expressing $\mathrm{H}^* \mathrm{B}G$ as the set of stable elements in the cohomology of a Sylow p-subgroup of G.

8.8. *Elementary abelian p-subgroups.* A finite abelian group is said to be an *elementary abelian p-group* if it is a module over $\mathbb{F}_p$. Suppose that p divides the order of G, and let $\mathcal{C}$ be the collection of all nontrivial elementary abelian p-subgroups of G, and $\mathcal{C}'$ the collection of all nontrivial p-subgroups of G. It is a theorem of Quillen that the natural map $K_{\mathcal{C}} \to K_{\mathcal{C}'}$ is a weak equivalence [2, 6.6.1].

One proof of Quillen's theorem goes like this. If $F : \mathbf{A} \to \mathbf{B}$ is a functor between small categories, and b is an object of $\mathbf{B}$, let $F{\downarrow}\, b$ denote the category whose objects consist of pairs (a, u) where a is an object of $\mathbf{A}$ and $u : F(a) \to b$ is a morphism in $\mathbf{B}$. A map $(a, u) \to (a', u')$ in this category is a morphism $f : a \to a'$ in $\mathbf{A}$ such that $u' = u \cdot F(f)$. Letting b vary gives a functor $F{\downarrow} - : \mathbf{B} \to \mathbf{Cat}$. Check that there are functors $\mathrm{Gr}(F{\downarrow} -) \leftrightarrow \mathbf{A}$ with the property that each composite is connected to the identity functor by a natural transformation. Conclude that the nerve of $\mathrm{Gr}(F{\downarrow} -)$ is weakly equivalent to the nerve of $\mathbf{A}$. Argue further that if each one of the categories $F{\downarrow}\, b$ has a contractible nerve, then the nerve of $\mathrm{Gr}(F{\downarrow} -)$ is equivalent to the nerve of $\mathbf{B}$. Apply this to the case in which F is the inclusion of posets $\mathcal{C} \to \mathcal{C}'$. Use 12.1 to show that for each object b of $\mathcal{C}'$, the identity functor of $F{\downarrow}\, b$ is connected to a constant functor by a zigzag of natural transformations.

Since $\mathcal{C}'$ is ample (8.7), so is $\mathcal{C}$. In fact, $\mathcal{C}$ is both centralizer-sharp and normalizer-sharp; by the duality between homology and cohomology, it cannot possibly be subgroup-sharp unless the mod p cohomology of G is detected on elementary abelian p-subgroups.

The mod p cohomology of G is *detected on elementary abelian subgroups* if for every nonzero $x \in \mathrm{H}^* \mathrm{B}G$ there is an elementary abelian p-subgroup V of G such that the restriction of x to $\mathrm{H}^* \mathrm{B}V$ is nonzero. Can you think of a finite group G such that $\mathrm{H}^* \mathrm{B}G$ is *not* detected on elementary abelian p-subgroups? Quillen showed [30] that if G is the group $\mathrm{GL}_n(\mathbb{F}_q)$ (where q is relatively prime to p) or if G is a symmetric group, then the mod p cohomology of $\mathrm{B}G$ *is* detected on elementary abelian subgroups.

We will discuss this and related collections in §13. The question of centralizer sharpness was first studied by Jackowski and McClure [19], later by Dwyer and Wilkerson [14] and by Benson [3]. The main ingredient necessary to prove normalizer sharpness was provided by Brown [7, p. 268].

Let $p = 2$, and let G be the group $\mathrm{SL}_3(\mathbb{F}_q)$, where q is odd. Show that up to conjugacy there are only two nontrivial elementary abelian p-subgroups of G. What are the corresponding homology decompositions?

8.9. *p-centric subgroups.* A p-subgroup P of G is said to be *p-centric* if the center of P is a Sylow p-subgroup of $C_G(P)$. This is equivalent to the condition that $C_G(P)$ is the direct product of the center of P and a group of order prime to p. Let $\mathcal{C}$ be the collection of all p-centric subgroups of G. Then $\mathcal{C}$ is ample [11] as well as subgroup-sharp [12].

Suppose that G is a group with the property that the Sylow p-subgroups of G are abelian. Argue that up to congugacy there is only one p-centric subgroup of G. What are the associated homology decompositions? Is it obvious that the subgroup decomposition is sharp? Recover a certain classical theorem of Swan.

8.10. *Subgroups both p-centric and p-stubborn.* A p-subgroup P of G is said to be *p-stubborn* or *p-radical* if $N_G(P)/P$ has no non-identity normal p-subgroups. Let $\mathcal{C}$ be the collection of all subgroups of G which are both p-centric (8.9) and p-stubborn. Then $\mathcal{C}$ is ample [11] and subgroup-sharp [12]. The role of p-stubborn subgroups was pointed out by Jackowski, McClure and Oliver [20], and independently (in a different context and with different terminology) by Bouc [2, 6.6.6].

The recent paper of Grodal [18] contains a lot of very interesting additional information about homology decompositions.

9. Reinterpreting the homotopy colimit spectral sequence

In this section we point out that the Bousfield-Kan homology spectral sequence (4.16) associated to a homotopy colimit can be interpreted as a Leray spectral sequence. For the particular homotopy colimits that give rise to the homology decompositions of §7, these Leray spectral sequences are the isotropy spectral sequences associated to G-spaces. This makes it possible to study the spectral sequences (e.g., in order to show that a decomposition is sharp) by making geometric manipulations with G-spaces.

We first introduce the Leray spectral sequence, then we specialize to the isotropy spectral sequence, and finally we explain the relationship of the isotropy spectral sequence to homology decompositions.

The Leray spectral sequence. The *n-skeleton* $\mathrm{sk}_n B$ ($n \geq 0$) of a space B is the subobject of B generated by all simplices of dimension $\leq n$.

9.1**. Definition.** Let $f : X \to B$ be a map of spaces, and let X_n denote $f^{-1}(\mathrm{sk}_n B)$. The *Leray spectral sequence* of f is the (mod p) homology spectral sequence associated to the filtration

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$$

of X.

This spectral sequence is usually indexed in such a way that $E^1_{i,j} = \mathrm{H}_{i+j}(X_i, X_{i-1})$. Since X_n contains $\mathrm{sk}_n X$, $E^1_{i,j} = 0$ for $j < 0$ and this is a first quadrant, strongly convergent homology spectral sequence. In particular, the differential d^r has bidegree $(-r, r-1)$. Here are a few examples.

9.2. *Collapse map.* If Y is a subspace of X, the E^2-term of the Leray spectral sequence of $f : X \to X/Y$ vanishes except for the groups $E^2_{0,0} = \mathrm{H}_0 X$, $E^2_{0,j} = \mathrm{H}_j(Y)$ ($j > 0$) and $E^2_{i,0} = \mathrm{H}_i(X/Y)$ ($i > 0$). The various differentials running from the horizontal axis to the vertical axis give the connecting homomorphisms in the long exact homology sequence of the pair (X, Y).

9.3. *Fibration.* If $f : X \to Y$ is a fibration of spaces [24], the Leray spectral sequence of f can be identified with the Serre spectral sequence [28] of $|f|$.

9.4. *Homotopy colimit.* Suppose that $\mathbf{D}$ is a small category and $F : \mathbf{D} \to \mathbf{Sp}$ is a functor. The unique natural transformation from F to the constant functor $*$ with value the one-point space induces a map

$$f : \mathrm{hocolim}\, F \to \mathrm{hocolim} * \approx \mathrm{N}(\mathbf{D})$$

The Leray spectral sequence of f can be identified from E^2 onwards with the Bousfield-Kan homology spectral sequence (4.16) of the homotopy colimit. This can be seen by inspecting the definitions (4.7, 9.1) and seeing that the two spectral sequences arise from the same filtration of $\mathrm{hocolim}\, F$.

The isotropy spectral sequence. We will be interested in a specific Leray spectral sequence associated to a G-space X.

9.5**. Definition.** The *isotropy spectral sequence* of a G-space X is the Leray spectral sequence of the map (6.2)

$$f : X_{\mathrm{h}G} = (\mathrm{E}G \times X)/G \to X/G\ .$$

9.6. *Remark.* Consider the diagram

$$
\begin{array}{ccccc}
 & & \{\mathrm{B}G_x\} & & \\
 & & \downarrow & & \\
X & \longrightarrow & X_{\mathrm{h}G} & \xrightarrow{q} & (*)_{\mathrm{h}G} = \mathrm{B}G \\
 & & {\scriptstyle f}\downarrow & & \\
 & & X/G & &
\end{array} \tag{9.7}
$$

The horizontal sequence is a fibration sequence, and the Leray spectral sequence of q is the usual Serre spectral sequence converging to $\mathrm{H}_*(X_{\mathrm{h}G}; M)$. Equivalently, this is the Bousfield-Kan homology spectral sequence associated to the description of $X_{\mathrm{h}G}$ as a homotopy colimit (6.1). The vertical sequence is meant to suggest that the geometric fibres of f in general differ from point to point, and that the fibre over a simplex $\bar{x} \in X/G$ can be identified up to homotopy with $\mathrm{B}G_x$, where G_x is the isotropy subgroup of a simplex $x \in X$ above $\bar{x}$. The Leray spectral sequence of f is the isotropy spectral sequence of X, and it too converges to $\mathrm{H}_*(X_{\mathrm{h}G})$.

9.8. *Examples.* If G acts freely on X then the map f of (9.7) is an equivalence with contractible fibres, and the isotropy spectral sequence of X collapses onto the horizontal axis. More generally, suppose that K is a normal subgroup of G which acts trivially on X, and that the quotient group $W = G/K$ acts freely on X. Then the map f of 9.7 is a fibration with fibre $\mathrm{B}K$, and the isotropy spectral sequence of X can be identified with the Serre spectral sequence of the fibration $\mathrm{B}K \to X_{\mathrm{h}G} \to X_{\mathrm{h}W}$.

9.9. *The E^2-page of the isotropy spectral sequence.* The isotropy spectral sequence of the G-space X can be constructed as the homology spectral sequence (4.7) of the simplicial space Y given by

$$Y_n = (X_n)_{\mathrm{h}G} \ .$$

Here we are treating X_n as a discrete space (3.19) with a G-action; the horizontal (4.5) face and degeneracy operators are induced by the face and degeneracy operators of X.

According to 4.8 and 3.23, the E^2-page of the isotropy spectral sequence can be identified as follows. Write $\mathrm{H}_j(X_i)_{\mathrm{h}G}$ for $\mathrm{H}_j((X_i)_{\mathrm{h}G})$. For each $j \geq 0$ one can form a chain complex $\langle \mathrm{H}_j(X_*)_{\mathrm{h}G}, d\rangle$ which in dimension $i \geq 0$ contains the group $\mathrm{H}_j(X_i)_{\mathrm{h}G}$ and has boundary map d induced by taking the alternating sum of the maps

$$\mathrm{H}_j(X_i)_{\mathrm{h}G} \to \mathrm{H}_j(X_{i-1})_{\mathrm{h}G}$$

induced by the $(i+1)$ face operators $X_i \to X_{i-1}$. The group in position $E^2_{i,j}$-term of the isotropy spectral sequence is then the i'th homology group of $\langle \mathrm{H}_j(X_*)_{\mathrm{h}G}, d\rangle$.

Homology decomposition spectral sequences. We now show that the Bousfield-Kan homology spectral sequence of any homotopy colimit constructed by the technique of 7.4 can be interpreted as the isotropy spectral sequence of an associated G-space. This will allow us to work with the Bousfield-Kan spectral sequence (which is pretty abstract) by manipulating G-spaces (which are much easier to handle). The basis for this is the following proposition.

9.10**. Proposition.** *Suppose that* $\mathbf{D}$ *is a small category and that* F *is a functor from* $\mathbf{D}$ *to the category of transitive* G*-sets. Let* $X(F)$ *denote the* G*-space* $\operatorname{hocolim} F$*. Then the Bousfield-Kan homology spectral sequence for the homology of* $\operatorname{hocolim}(F_{hG})$ *can be identified in a natural way with the isotropy spectral sequence of* $X(F)$*.*

Proof. There is a commutative diagram

$$\begin{array}{ccc} \operatorname{hocolim}(F_{\mathrm{h}G}) & \longrightarrow & \operatorname{hocolim}(F/G) \approx \mathrm{N}(\mathbf{D}) \\ \approx\big\downarrow & & \approx\big\downarrow \\ (\operatorname{hocolim} F)_{\mathrm{h}G} & \longrightarrow & (\operatorname{hocolim} F)/G \end{array}$$

in which both vertical arrows are isomorphisms (see 6.5 for the left arrow). The Bousfield-Kan homology spectral sequence of F is the Leray spectral sequence of the upper map (9.4) and the isotropy spectral sequence of $X(F)$ is the Leray spectral sequence of the lower one. □

We now establish notation for the G-spaces associated to the decompositions of §7.

9.11*. The centralizer decomposition.* Let $X^\alpha_{\mathcal{C}}$ denote the G-space given by $\operatorname{hocolim} \tilde{\alpha}_{\mathcal{C}}$. As in the proof of 7.12, $X^\alpha_{\mathcal{C}}$ is the nerve of the category $\mathbf{X}^\alpha_{\mathcal{C}}$ whose objects are pairs (H, i), where H is a group and $i : H \to G$ is a monomorphism with $i(H) \in \mathcal{C}$. The Bousfield-Kan homology spectral sequence associated to $\alpha_{\mathcal{C}}$ is the isotropy spectral sequence of the action of G on $X^\alpha_{\mathcal{C}}$.

9.12*. The subgroup decomposition.* Let $X^\beta_{\mathcal{C}}$ denote the G-space given by $\operatorname{hocolim} \tilde{\beta}_{\mathcal{C}}$. As in the proof of 7.14, this is the nerve of a category $\mathbf{X}^\beta_{\mathcal{C}}$ whose objects consist of pairs $(G/H, x)$ where $H \in \mathcal{C}$ and $x \in G/H$. The Bousfield-Kan homology spectral sequence associated to $\beta_{\mathcal{C}}$ is the isotropy spectral sequence of the action of G on $X^\beta_{\mathcal{C}}$.

9.13. *The normalizer decomposition.* Let $X^\delta_{\mathcal{C}}$ denote the space $K_{\mathcal{C}}$. This is the nerve of the category $\mathbf{X}^\delta_{\mathcal{C}}$ associated to the poset $\mathcal{C}$. It is possible to prove that $X^\delta_{\mathcal{C}}$ is weakly G-equivalent (10.5) to the space $\operatorname{hocolim} \tilde{\delta}_{\mathcal{C}}$ (see the remarks in the proof of 7.17), and so the Bousfield-Kan homology spectral sequence associated to $\delta_{\mathcal{C}}$ is the isotropy spectral sequence of the action of G on $X^\delta_{\mathcal{C}}$.

10. Bredon homology and the transfer

The previous section identified the Bousfield-Kan spectral sequence derived from a homology decomposition of BG as the isotropy spectral sequence of an associated G-space X. The E^2-term of this isotropy spectral sequence is a type of homological functor of X; in this section we point out exactly what sort of construction this homological functor is, and describe some of its properties.

10.1. **Definition.** Let $\mathcal{H}$ be a functor from the category of $\mathbb{F}_p[G]$-modules to the category of $\mathbb{F}_p$ vector spaces. The functor $\mathcal{H}$ is said to be *a coefficient functor* for G if $\mathcal{H}$ preserves arbitrary direct sums. If K is a subgroup of G, the *restriction of* $\mathcal{H}$ *to* K, denoted $\mathcal{H}|_K$, is the coefficient functor for K given by $\mathcal{H}|_K(A) = \mathcal{H}(\mathbb{F}_p[G] \otimes_{\mathbb{F}_p[K]} A)$.

The functor $\mathcal{H}$ is said to *preserve arbitrary direct sums* if, for any set $\{M_\alpha\}$ of $\mathbb{F}_p[G]$-modules, the natural map

$$\oplus_\alpha \mathcal{H}(M_\alpha) \to \mathcal{H}(\oplus_\alpha M_\alpha)$$

is an isomorphism.

10.2. *Example.* For each $j \geq 0$ there is a coefficient functor $\mathcal{H}^G_j$ given by $\mathcal{H}^G_j(A) = \mathrm{H}_j(G; A)$ (see 2.8). These are the coefficient functors we will interested in. If K is a subgroup of G and $\mathcal{H} = \mathcal{H}^G_j$, then $\mathcal{H}|_K \approx \mathcal{H}^K_j$.

Prove the final statement; it's a form of Shapiro's lemma [38, 6.3.2].

10.3. **Definition.** Suppose that $\mathcal{H}$ is a coefficient functor for G and that X is a G-space. Let $(C^G_*(X;\mathcal{H}), d)$ be the chain complex with $C^G_n(X;\mathcal{H}) = \mathcal{H}(\mathbb{F}_p[X_n])$ (see 3.24) and with boundary d induced by the alternating sum of the face maps in X. The *Bredon homology groups* of X with coefficients in $\mathcal{H}$, denoted $\mathrm{H}^G_*(X;\mathcal{H})$, are defined to be the homology groups of $C^G_*(X;\mathcal{H})$.

10.4. *Example.* Suppose that X is a G-space, and let $\{\mathcal{H}^G_j\}$ be the coefficient functors from 10.2. The E^2 page of the isotropy spectral sequence for X is then given by

$$E^2_{i,j} = \mathrm{H}^G_i(X;\mathcal{H}^G_j)\ .$$

This follows immediately from 9.10, 9.9, and the observation that for any G-set S, there is a natural isomorphism

$$\mathcal{H}_j^G(\mathbb{F}_p[S]) \approx \mathrm{H}_j(S_{\mathrm{h}G}) \ .$$

Prove this last statement, by combining additivity of $\mathcal{H}_j^G$ with the final sentence of 10.2. It is possible to use coefficients for Bredon homology more general than the functors allowed above; in fact, any functor from the orbit category of G to abelian groups can be used to construct a Bredon homology theory. For practical purposes, the "coefficient functors" above are the same as the "cohomological Mackey functors" of [39] or [34, p. 1928].

A key property of $\mathrm{H}_*^G(X;\mathcal{H})$. The groups $\mathrm{H}_*^G(X;\mathcal{H})$ have a certain basic invariance property.

10.5. **Definition.** A map $f : X \to Y$ of G-spaces is said to be *a weak G-equivalence* if $f^H : X^H \to Y^H$ is a weak equivalence for every subgroup H of G.

10.6. *Remark.* If X is a G-space, let $\mathrm{Iso}(X)$ denote the set of subgroups of G which appear as isotropy groups of simplices of X. By [11, 4.1], a map $f : X \to Y$ of G-spaces is a weak G-equivalence if and only if it induces a weak equivalence $f^H : X^H \to Y^H$ for all $H \in \mathrm{Iso}(X)\cup\mathrm{Iso}(Y)$.

10.7. **Proposition.** *Suppose that $f : X \to Y$ is a a weak G-equivalence and that $\mathcal{H}$ is a coefficient functor (10.1). Then f induces isomorphisms $\mathrm{H}_*^G(X;\mathcal{H}) \cong \mathrm{H}_*^G(Y;\mathcal{H})$.*

A proof of this is sketched in §14.

10.8. **The transfer.** All of our techniques for dealing with the G-spaces from 9.11, 9.12, and 9.13 are based in one way or another on the transfer. We assume from now on that $\mathcal{H}$ is a coefficient functor (10.1), in practice one of the coefficient functors $\mathcal{H}_j^G$ (10.2).

Suppose that $f : S \to T$ is a map of G-sets. There is an induced map $\mathbb{F}_p[S] \to \mathbb{F}_p[T]$, also denoted f, as well as a map

$$f_* = \mathcal{H}(f) : \mathcal{H}(\mathbb{F}_p[S]) \to \mathcal{H}(\mathbb{F}_p[T])\,.$$

Say that f is *finite-to-one* if for each $x \in T$ the set $f^{-1}(x)$ is finite. For such an f there is a G-map $\tau(f) : \mathbb{F}_p[T] \to \mathbb{F}_p[S]$, called the *pretransfer*, which sends $x \in T$ to $\sum_{y\in f^{-1}(x)} y$. The induced map

$$\tau_*(f) : \mathcal{H}(\mathbb{F}_p[T]) \to \mathcal{H}(\mathbb{F}_p[S])$$

is the *transfer* associated to f.

10.9. *Example.* Suppose that H and K are subgroups of G with $H \subset K$, and let $\mathcal{H} = \mathcal{H}_j^G$ (10.2). Let $f : G/H \to G/K$ be the projection map. By Shapiro's lemma [38, 6.3.2] there are isomorphisms

$$\mathcal{H}(\mathbb{F}_p[G/H]) \cong \mathrm{H}_j(H) \qquad \mathcal{H}(\mathbb{F}_p[G/K]) \cong \mathrm{H}_j(K)\ .$$

Under these identifications, $f_* : \mathrm{H}_j(H) \to \mathrm{H}_j(K)$ is the map induced by the inclusion $H \subset K$ and $\tau_*(f) : \mathrm{H}_j(K;M) \to \mathrm{H}_j(H;M)$ is the associated group homology transfer map [38, 6.3.9] [2, p. 67].

The transfer has the following basic properties, which are easy to verify by calculations with pretransfers. Recall that $\mathcal{H}$ is assumed to commute with direct sums.

10.10. **Lemma.** *Suppose that $f_1 : S_1 \to T_1$ and $f_2 : S_2 \to T_2$ are maps of G-sets. If f_1 and f_2 are finite-to-one, then so is $f_1 \amalg f_2 : S_1 \amalg S_2 \to T_1 \amalg T_2$, and $\tau_*(f_1 \amalg f_2) = \tau_*(f_1) \oplus \tau_*(f_2)$.*

10.11. **Lemma.** *Suppose that $f_1 : S_1 \to T$ and $f_2 : S_2 \to T$ are maps of G-sets. If f_1 and f_2 are finite-to-one, the so is $f_1 + f_2 : S_1 \amalg S_2 \to T$, and $\tau_*(f_1 + f_2) = (\tau_*(f_1), \tau_*(f_2))$*

10.12. *Remark.* It follows from 10.9, 10.10, and 10.11 that if $f : S \to T$ is a map of G-sets which is finite-to-one, then $\tau_*(f)$ can be computed in terms of a sum of transfers associated to the projections $G/G_x \subset G/G_{f(x)}$, $x \in S$. For convenience, we will sometimes call such a transfer *the transfer associated to the inclusion $G_x \to G_{f(x)}$*.

10.13. **Lemma.** *Suppose that $f : S \to T$ and $g : T \to R$ are maps of G-sets. If f and g are finite-to-one then so is $g \cdot f$, and $\tau_*(g \cdot f) = \tau_*(f) \cdot \tau_*(g)$.*

10.14. **Lemma.** *Suppose that*

$$\begin{array}{ccc} S' & \xrightarrow{s} & S \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{t} & T \end{array}$$

is a pullback square of G-sets (i.e. a commutative diagram which induces an isomorphism from S' to the pullback of S and T' over T). Then if f is finite-to-one, so is f', and $\tau_*(f) \cdot t_* = s_* \cdot \tau_*(f')$.

10.15. **Definition.** A map $f : S \to T$ of G-sets is said to be *an even covering mod p* if it is finite-to-one and the cardinality mod p of $f^{-1}(x)$ does not depend on the choice of $x \in T$. The common value mod p of these inverse image cardinalities is called the *degree* of f and denoted $\deg(f)$.

10.16. **Lemma.** *Suppose that $f : S \to T$ is a map of G-sets which is an even covering mod p. Then the composite $f_* \cdot \tau_*(f)$ is the endomorphism of $\mathcal{H}(\mathbb{F}_p[T])$ given by multiplication by* $\deg(f)$*.*

10.17. *Example.* Suppose that H is a subgroup of G and that S is a G-set. The action map $a : G \times_H S \to S$ is finite-to-one and has degree given by the index of H in G. Moreover, if $S' \to S$ is a map of G-sets, the diagram

$$\begin{array}{ccc} G \times_H S' & \longrightarrow & G \times_H S \\ {\scriptstyle a}\big\downarrow & & {\scriptstyle a}\big\downarrow \\ S' & \longrightarrow & S \end{array}$$

is a pullback square. It follows that the maps $\tau_*(a)$ give a natural map

$$\mathcal{H}(\mathbb{F}_p[S]) \xrightarrow{\tau_*(a)} \mathcal{H}(\mathbb{F}_p[G \times_H S])$$

on the category of G-sets. Moreover, the composite $a_* \cdot \tau_*(a)$ is the endomorphism of $\mathcal{H}(\mathbb{F}_p[S])$ given by multiplication by the index of H in G.

The notation "$G \times_H S$" above stands for the quotient of $G \times S$ by the equivalence relation "$\sim$" given by $(g, hs) \sim (gh, s)$ for $g \in G$, $s \in S$, and $h \in H$. The action of G on this quotient is induced by the action of G on $G \times S$ obtained by setting $g' \cdot (g, s) = (g'g, s)$. There is an isomorphism $\mathbb{F}_p[G \times_H S] \approx \mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} \mathbb{F}_p[S]$ of $\mathbb{F}_p[H]$-modules. Show that there is also an isomorphism of G-sets $G \times_H S \approx (G/H) \times S$, where G acts diagonally on the product $(G/H) \times S$.

11. Acyclicity for G-spaces

In this section we translate the question of whether a homology decomposition of $\mathrm{B}G$ is sharp into a question about the Bredon homology of the associated G-space (§9). We then study this second question. The symbol $\mathcal{H}$ denotes a coefficient functor (10.1) for G. A lot of the material in this section first appeared (in a slightly different form) in a paper of Webb [37].

11.1. **Definition.** A G-space X is said to be *acyclic for* $\mathcal{H}$ if the map $X \to *$ induces an isomorphism $\mathrm{H}^G_*(X; \mathcal{H}) \to \mathrm{H}^G_*(*; \mathcal{H})$.

11.2. *Remark.* Note that $\mathrm{H}^G_i(*; \mathcal{H})$ vanishes for $i > 0$ and $\mathrm{H}^G_0(*; \mathcal{H}) = \mathcal{H}(\mathbb{F}_p)$. Let $\mathcal{C}$ be a collection of subgroups of G. By §9 and 10.4, the centralizer, subgroup and normalizer decompositions associated to $\mathcal{C}$ are sharp (8.1) if and only if the G-spaces $X^\alpha_{\mathcal{C}}$, $X^\beta_{\mathcal{C}}$, and $X^\delta_{\mathcal{C}}$ (respectively) are acyclic for the functors $\mathcal{H}^G_j$, $j \geq 0$. In fact, more is true: if any *one* of these three G-spaces is acyclic for all of the functors $\mathcal{H}^G_j$

($j \geq 0$), then $\mathcal{C}$ is ample, all three spaces correspond to homology decompositions for BG, and the particular decomposition corresponding to the acyclic G-space is sharp.

Note that there are functors

$$\mathbf{X}^{\alpha}_{\mathcal{C}} \to \mathbf{X}^{\delta}_{\mathcal{C}} \leftarrow \mathbf{X}^{\beta}_{\mathcal{C}}$$

which induce G-maps

$$X^{\alpha}_{\mathcal{C}} \to X^{\delta}_{\mathcal{C}} \leftarrow X^{\beta}_{\mathcal{C}} \ . \tag{11.3}$$

According to the arguments in the proofs of 7.12 and 7.14, these maps are weak equivalences. This reflects the fact that when it comes to the centralizer, subgroup, and normalizer diagrams, either all three give homology decomposition of BG or none of them do. However, the two maps of (11.3) are *not* usually weak G-equivalences. This reflects (10.7) the fact that the three decompositions can have different sharpness properties.

We have two ways to show that a G-space X is acyclic for $\mathcal{H}$. In fact, the second method is a refinement of the first one.

The direct transfer method. This uses the fact that if K is a subgroup of G of index prime to p then the transfer exhibits $\mathrm{H}^G_*(X;\mathcal{H})$ as a retract of $\mathrm{H}^K_*(X;\mathcal{H}|_K)$.

11.4**. Theorem.** *Suppose that X is a G-space, $\mathcal{H}$ is a coefficient functor for G, and K is a subgroup of G of index prime to p. If X is acyclic as a K-space for $\mathcal{H}|_K$, then X is acyclic as a G-space for $\mathcal{H}$.*

Proof. The transfers (10.8) associated to the maps $q : G \times_K X_n \to X_n$ provide a map $t : C^G_*(X;\mathcal{H}) \to C^K_*(X;\mathcal{H}|_K)$ (9.9). By 10.17 this map commutes with differentials, and the index assumption implies that the composite $C^G_*(X;\mathcal{H}) \xrightarrow{t} C^K_*(X;\mathcal{H}|_K) \xrightarrow{q} C^G_*(X;\mathcal{H})$ is an isomorphism. By naturality, then, the map $\mathrm{H}^G_*(X;\mathcal{H}) \to \mathrm{H}^G_*(*;\mathcal{H})$ is a retract of $\mathrm{H}^K_*(X;\mathcal{H}|_K) \to \mathrm{H}^K_*(*;\mathcal{H}|_K)$, and the theorem follows from the fact that a retract of an isomorphism is an isomorphism. □

11.5**. The method of discarded orbits.** This is a more sophisticated version of the direct transfer method which exploits the fact that K-orbits can be discarded if they do not contribute to the transfer.

11.6**. Theorem.** *Let X be a G-space, K a subgroup of G of index prime to p, and Y a subspace of X which is closed under the action of K. Assume that Y is acyclic for $\mathcal{H}|_K$, and that for each simplex $x \in X \setminus Y$ the transfer map $\mathcal{H}(\mathbb{F}_p[G/G_x]) \to \mathcal{H}(\mathbb{F}_p[G/K_x])$ is zero (cf. 10.12). Then X is acyclic for $\mathcal{H}$.*

Proof. The transfers associated to the maps $q : G\times_K X_n \to X_n$ provide a map $t : C_*^G(X;\mathcal{H}) \to C_*^K(X;\mathcal{H}|_K)$ (§10). By 10.17 this map commutes with differentials, and the index assumption implies that the composite $C_*^G(X;\mathcal{H}) \xrightarrow{t} C_*^K(X;\mathcal{H}|_K) \xrightarrow{q} C_*^G(X;\mathcal{H})$ is an isomorphism. The transfer hypothesis shows that the image of t is actually in the subcomplex $C_*^K(Y;\mathcal{H}|_K)$ of $C_*^K(X;\mathcal{H}|_K)$. By naturality, the homology map $\mathrm{H}_*^G(X;\mathcal{H}) \to \mathrm{H}_*^G(*;\mathcal{H})$ is a retract of $\mathrm{H}_*^K(Y;\mathcal{H}|_K) \to \mathrm{H}_*^K(*;\mathcal{H}|_K)$, and the theorem follows from the fact that a retract of an isomorphism is an isomorphism. □

11.7. *Example.* Let X be a G-space, K a subgroup of G of index prime to p, and Y a subspace of X which is closed under the action of K. Suppose that Y is acyclic for the functors $\mathcal{H}_j^G$, $j \geq 0$. Assume finally that for each $x \in X \setminus Y$ the transfer map $\mathrm{H}_*(G_x) \to \mathrm{H}_*(K_x)$ is zero. In light of 10.2, Theorem 11.6 implies that X is acyclic for the functors $\mathcal{H}_j^G$, $j \geq 0$.

A special case of the above is due to Webb [37] [1, V.3]; the proof here is essentially the same as his.

11.8. **Corollary.** *Let X be a G-space and P a Sylow p-subgroup of G. Suppose that for every nonidentity subgroup Q of P the fixed point space X^Q is contractible, and that for any $x \in X$ there is a (nonidentity) element of order p in G_x. Then X is acyclic for the functors $\mathcal{H}_j^G$, $j \geq 0$.*

Proof. Let Y be the P-subspace of X consisting of simplices which are fixed by a nonidentity element of P. By 10.6, the map $Y \to *$ is a weak P-equivalence, and so by 10.7 the space Y is acyclic for $\mathrm{H}_j(P;–)$. Moreover, for any $j \geq 0$ and $x \in X \setminus Y$ the transfer map

$$\mathrm{H}_j(G_x) \to \mathrm{H}_j(P_x) = \mathrm{H}_j(\{e\})$$

is trivial. This is true for $j > 0$ because the target group is zero, and true for $j = 0$ because by assumption the norm map $\sum_{g\in G_x} g : \mathbb{F}_p \to \mathbb{F}_p$ is trivial. The result follows from 11.6 (cf. 11.7). □

12. Non-identity p-subgroups

Let $\mathcal{C}$ be the collection of all non-identity p-subgroups of G. Assume that $\mathcal{C}$ is nonempty, i.e., that the order of G is divisible by p. In this section we will show that $\mathcal{C}$ is ample and that the three homology decompositions derived from $\mathcal{C}$ are sharp. In all three cases we use the method of Webb (11.8) to show that the spaces $X_{\mathcal{C}}^{\delta}$, $X_{\mathcal{C}}^{\beta}$, and $X_{\mathcal{C}}^{\alpha}$ are acyclic for the coefficient functors $\mathcal{H}_j^G$ $(j \geq 0)$ (11.2).

We first recall a property of finite p-groups.

12.1. **Lemma.** *Any nontrivial finite p-group has a nontrivial center.*

Proof. Let P be a nontrivial finite p-group, and consider the action of P on itself by conjugation. The fixed point set of the action is the center C of P. Since each nontrivial orbit has order a power of p, card(C) is congruent mod p to card(P), so in particular card(C) is divisible by p. Since C certainly contains the identity element, it follows that C also contains nonidentity elements of P. □

Let P denote a Sylow p-subgroup of G.

12.2. *The normalizer decomposition.* See Webb [37, 2.2.2]. We will show that $X^\delta_{\mathcal{C}}$ is acyclic for the coefficient functors $\mathcal{H}^G_j$. The first step is to show that for each nonidentity subgroup Q of P the space $(X^\delta_{\mathcal{C}})^Q$ is contractible. By 9.13 and 5.11 $(X^\delta_{\mathcal{C}})^Q$ is the nerve of the full subcategory $\mathbf{D}$ of $\mathbf{X}^\delta_{\mathcal{C}}$ generated by the objects H of $\mathbf{X}^\delta_{\mathcal{C}}$ (equivalently, elements $H \in \mathcal{C}$) such that $Q \subset N_G(H)$. The inclusions

$$H \subset H \cdot Q \supset Q$$

provide a zigzag of natural transformations between the identity functor of $\mathbf{D}$ and the constant functor with value Q. The existence of this zigzag implies that $\mathrm{N}(\mathbf{D})$ is contractible (5.6).

A typical n-simplex $Q_0 \subset \cdots \subset Q_n$ ($Q_i \in \mathcal{C}$) of $X^\delta_{\mathcal{C}}$ has isotropy subgroup $\cap_i N_G(Q_i)$. It is clear that there is an element of order p in this isotropy subgroup; any element of Q_0 will do. Now use 11.8.

12.3. *Remark.* The above result implies that the collection $\mathcal{C}$ of nontrivial p-subgroups of G is ample (see 11.2).

12.4. *The centralizer decomposition.* We again show that $X^\alpha_{\mathcal{C}}$ is acyclic for the coefficient functors $\mathcal{H}^G_j$. We first prove that for any nonidentity subgroup Q of P the space $(X^\alpha_{\mathcal{C}})^Q$ is contractible. By 9.11 and 5.11, $(X^\alpha_{\mathcal{C}})^Q$ is the nerve of the full subcategory $\mathbf{D}$ of $\mathbf{X}^\alpha_{\mathcal{C}}$ generated by the objects (H, i) with the property that $Q \subset C_G(i(H))$. Let Z be the center of Q and $j : Z \to G$ the inclusion. For an object (H, i) of $\mathbf{D}$, let H' denote the image of the product map $H \times Z \to G$ and $i' : H' \to G$ the inclusion. The maps

$$(H, i) \to (H', i') \leftarrow (Z, j)$$

give a zigzag of natural transformations between the identity functor of $\mathbf{D}$ and the constant functor with value (Z, j). As above, then, $\mathrm{N}(\mathbf{D})$ is contractible.

The isotropy subgroup of a typical n-simplex

$$H_0 \to H_1 \to \cdots H_n \to G \tag{12.5}$$

of $X^\alpha_{\mathcal{C}}$ has the form $C_G(Q)$ for some $Q \in \mathcal{C}$. It follows from 12.1 that such an isotropy subgroup contains a nonidentity element of order p. Now use 11.8.

12.6. *The subgroup decomposition.* Again, we show that $X^\beta_{\mathcal{C}}$ is acyclic for the coefficient functors $\mathcal{H}^G_j$. As above the first problem is to show that for any non-identity subgroup Q of P the space $(X^\beta_{\mathcal{C}})^Q$ is contractible. By 9.12 and inspection $(X^\beta_{\mathcal{C}})^Q$ is the nerve of the full subcategory $\mathbf{D}$ of $\mathbf{X}^\beta_{\mathcal{C}}$ generated by the pairs $(x, G/H)$ with $Q \subset G_x$. The category $\mathbf{D}$ has $(eQ, G/Q)$ as an initial element; in other words, for any object $(x, G/H)$ of $\mathbf{D}$ there is a unique map $(eQ, G/Q) \to (x, G/H)$. This implies (5.4) that $\mathrm{N}(\mathbf{D})$ is contractible.

A typical simplex of $X^\beta_{\mathcal{C}}$ has as its isotropy subgroup a group of the form Q for some $Q \in \mathcal{C}$. Any such isotropy subgroup contains an element of order p. Now, again, use 11.8.

13. Elementary abelian p-subgroups

In this section we prove several sharpness statements about collections of elementary abelian p-subgroups of G.

Non-identity elementary abelian p-subgroups. Let $\mathcal{C}$ be the collection of all non-identity elementary abelian p-subgroups of G. We will show that $\mathcal{C}$ is both centralizer-sharp and normalizer-sharp. The arguments mimic the ones in §12. Let P be a Sylow p-subgroup of G. In each case we use the method of Webb (11.8) to show that the spaces $X^\delta_{\mathcal{C}}$ and $X^\alpha_{\mathcal{C}}$ are acyclic for the coefficient functors $\mathcal{H}^G_j$.

Again, we begin with an elementary property of finite p-groups. It is proved by a counting argument like the one in the proof of 12.1

13.1. **Lemma.** *Let P and Q be finite p-groups with $Q \neq \{e\}$, and suppose that P acts on Q via group automorphisms. Then there exists a nonidentity element x in the center of Q such that x is fixed by the action of P.*

13.2. *The normalizer decomposition.* Again, see [37, 2.2.2]. We follow 12.2. The first step is to show that for any non-identity subgroup Q of P the space $(X^\delta_{\mathcal{C}})^Q$ is contractible. By 9.13 and 5.11, $(X^\delta_{\mathcal{C}})^Q$ is the nerve of the full subcategory $\mathbf{D}$ of $\mathbf{X}^\delta_{\mathcal{C}}$ determined by the objects H of $\mathbf{X}^\delta_{\mathcal{C}}$ (equivalently, elements $H \in \mathcal{C}$) such that $Q \subset N_G(H)$. Let Z be the group of elements of exponent p in the center of Q, and given an object H of $\mathbf{D}$, let H' be the group of elements of exponent p in the center of QH. The inclusions

$$H \supset H \cap H' \subset H'Z \supset Z$$

give a zigzag of natural transformations between the identity functor of $\mathbf{D}$ and the constant functor with value Z. Lemma 13.1 implies that $H \cap H'$ is not the identity subgroup of G. By 5.6, $\mathrm{N}(\mathbf{D})$ is contractible.

As in 12.2, the isotropy subgroup of any simplex of $X^{\delta}_{\mathcal{C}}$ contains an element of order p.

13.3. *The centralizer decomposition.* See [19], where the authors prove a similar theorem for compact Lie groups by a somewhat different argument. We follow the argument of 12.4. The first step is to show that if Q is a non-identity subgroup of P, then $(X^{\alpha}_{\mathcal{C}})^Q$ is contractible. By 9.11 and 5.11, $(X^{\alpha}_{\mathcal{C}})^Q$ is the nerve of the full subcategory $\mathbf{D}$ of $\mathbf{X}^{\alpha}_{\mathcal{C}}$ generated by objects (H, i) such that $Q \subset C_G(i(H))$. Let Z denote group of elements of exponent p in the center of Q and $j : Z \to G$ the inclusion. For an object (H, i) of $\mathbf{D}$, let H' denote the image of the product map $H \times Z \to G$ and $i' : H' \to G$ the inclusion. The maps

$$(H, i) \to (H', i') \leftarrow (Z, j) \tag{13.4}$$

provide a zigzag of natural transformations between the identity functor of $\mathbf{D}$ and the constant functor with value (Z, j). As above, this implies that $\mathrm{N}(\mathbf{D})$ is contractible.

As in 12.4, the isotropy subgroup of any simplex of $X^{\alpha}_{\mathcal{C}}$ contains an element of order p.

Smaller collections. If $\mathcal{C}$ is a collection of subgroups of G and K is a subgroup of G, let $\mathcal{C} \cap 2^K$ denote the set of all elements of $\mathcal{C}$ which are subgroups of K. Clearly $\mathcal{C} \cap 2^K$ is a collection of subgroups of K. We are aiming at the following theorem, as an illustration of how it is sometimes possible to work with smaller collections than the collection of all nontrivial elementary abelian p-subgroups.

13.5. **Theorem.** *Let K be a subgroup of G of index prime to p, and $\mathcal{C}$ a collection of elementary abelian p-subgroups of G. If $\mathcal{C} \cap 2^K$ is centralizer-sharp (as a K-collection) then $\mathcal{C}$ is centralizer-sharp (as a G-collection).*

13.6. *Example.* Theorem 13.5 can be used as a substitute for the argument of 13.3 in showing that the collection $\mathcal{C}$ of all non-identity elementary abelian p-subgroups of G is centralizer-sharp (for the trivial module $\mathbb{F}_p$). To see this, let P be a Sylow p-subgroup of G. It is enough to prove that the collection $\mathcal{C}' = \mathcal{C} \cap 2^P$ of all non-identity elementary abelian p-subgroups of P is centralizer-sharp as a P-collection. We can derive this from 10.7 by showing that $X^{\alpha}_{\mathcal{C}'}$ is P-equivariantly equivalent to a point. Let $j : Z \to P$ be the inclusion of the group of elements of exponent p in the center of P. If (H, i) is an object of $\mathbf{X}^{\alpha}_{\mathcal{C}'}$, let H'

denote the image of the product map $H \times Z \to P$ and $i' : H' \to P$ the inclusion. The maps

$$(H, i) \to (H', i') \leftarrow (Z, j)$$

provide a zigzag of natural transformations between the identity functor of $\mathbf{X}^{\alpha}_{\mathcal{C}'}$ and the constant functor with value (Z, j). This zigzag respects the action of P on $\mathbf{X}^{\alpha}_{\mathcal{C}'}$, and so, according to the arguments of §5, gives an equivariant contraction of $X^{\alpha}_{\mathcal{C}'}$.

13.7. *Example.* [3] It is possible to do better than the above. Let P be a Sylow p-subgroup of G, and let Z be any non-identity central elementary abelian p-subgroup of P. Let $\mathcal{C}$ be the smallest collection of elementary abelian p-subgroups of G which contains Z and has the property that if $V \in \mathcal{C}$ and V commutes with Z then $\langle Z, V\rangle \in \mathcal{C}$. An argument virtually identical to the one in 13.6 shows that if $\mathcal{C}' = \mathcal{C} \cap 2^P$, then $X^{\alpha}_{\mathcal{C}'}$ is P-equivariantly contractible. It follows from 13.5 and 10.7 that $\mathcal{C}$ is centralizer-sharp.

The proof of 13.5 depends on the following observation.

13.8. **Lemma.** *Suppose that K is a subgroup of G and that V is an elementary abelian subgroup of G not entirely contained in K. Then the transfer map*

$$\mathrm{H}_*(C_G(V); \mathbb{F}_p) \to \mathrm{H}_*(C_G(V) \cap K; \mathbb{F}_p)$$

associated to $C_G(V) \cap K \to C_G(V)$ (see 10.12) is zero.

Proof. Let $C_1 = C_G(V) \cap K$ and $C_2 = C_G(V)$. Choose $v \in V$ with $v \notin K$, and let $C_1' \cong C_1 \times \langle v\rangle$ be the subgroup of C_2 generated by C_1 and v. The inclusion $C_1 \to C_2$ factors as the composite of $f' : C_1' \to C_2$ with $f : C_1 \to C_1'$, so the transfer in question factors (10.13) as a parallel composite $\tau_*(f)\tau_*(f')$. However, the map $\tau_*(f)$ is zero; this follows from the fact that the map

$$f_* : \mathrm{H}_*(C_1; \mathbb{F}_p) \to \mathrm{H}_*(C_1'; \mathbb{F}_p)$$

is a monomorphism (f has a left inverse) and the fact that the composite $\tau_*(f) \cdot f_*$ is multiplication by p (10.16). □

Proof of 13.5. Let X be the G-space $X^{\alpha}_{\mathcal{C}} = \mathrm{N}(\mathbf{X}^{\alpha}_{\mathcal{C}})$; we have to show that X is acyclic for the functors $\mathrm{H}_i(G; –)$. The strategy is to use the method of discarded orbits (11.5). Let $\mathbf{Y}$ be the full subcategory of $\mathbf{X}^{\alpha}_{\mathcal{C}}$ determined by the objects (H, i) such that $i(H)$ is a subgroup of K, and let $Y = \mathrm{N}(\mathbf{Y})$, so that Y is a subspace of $X^{\alpha}_{\mathcal{C}}$. The action of G on $X^{\alpha}_{\mathcal{C}}$ restricts to an action of K on Y, and it is clear that Y is equivalent as a K-space to $X^{\alpha}_{\mathcal{C}'}$, where $\mathcal{C}' = \mathcal{C} \cap 2^K$. In particular, Y is by hypothesis acyclic for the functors $\mathrm{H}_i(K;-)$, $i \geq 0$. Let x be a simplex of $X \setminus Y$ as in 12.5, and let V be the image of H_n in G. Since V is not contained in K, Lemma 13.8 guarantees that the homology transfer map associated to the inclusion

$$K_x = C_G(V) \cap K \to G_x = C_G(V)$$

is zero. The desired result follows from 11.7.

14. Appendix

In this section, we sketch the proof of 10.7.

If (X, A) is a pair of G-spaces (i.e., A is a subspace of X), let $C_*^G(X, A; \mathcal{H})$ denote the quotient complex $C_*^G(X; \mathcal{H})/C_*^G(A; \mathcal{H})$ and $H_*^G(X, A; \mathcal{H})$ its homology. It is clear that there is a long exact sequence relating $H_*^G(A; \mathcal{H})$, $H_*^G(X; \mathcal{H})$, and $H_*^G(X, A; \mathcal{H})$.

Let K be a normal subgroup of G. A pair (X, A) is said to be *relatively free mod* K if K acts trivially on the simplices of X not in A and G/K acts freely on these simplices. Let $R = \mathbb{F}_p[G/K]$. If (X, A) is relatively free mod K, then the relative simplicial chain complex $C_*(X, A; \mathbb{F}_p)$ is a chain complex of free R-modules, and there is an evident isomorphism $C_*^G(X, A; \mathcal{H}) \cong \mathcal{H}(R) \otimes_R C_*(X, A)$. The next lemma follows from basic homological algebra.

14.1. **Lemma.** *Suppose that $f : (X, A) \to (Y, B)$ is a map between pairs of G-spaces which are relatively free mod K, and that f induces an isomorphism $\mathrm{H}_*(X, A; \mathbb{F}_p) \cong \mathrm{H}_*(Y, B; \mathbb{F}_p)$. Then f induces an isomorphism $H_*^G(X, A; \mathcal{H}) \cong H_*^G(Y, B; \mathcal{H})$.*

Prove this. It follows from the fact that if R is a ring and $f : C \to C'$ is a map of nonnegatively graded chain complexes over R such that

- both C and C' are chain complexes of projective modules, and
- f induces an isomorphism on homology groups,

then f is a chain homotopy equivalence.

We can now prove 10.7. Pick representatives $\{K_i\}_{i=0}^m$ for the conjugacy classes of subgroups of G and label the representatives in such a way that if K_i is conjugate to a subgroup of K_j then $i \geq j$. If Z is a G-space, write $Z^{(n)}$ for the subspace of Z consisting of all $z \in Z$ such that G_z is conjugate to one of the groups K_i for $i \leq n$. Let the height of Z be the least integer n such that $Z = Z^{(n)}$. The proof will be by induction on the heights of the spaces X and Y involved. The result is easy to check if G acts trivially on X and Y, i.e., if the heights of these spaces are ≤ 0.

Assume by induction that the statement of 10.7 is true if the G-spaces involved have height $\leq n-1$. Suppose that X and Y are G-spaces of height $\leq n$ and that $f : X \to Y$ is a map which induces weak equivalences $X^H \to Y^H$ for all subgroups H of G. Let $A = X^{(n-1)}$, $B = Y^{(n-1)}$, $K = K_n$ and $N = N_G(K)$. We must prove that the map $H_*^G(X;\mathcal{H}) \to H_*^G(Y;\mathcal{H})$ is an isomorphism.

It is easy to check that there is a map of pushout squares

$$\begin{array}{ccccccc} G \times_N A^K & \longrightarrow & A & & G \times_N B^K & \longrightarrow & B \\ \downarrow & & \downarrow & \longrightarrow & \downarrow & & \downarrow \\ G \times_N X^K & \longrightarrow & X & & G \times_N Y^K & \longrightarrow & Y \end{array}$$

and that in these squares the vertical arrows are monic. By 10.6 the map $A \to B$ is a weak G-equivalence, so by induction and a long exact sequence argument it is enough to show that the map $H_*^G(X, A;\mathcal{H}) \to H_*^G(Y, B;\mathcal{H})$ is an isomorphism. Given the above diagram of squares, this is equivalent to showing that the map

$$H_*^N(X^K, A^K; \mathcal{H}|_N) \to H_*^N(Y^K, B^K; \mathcal{H}|_N)$$

is an isomorphism. Since the maps $A^K \to B^K$ and $X^K \to Y^K$ are weak equivalences of spaces, the map $\mathrm{H}_*(X^K, A^K; \mathbb{F}_p) \to \mathrm{H}_*(Y^K, B^K; \mathbb{F}_p)$ is an isomorphism. The desired result now follows from 14.1, since the N-space pairs (X^K, A^K) and (Y^K, B^K) are relatively free mod K. This last statement follows from the fact that all of the simplices which are added in going from A to X or from B to Y have isotropy group conjugate to K.

Can you use similar ideas to give a proof of 10.6?

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