

Algebraic K-Theory Eventually Surjects onto Topological K-Theory

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We are going to prove the following theorem. Its hypotheses are satisfied by regular quasiprojective algebraic varieties over algebraically closed or finite fields of characteristic not ℓ , by localizations away from ℓ of rings of integers in number fields, and in fact by virtually all regular schemes over $\mathbb{Z}(\ell^{-1})$ that are not recognized as pathological in some way.

Theorem 1. *Let X be a regular scheme, quasiprojective over some noetherian ring of finite Krull dimension. Let $\ell > 3$ be a prime invertible in that ring, and suppose that all the residue fields of X have uniformly bounded etale cohomological dimension for ℓ -torsion sheaves. Then for any positive integer n and any p the natural map*

$$(1.0) \quad \rho: K_p(X; \mathbb{Z}/\ell^n)(\beta_n^{-1}) \rightarrow K_p^{\text{et}}(X; \mathbb{Z}/\ell^n)$$

is a split surjection.

Here $K_*(X; \mathbb{Z}/\ell^n)(\beta_n^{-1})$ is mod ℓ^n algebraic K-theory of X localized by inverting a Bott element, as studied in [14]. $K^{\text{et}}(X; \mathbb{Z}/\ell^n)$ is the mod ℓ^n topological or etale K-theory of X , as studied in [5, 6], and [7]. The map ρ is constructed in these references.

Background. For X a smooth scheme over \mathbb{C} , $K_*^{\text{et}}(X; \mathbb{Z}/\ell^n)$ is isomorphic to the usual mod ℓ^n topological K-theory of X as a complex manifold ([6]). In this case, Theorem 1 essentially answers a question of Fulton, [9], §5. For many such X the map $\rho: K_0(X) \rightarrow K_0^{\text{et}}(X)$ cannot be a surjection, as non-trivial Hodge conditions on the Chern classes prevent some topological bundles from being stably algebraic. Nevertheless, Theorem 1 shows that the existence of these non-algebraic topological bundles is reflected in the *higher-dimensional* algebraic K-groups of X .

There are several conjectures in this area which are part of the Lichten-

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baum conjecture complex. In particular, it is thought that the split surjection ρ of (1.0) is an isomorphism, and that for large i the domain group $K_i(X; \mathbb{Z}/\ell^n)(\beta_n^{-1})$ of ρ is in fact isomorphic to $K_i(X; \mathbb{Z}/\ell^n)$. These conjectures are interesting because the range groups of ρ can usually be computed. (In fact the range groups are essentially the topological K -groups, with possibly twisted coefficients, of a certain space associated to X [5].) The conjectures are known to be true for certain specific varieties over the algebraic closure of a finite field, e.g., projective spaces minus planes [7], complete rational surfaces, reductive group schemes G and their homogeneous spaces G/P , P a parabolic subgroup [15].

Theorem 1 has an immediate corollary the following (much weaker) statement.

Corollary. *Let X be as in Theorem 1, and assume that $K_j^{\text{ét}}(X; \mathbb{Z}/\ell)$ is finite for all $j \geq 0$. Then for any positive integer n there exists a J such that for all $j \geq J$ the natural map.*

$$K_j(X; \mathbb{Z}/\ell^n) \rightarrow K_j^{\text{ét}}(X; \mathbb{Z}/\ell^n)$$

is surjective.

This is the statement referred to in the title of the paper. Its hypotheses are satisfied if X is regular and quasiprojective over an algebraically closed field of characteristic not ℓ .

Remark. That a localized version of algebraic K -theory might be better behaved than algebraic K -theory itself was suggested by Max Karoubi. Theorem 1 was proved for X over an algebraically closed field of finite characteristic in [14]; some cases were treated in [12]. Surjectivity of the map $K_j(X; \mathbb{Z}/\ell^n) \rightarrow K_j^{\text{ét}}(X; \mathbb{Z}/\ell^n)$ (no Bott element inverted) has been proved for some values of j and some low-dimensional X , cf. [5, 7, 13–15]. C. Soulé was a pioneer in this work.

Sketch of the Proof. The idea behind the proof of Theorem 1 is to construct an explicit splitting of the map ρ of (1.0) by using Snaith’s approximation to algebraic K -theory [12]. We show by calculation that this succeeds strictly locally in the étale topology, i.e., when X is the spectrum of a strict local Hensel ring. The étale local-to-global principle of [14] for localized algebraic K -theory and its easier analogue for étale K -theory then yield the general result by a formal argument.

Note that in what follows we will usually make no notational distinction between a ring A and the corresponding affine scheme $\text{Spec } A$.

Bott Elements

The first step in the proof of Theorem 1 is to construct various Bott elements. The procedure is complicated by the need to find a Bott element in the mod ℓ^n algebraic K -theory of a ring R even if R does not contain the ℓ^n th roots of

unity. For a ring R , let $S_*(R; \mathbb{Z}/\ell^n)$ denote the mod ℓ^n stable homotopy groups of the ring spectrum $\Sigma^\infty(BGL_1(R)_{\perp\perp *})$. This construction was studied by Snaith in [11, 12].

Let $A = \mathbb{Z}(\ell^{-1}, \zeta)$ where ζ is a primitive ℓ 'th root of unity. The cyclic subgroup of $GL_1(A)$ generated by ζ can be identified by means of a Bockstein homomorphism with $\pi_2(BGL_1(A); \mathbb{Z}/\ell)$. Let b_1 be the image under the natural map $\pi_2(BGL_1(A); \mathbb{Z}/\ell) \rightarrow S_2(A; \mathbb{Z}/\ell)$ of the element corresponding to ζ .

Note that A is finite and etale over $\mathbb{Z}(\ell^{-1})$ with Galois group π cyclic of order $\ell - 1$. Since π acts via ring automorphisms on A , it also acts via ring automorphisms on $S_*(A; \mathbb{Z}/\ell^n)$ ($n \geq 1$). The action of π on the cyclic subgroup of $GL_1(A)$ generated by ζ gives a canonical isomorphism $\chi: \pi \rightarrow (\mathbb{Z}/\ell)^*$, and the construction of b_1 immediately implies that π also acts through χ on the cyclic subgroup of $S_2(A; \mathbb{Z}/\ell)$ generated by b_1 . It follows that the $(\ell - 1)$ 'st cup power β_1 of b_1 is invariant under the action of π .

Lemma 2. *For each $n > 1$, the ℓ^{n-1} cup power of β_1 in $S_*(A; \mathbb{Z}/\ell)$ is the reduction mod ℓ of an element β_n in $S_*(A; \mathbb{Z}/\ell^n)$. Moreover, β_n can be chosen to be invariant under the action of π on $S_*(A; \mathbb{Z}/\ell^n)$.*

Proof. The first statement is proved by induction on n , with the help of the derivation properties of the Bockstein spectral sequence ([3]). The second statement follows easily from the fact that the order of π is prime to ℓ .

Pick a collection of β_n 's as in Lemma 2, so that β_n is invariant under the action of π . There is a certain ambiguity here, but it is unimportant. Any two candidates β'_n, β''_n for β_n differ by an element which is divisible by ℓ . It follows that this difference element $\beta'_n - \beta''_n$ is nilpotent in the ring $S_*(A; \mathbb{Z}/\ell^n)$, so that localizing a module over $S_*(A; \mathbb{Z}/\ell^n)$ by inverting β'_n yields the same result as localizing the module by inverting β''_n .

There is a natural map $\gamma: S_*(A; \mathbb{Z}/\ell^n) \rightarrow K_*(A; \mathbb{Z}/\ell^n)$, [12]. We will denote $\gamma(\beta_n)$ also by β_n .

Lemma 3. *For each $n \geq 1$, the element β_n in $K_*(A; \mathbb{Z}/\ell^n)$ is the image under the natural map of a unique element in $K_*(\mathbb{Z}(\ell^{-1}), \mathbb{Z}/\ell^n)$.*

Proof. Since the order of the Galois group π is prime to ℓ , a transfer argument shows that the natural map from $K_*(\mathbb{Z}(\ell^{-1}); \mathbb{Z}/\ell^n)$ to the Galois invariant elements of $K_*(A; \mathbb{Z}/\ell^n)$ is an isomorphism.

We will continue to use β_n to denote the elements in $K_*(\mathbb{Z}(\ell^{-1}); \mathbb{Z}/\ell^n)$ provided by Lemma 3. This concludes the construction of Bott elements. Note that if X is a scheme over $\mathbb{Z}(\ell^{-1})$ then $K_*(X; \mathbb{Z}/\ell^n)$ is an algebra over $K_*(\mathbb{Z}(\ell^{-1}); \mathbb{Z}/\ell^n)$ and so it is possible to form the localized K-theory $K_*(X; \mathbb{Z}/\ell^n)(\beta_n^{-1})$. The groups $S_*(A; \mathbb{Z}/\ell^n)$, $K_*(X; \mathbb{Z}/\ell^n)$, $K_*^{\text{et}}(X; \mathbb{Z}/\ell^n)$ are all mod ℓ^n stable homotopy groups of spectra, and are also the ordinary stable homotopy groups of spectra which have been reduced mod ℓ^n ([15, 6, 7, 11, 12, 14]). We note that those mod ℓ^n spectra are all ring spectra ($\ell \neq 2, 3$); and as in [14] 3.1-3.2 we may localize them to form $K(X; \mathbb{Z}/\ell^n)(\beta_n^{-1})$ naturally for X over $\mathbb{Z}(\ell^{-1})$ and $S(R; \mathbb{Z}/\ell^n)(\beta_n^{-1})$ naturally for R over $\mathbb{Z}(\ell^{-1}, \zeta)$. Since we are dealing with the non-connective version of K^{et} , the localization

map $K^{et}(X; \mathbb{Z}/\ell^n) \rightarrow K^{et}(X; \mathbb{Z}/\ell^n)(\beta_n^{-1})$ is a homotopy equivalence, and so we do not distinguish these spectra notationally.

In [12], Snaith describes a map of ring spectra, natural in the ring R

$$(1.2) \quad \gamma: S(R; \mathbb{Z}/\ell^n) \rightarrow K(R, \mathbb{Z}/\ell^n).$$

This map is ultimately induced by the monomial matrix inclusions $\Sigma_n \int GL_1(R) \rightarrow GL_n(R)$ and it itself induces on homotopy groups the map which was called γ above. We may invert β_n on both sides of (1.2) if R is an algebra over $\mathbb{Z}(\gamma^{-1}, \zeta)$.

In [5-7], Friedlander and also Dwyer describe a map for many X :

$$(1.3) \quad \rho: K(X; \mathbb{Z}/\ell^n) \rightarrow K^{et}(X; \mathbb{Z}/\ell^n)$$

This is a natural map of ring spectra for X affine. In general, it is only well-defined and natural up to homotopy. For X over $\mathbb{Z}(\ell^{-1})$, we may naturally invert β_n on both sides of (1.3), although this procedure has no effect on the right-hand side. The resulting map of localized spectra induces on homotopy the map called ρ in Theorem 1.

The Strictly Local Case

The next step in proving Theorem 1 is to prove it strictly locally in the etale topology, i.e. for $X = \text{Spec}(R)$, R a regular strictly local hensel ring.

Lemma 4. *Let R be a regular noetherian strict local hensel ring containing ℓ^{-1} . Then*

$$S_*(R; \mathbb{Z}/\ell)(b_1^{-1}) \cong \mathbb{Z}/\ell[b_1, b_1^{-1}].$$

Pf: Under the hypotheses, R is an integral domain containing all ℓ -power roots of unity, and R^* is ℓ -divisible. Let $\mathbb{Q}/\mathbb{Z}_{(\ell)} = \mu$ be the subgroup of all ℓ -power roots of unity in R^* . Then the cokernel R^*/μ is uniquely ℓ -divisible, and $B(R^*/\mu)$ has the mod ℓ^n stable homotopy type of a point. Thus we have isomorphisms (1.4).

$$(1.4) \quad \pi_*^s(B\mu \perp \ast; \mathbb{Z}/\ell^n) \cong \pi_*^s(BR^* \perp \ast; \mathbb{Z}/\ell^n) = S_*(R; \mathbb{Z}/\ell^n)$$

The canonical class in $H^2(\mathbb{Q}/\mathbb{Z}_{(\ell)}, \mathbb{Z})$ yields a map $B\mu = B\mathbb{Q}/\mathbb{Z}_{(\ell)} \rightarrow K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$ into the Eilenberg-MacLane space which becomes an equivalence upon ℓ -adic completion. Thus we have isomorphisms (1.5).

$$(1.5) \quad S_*(R; \mathbb{Z}/\ell^n) \cong \pi_*^s(B\mu \perp \ast; \mathbb{Z}/\ell^n) \cong \pi_*^s(\mathbb{C}P^\infty \perp \ast; \mathbb{Z}/\ell^n)$$

By [11], 9.1.1, $\pi_*^s(\mathbb{C}P^\infty \perp \ast; \mathbb{Z}/\ell)(b_1^{-1})$ maps isomorphically to $\pi_*(BU)$, where BU denote the spectrum for topological K -theory. Thus yields the result.

Lemma 4 was first proved by Snaith in [12] II §2.

Proposition 5. *Let R be a regular noetherian strict local hensel ring containing ℓ^{-1} . Then the composite $\rho\gamma: S(R; \mathbb{Z}/\ell^n)(\beta_n^{-1}) \rightarrow K^{et}(\text{Spec}(R); \mathbb{Z}/\ell^n)(\beta_n^{-1})$ is a weak homotopy equivalence.*

Pf: It suffices to show $\rho\gamma$ induces an isomorphism on $\pi_*, S_*(R; \mathbb{Z}/\ell^n)(\beta_n^{-1}) \cong K_*^{et}(R; \mathbb{Z}/\ell^n)(\beta_n^{-1})$. By the usual Bockstein argument, it suffices to do the case $n=1$, (e.g., [14] (3.7)–(3.10) and ambient discussion). Then by Lemma 4, both rings are abstractly isomorphic to $\mathbb{Z}/\ell[x, x^{-1}]$. Working through the definitions of γ and ρ in [5] and [12] one checks that b_1 goes to a generator in $K_2^{et}(R; \mathbb{Z}/\ell)$.

This may be done as follows. Note that maps between strict local hensel rings induce isomorphisms on $S_*(; \mathbb{Z}/\ell^n)(b_n^{-1})$ and on $K_*^{et}(; \mathbb{Z}/\ell^n)$. Naturality of $\rho\gamma$ with respect to the map to the residue field $R \rightarrow R/m$ reduces the problem to the case where R is a separably closed field. The usual Witt vector lifting then reduces the problem to characteristic 0, hence to the special case $R = \mathbb{Q}$, or even $R = \mathbb{C}$. For $R = \mathbb{C}$, $\rho\gamma$ is induced by the obvious maps $BGL_1(\mathbb{C}) \rightarrow \mathbb{C}P^\infty \rightarrow BU$. Then Snaith’s theorem ([11] 9.1.1) yields the result, as $BGL_1(\mathbb{C})$ and $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$ are ℓ -adically equivalent as in Lemma 4.

Descent for Algebraic K-Theory

We next use the cohomological descent machinery of Thomason [14] to reduce Theorem 1 to this strictly local case.

Consider the restricted etale site of X . This is the category of affine schemes etale over X . The Grothendieck topology has as covers the faithfully flat, possibly infinite families of etale maps. The maps γ and ρ are natural transformations of presheaves of spectra on this site.

We will apply the hypercohomology spectrum functor $\mathbb{H}(X;)$. Recall from [14] its construction. Let F be a presheaf of spectra on the site of X . For $\mathfrak{U} = \{U_i \rightarrow X | i \in I\}$ a cover, let $F_{\mathfrak{U}}^*$ be the Čech cosimplicial spectrum (1.6).

$$(1.6) \quad F_{\mathfrak{U}}^* = \prod_{i_0 \in I} F(U_{i_0}) \overset{\leftarrow}{\rightrightarrows} \prod_{(i_0, i_1) \in I^2} F(U_{i_0} \times_{\chi} U_{i_1}) \overset{\leftarrow}{\rightrightarrows}$$

Let $\text{colim}_{\mathfrak{U}} F_{\mathfrak{U}}^*$ be the direct limit over all Godement covers \mathfrak{U} . Then $\mathbb{H}(X; F)$ is the homotopy limit over Δ of the cosimplicial spectrum $\text{colim}_{\mathfrak{U}} F_{\mathfrak{U}}^*$. It is a spectrum, and there is a natural augmentation (1.7).

$$(1.7) \quad \eta: F(x) \rightarrow \mathbb{H}(X; F)$$

Lemma 6. *For X a separated scheme, the Godement covers \mathfrak{U} in the restricted etale site form a cofinal direct subsystem of the Godement covers in the full etale site. Thus $\text{colim}_{\mathfrak{U}} F_{\mathfrak{U}}^*$ and $\mathbb{H}(X; F)$ formed in the two sites are naturally isomorphic.*

Pf: Clear. \square

The maps ρ and γ are defined as maps of presheaves of spectra only on the restricted etale site. On the other hand, it is only the full site which is natural in X . Lemma 6 resolves the difficulties caused by this, and allows us to form $\mathbb{H}(X; \rho)$ and $\mathbb{H}(X; \gamma)$ natural even in non-affine maps of X .

Theorem 7 (cf. [14]). *Let X be a regular scheme, quasiprojective over some noetherian ring of finite Krull dimension. Let $\ell > 3$ be a prime invertible in that ring. Suppose all residue fields of X have uniformly bounded etale cohomological*

dimension for ℓ -torsion sheaves. Then the natural augmentation (1.8) is a weak equivalence.

$$(1.8) \quad \eta(x): K(X; \mathbb{Z}/\ell^n)(\beta_n^{-1}) \xrightarrow{\sim} \mathbb{H}(X; K(\ ; \mathbb{Z}/\ell^n)(\beta_n^{-1}))$$

Corollary 8. Under the hypotheses of Theorem 7, there is a strongly convergent spectral sequence (1.9).

$$(1.9) \quad E_2^{p,q} = H_{\text{et}}^p(X; K_q(\ ; \mathbb{Z}/\ell^n)(\beta_n^{-1})) \Rightarrow K_{q-p}(X; \mathbb{Z}/\ell^n)(\beta_n^{-1})$$

Pf: For X over a ring with primitive ℓ^n th roots of unity, this is Thomason’s main theorem, [14] 3.4. As remarked in [14], now that $K(\ ; \mathbb{Z}/\ell^n)(\beta_n^{-1})$ has been defined for more general X , it is easy to extend the result.

First, arguing via the Bockstein exact sequence as in [14] 3.4, one reduces to the case $n = 1$.

Now for X over $\text{Spec}(\mathbb{Z}(\ell^{-1}))$, let X' be the pull-back of X to $\mathbb{Z}(\ell^{-1}, \zeta)$, where ζ is a primitive ℓ th root of one. Then $X' \rightarrow X$ is finite etale with Galois group $(\mathbb{Z}/\ell)^* = \mathbb{Z}/\ell^{-1}$. The map $\eta(X'): K(X; \mathbb{Z}/\ell)(\beta_1^{-1}) \rightarrow \mathbb{H}(X'; K(\ ; \mathbb{Z}/\ell)(\beta_1^{-1}))$ is a weak equivalence as X' is over $\mathbb{Z}(\ell^{-1}, \zeta)$. The maps $\eta(X)$ and $\eta(X')$ are compatible under the transfer map from X' to X by [14] 2.14–2.20. As the order of $(\mathbb{Z}/\ell)^*$ is prime to ℓ , $K_*(X; \mathbb{Z}/\ell)(\beta_1^{-1})$ and $\pi_* \mathbb{H}(X; K(\ ; \mathbb{Z}/\ell)(\beta_1^{-1}))$ are identified to the Galois invariant elements in $K_*(X; \mathbb{Z}/\ell)(\beta_1^{-1})$ and $\pi_* \mathbb{H}(X'; K(\ ; \mathbb{Z}/\ell)(\beta_1^{-1}))$ by the usual transfer argument. The map $\pi_* \eta(X)$ is induced by $\pi_* \eta(X')$ under this identification. Thus $\pi_* \eta(X)$ is an isomorphism and $\eta(X)$ is a weak equivalence, as $\pi_* \eta(X')$ is a Galois equivariant isomorphism.

The spectral sequence follows as in [14].

Descent for Etale K-Theory

Theorem 9. Let X be a noetherian separated scheme over $\mathbb{Z}(\ell^{-1})$. Suppose X has finite Krull dimension, and all its residue fields have uniformly bounded etale cohomological dimension for ℓ -torsion sheaves. Then the natural augmentation map (1.10)

$$(1.10) \quad \eta(X): K^{\text{et}}(X; \mathbb{Z}/\ell^n) \rightarrow \mathbb{H}(X; K^{\text{et}}(\ ; \mathbb{Z}/\ell^n))$$

is a weak equivalence.

Pf: Recall from [5] that for a simplicial scheme U , $K^{\text{et}}(U; \mathbb{Z}/\ell^n)$ is the β_n^{-1} localization of the mod ℓ^n reduction of a spectrum constructed from a Γ -space. This Γ -space is $\coprod \text{fib}(p_m)$ constructed from the connected components of the fibres of (1.11).

$$(1.11) \quad p_m: \text{holim Hom}((U)_{\text{et}}, \# \mathbb{Z}/\ell_{\infty}^*(BGL_{m, R})_{\text{et}}) \rightarrow \text{holim Hom}((U)_{\text{et}}, \# (\text{Spec}(R))_{\text{et}})$$

over the point determined by the structure map $U \rightarrow \text{Spec}(R)$. The symbol $\#$ indicates the tower of coskeleta. Here $R = \mathbb{Z}(\ell^{-1})$. The holims are taken over

the indexing category for the Hom shapes. $\mathbb{Z}/\ell_\infty^\bullet$ is the Bousfield-Kan fibrewise completion functor [2]. Consult Dwyer-Friedlander [5] for details. Let $F(U)$ denote the connective spectrum obtained from this Γ -space, and let $F\{k\}(U)$ denote the spectrum obtained from the analogous Γ -space constructed from fibres of $p_m\{k\}$. Here $p_m\{k\}$ is obtained by replacing $\# \mathbb{Z}/\ell_\infty^\bullet(BGL_{m,R})_{\text{et}}$ and $\#(\text{Spec}(R))_{\text{et}}$ in (1.11) by their k -th coskeleta.

To prove the theorem, we exhibit the following diagram of weak equivalences, (1.12).

$$(1.12) \quad \begin{array}{ccc} F(X) & \xrightarrow{\eta} & \mathbb{H}(X; F) \\ \downarrow \varphi & & \downarrow \psi \\ \text{holim}_k F\{k\}(X) & \xrightarrow{\varepsilon} & \text{holim}_k \text{colim}_U \text{holim}_\Delta F\{k\}_U \xrightarrow{\theta} \text{holim}_k \text{holim}_\Delta \text{colim}_U F\{k\}_U \end{array}$$

The indicated maps are all the canonical ones. The point is to show that $\varphi, \varepsilon, \theta$, and ψ are weak equivalences, or at least induce isomorphisms on π_q for q not small. Then η will be a weak equivalence after inverting β_n .

The hypotheses on X imply that X and each component $U_{i_0} \times \dots \times U_{i_n}$ of the nerve of a cover \mathbb{U} have bounded etale cohomological dimension ([1] X 4.1, [4]). The construction of an Atiyah-Hirzebruch type spectral sequence in [5] generalizes to yield a strongly converging fringed spectral sequence for $\pi_* F\{k\}(X)$ and $\pi_* F(X)$. It follows that $F\{k\}(X)$ and $F\{k\}(U_{i_0} \times \dots \times U_{i_n})$ have only finitely many non-zero homotopy groups for each k , and $\pi_i F\{k\}(U_{i_0} \times \dots \times U_{i_n}) = \pi_i F(U_{i_0} \times \dots \times U_{i_n})$ for $k \gg i$, uniformly for $U_{i_0} \times \dots \times U_{i_n}$. Consequently, ψ is a weak equivalence as $\pi_* \psi$ is an isomorphism by the Milnor sequence and the Mittag-Leffler condition (c.f. [2] IX §3). Similarly, φ is a weak equivalence; in this case almost by definition. Compare the argument in [14] 1.44.

To show θ is a weak equivalence, it suffices to check that for each k

$$(1.13) \quad \theta: \text{colim}_U \text{holim}_\Delta F\{k\}_U \rightarrow \text{holim}_\Delta \text{colim}_U F\{k\}_U$$

is a weak equivalence. As $\pi_n F\{k\}(\) = 0$ for $n \gg k$ uniformly, this is [14] Lemma 1.18.

To show ε is a weak equivalence, it suffices to show for each k and \mathbb{U} that

$$(1.14) \quad \varepsilon(\mathbb{U}): F\{k\}(X) \rightarrow \text{holim}_U F\{k\}_U$$

is a weak equivalence. Let the simplicial scheme $U_\bullet = N_X(\mathbb{U})$ be the nerve of the cover \mathbb{U} . There is a weak equivalence of cosimplicial spectra (1.15).

$$(1.15) \quad (n \rightarrow F\{k\}(U_n)) \simeq F\{k\}_U$$

Remember $F\{k\}(\)$ is defined on simplicial schemes such as U_\bullet . As $U_\bullet \rightarrow X$ induces a weak equivalence of etale topological types ([8]), the natural map $F\{k\}(X) \rightarrow F\{k\}(U_\bullet)$ is a weak equivalence. Thus we need only show (1.16) is a weak equivalence.

$$(1.16) \quad F\{k\}(U_\bullet) \rightarrow \text{holim}_\Delta (n \rightarrow F\{k\}(U_n))$$

As in [5, 7], group completion of the Γ -space $\coprod \text{fib}(p\{k\})$ doesn't change the homotopy type of the 0th space $\text{colim}_m \text{fib}(p_m\{k\})$. This is quite different from what happens in algebraic K -theory. It allows us to commute group completion and taking holim . (Note that $\text{holim}(n \rightarrow G(U_n))$ commutes with direct limits of G such that $\text{cosk}_k G = G$. To see this, first replace U_\bullet by a refining and refined noetherian hypercover to see $\tilde{H}^*(U_\bullet; _)$ commutes with direct limits. Then appeal to the strongly converging spectral sequence analogous to [14] 1.17.) At least this is true modulo problems with π_0 resulting from the fringe effect in the spectral sequence. But $\pi_q \varepsilon$ will be an isomorphism for $q > 0$ and so η will become a weak equivalence after inverting β_n . Thus it suffices to find a weak equivalence (1.17):

$$(1.17) \quad \text{fib}(p_m\{k\}) \rightarrow \text{holim}_\Delta (n \rightarrow \text{fib}(q_{mn}\{k\}))$$

Here $q_{mn}\{k\}$ is the map (1.18).

$$(1.18) \quad q_{m,n}\{k\}: \text{holim Hom}((U_n)_{\text{et}}, \# \mathbb{Z}/\ell_\infty^*(BGL_{m,R}) \rightarrow \text{holim Hom}((U_n)_{\text{et}}, \# (\text{Spec } R)_{\text{et}}).$$

Because $(U_\bullet)_{\text{et}} = \text{diag}\{V_{\bullet,i}\}_{i \in I}$, with $\{V_{m,\bullet,i}\}_{i \in I} = (U_m)_{\text{et}}$ by [8], the required weak equivalence is provided by the following lemma.

Lemma 10. *Let $\{V_{\bullet,i}\}_{i \in I}$ be a probisimplicial set and $\{W_\bullet(j)\}_{j \in J}$ be a pro-simplicial set such that each $W_\bullet(j)$ is a Kan complex which is isomorphic to a finite coskeleton of itself via the canonical map. Then there is a natural weak equivalence (1.19).*

$$(1.19) \quad \begin{aligned} & \text{holim}_J \text{colim}_I \text{Hom}(\text{diag } V_{\bullet,i}, W_\bullet(j)) \xrightarrow{\sim} \\ & \xrightarrow{\sim} \text{holim}_{m \in \Delta} \text{holim}_J \text{colim}_I \text{Hom}(V_{m,\bullet,i}, W_\bullet(j)) \end{aligned}$$

Pf: This results from the sequence of canonical isomorphisms and weak equivalences ([2]) (1.20).

$$(1.20) \quad \begin{aligned} & \text{holim}_J \text{colim}_I \text{Hom}(\text{diag } V_{\bullet,i}, W_\bullet(j)) \\ & \cong \text{holim}_J \text{colim}_I \text{Tot Hom}(V_{m,\bullet,i}, W_\bullet(j)) \\ & \cong \text{holim}_J \text{Tot}(\text{colim}_m \text{Hom}(V_{m,\bullet,i}, W_\bullet(j))) \\ & \cong \text{holim}_J \text{holim}_{m \in \Delta} \text{colim}_I \text{Hom}(V_{m,\bullet,i}, W_\bullet(j)) \\ & = \text{holim}_{m \in \Delta} \text{holim}_J \text{colim}_I \text{Hom}(V_{m,\bullet,i}, W_\bullet(j)) \end{aligned}$$

The first isomorphism is elementary, as in the proof of [2] XII 4.3. The third weak equivalence and the fourth isomorphism are [2] XI 4.4 and XI 4.3 respectively: the fibrancy conditions are easy to check. The second isomorphism is a bit trickier. We show the inducing map is an isomorphism for each

fixed $j \in J$. Fix j and pick N such that $W(j) = \cos k_N W(j)$. Then for all Y ,

$$\begin{aligned} \text{Hom}(Y, W(j)) &= \text{Hom}(Y, \cos k_N W(j)) \cong \cos k_N \text{Hom}(Y, \cos k_N W(j)) \\ &\cong \cos k_N \text{Hom}(Y, W(j)). \end{aligned}$$

Thus it suffices to show if $Z(i)$ is a direct system of cosimplicial sets such that for each $p \in \Delta$ and $i \in I$, $Z(i)^p = \cos k_N Z(i)^p$, then there is an isomorphism (1.21).

$$(1.21) \quad \text{colim}_I \text{Tot}(p \rightarrow Z(i)^p) \cong \text{Tot}(p \rightarrow \text{colim}_I Z(i)^p)$$

For we take $Z(i)^p = \text{Hom}(X_{p_i}(i), W(j))$.

A q simplex of $\text{Tot}(Z(i))$ is a family of maps (1.22)

$$(1.22) \quad f(p): \Delta(p) \times \Delta(q) \rightarrow Z(i)^p$$

which form a natural transformation of functors as p varies over Δ . As $Z(i)^p = \cos k_N Z(i)^p$, families (1.22) correspond to families (1.23) natural in p .

$$(1.23) \quad \tilde{f}(p): s k^N(\Delta(p) \times \Delta(q)) \rightarrow Z(i)^p$$

But one calculates that a natural family of $\tilde{f}(p)$ is uniquely determined by its restriction to the subcategory of $p \leq N$. Conversely any natural family of $\tilde{f}(p)$ defined for $p \leq N$ extends to a natural family (1.23). The set of natural transformations on the finite subcategory of $p \leq N$ is a finite end, and so computed by finite limits. Thus it commutes with direct limits like colim_I . Tracing back along the bijections, one gets (1.21) for q -simplices. This proves the lemma.

Lemma 11. *Let ℓ, n, X be as in Theorem 1. Then if X is affine, the diagram (1.24) strictly commutes. In those cases where $\rho(x)$ is defined by Jouanolou's device ([5-7, 10]) (1.24) commutes up to homotopy.*

$$(1.24) \quad \begin{array}{ccc} \mathbb{H}(X; K(\ ; \mathbb{Z}/\ell^n)(\beta_n^{-1})) & \xrightarrow{\mathbb{H}(X; \rho)} & \mathbb{H}(X; K^{et}(\ ; \mathbb{Z}/\ell^n)) \\ \uparrow \eta \wr & & \uparrow \eta \wr \\ K(X; \mathbb{Z}/\ell^n)(\beta_n^{-1}) & \xrightarrow{\rho(x)} & K^{et}(X; \mathbb{Z}/\ell^n)(\beta_n^{-1}) \end{array}$$

Pf: For X affine, this is just the naturality of the augmentations η . If Jouanolou's device applies to X , it follows as the above diagram is mapped via weak equivalences to the diagram for the affine resolution \tilde{X} of X .

Note as the augmentations $\eta(X)$ are weak equivalences by Theorem 7 and 9, we may use (1.24) to define $\rho(X)$ up to homotopy for general X .

Proof of Theorem 1

We now prove Theorem 1 globally for X over $\mathbb{Z}(\ell^{-1}, \zeta)$.

Proposition 12. *Let ℓ, n, X be as in Theorem 1. Suppose also that X is over $\mathbb{Z}(\ell^{-1}, \zeta)$. Then the conclusion of Theorem 1 holds for X ; i.e., $\rho(X)$ of (1.0) is a*

split surjection. In fact, $\rho(X)$ of (1.25) splits up to homotopy.

$$(1.25) \quad \rho(X): K(X; \mathbb{Z}/\ell^n)(\beta_n^{-1}) \xrightarrow{\rho} K^{et}(X; \mathbb{Z}/\ell^n)$$

Pf: As X is over $\mathbb{Z}(\ell^{-1}, \zeta)$, there is a sequence of natural transformations of presheaves of spectra on X , (1.26)

$$(1.26) \quad \begin{array}{ccc} S(; \mathbb{Z}/\ell^n)(\beta_n^{-1}) & \xrightarrow{\gamma} & K(; \mathbb{Z}/\ell^n)(\beta_n^{-1}) \rightarrow K^{et}(; \mathbb{Z}/\ell^n) \\ & & \downarrow \wr \\ & & K(; \mathbb{Z}/\ell^n)(\beta_n^{-1}) \end{array}$$

Consider the diagram (1.27).

$$(1.27) \quad \begin{array}{ccccc} \mathbb{H}(X; S(; \mathbb{Z}/\ell^n)(\beta_n^{-1})) & \xrightarrow{\mathbb{H}(X; \gamma)} & \mathbb{H}(X; K(; \mathbb{Z}/\ell^n)(\beta_n^{-1})) & \xrightarrow{\mathbb{H}(X; \rho)} & \mathbb{H}(X; K^{et}(; \mathbb{Z}/\ell^n)) \\ & & \uparrow \wr \eta & & \uparrow \wr \eta \\ & & K(X; \mathbb{Z}/\ell^n)(\beta_n^{-1}) & \xrightarrow{\rho(X)} & K^{et}(X; \mathbb{Z}/\ell^n) \end{array}$$

Contemplation reveals it suffices to show $\mathbb{H}(X; \rho\gamma)$ is a weak equivalence; i.e. that $\pi_* \mathbb{H}(X; \rho\gamma)$ is an isomorphism. By the strongly converging spectral sequences of [14] 1.31, it will be enough to show the $\rho\gamma$ of (1.26) induces an isomorphism of the sheafifications of π_* . We check this at each stalk. By [14] 1.25, this amounts to showing (1.26) is a weak equivalence when evaluated at each strict local henselization of X . But this was done in Proposition 5. (Note $K^{et}(; \mathbb{Z}/\ell^n)$ is “continuous” in the sense of [14] 1.25 because of the strongly converging Atiyah-Hirzebruch spectral sequence [5] and the “continuity” of etale cohomology ([1], VII§5); $S(; \mathbb{Z}/\ell^n)$ is obviously “continuous”).

It remains to prove Theorem 1 for X over $\mathbb{Z}(\ell^{-1})$. Without ζ , we cannot for m $S(; \mathbb{Z}/\ell^n)(\beta_n^{-1})$ as a presheaf (i.e., naturally) on the etale site of X . We dodge this as follows.

Let X' be the pullback of X and $\mathbb{Z}(\ell^{-1}, \zeta)$ over $\mathbb{Z}(\ell^{-1})$. As before, $p: X' \rightarrow X$ is finite etale with Galois group $(\mathbb{Z}/\ell)^*$. $K_*(\mathbb{Z}; \mathbb{Z}/\ell^n)(\beta_n^{-1})$ and $K_*^{et}(X; \mathbb{Z}/\ell^n)$ are the $(\mathbb{Z}/\ell)^*$ invariant elements of $K_*(X'; \mathbb{Z}/\ell^n)(\beta_b^{-1})$ and $K_*^{et}(X'; \mathbb{Z}/\ell^n)$ respectively. The first follows from a transfer argument, the second from the descent spectral sequence for K_*^{et} of the cover $X' \rightarrow X$ (Theorem 9, [14, 1.42]). The desired result now follows easily from the fact that the order of $(\mathbb{Z}/\ell)^*$ is prime to ℓ , and the fact that a retract of a split epimorphism is a split epimorphism.

References

1. Artin, M., Grothendieck, A., Verdier, J.-L.: Theorie des topos et cohomologie etale des schemas. Lecture Notes in Math. Vols. 269, 270, pp.305. Berlin-Heidelberg-New York: Springer 1972-73

2. Bousfield, A.K., Kan, D.M.: Homotopy limits, completions, and localizations. Lecture Notes in Math. Vol. 304. Berlin-Heidelberg-New York: Springer 1973
3. Browder, W.: Algebraic K-theory with coefficients \mathbb{Z}/p . Springer Lecture Notes in Math. Vol. 651, pp. 40-84. Berlin-Heidelberg-New York: Springer 1973
4. Deligne, P.: Cohomologie étale: SGA 4 1/2. Lecture Notes in Math. Vol. 569. Berlin-Heidelberg-New York: Springer 1977
5. Dwyer, W., Friedlander, E.: Étale K-theory and arithmetic, Bull. A.M.S. (in press 1982)
6. Friedlander, E.M.: Étale K-theory I: Connections with étale cohomology and algebraic vector bundles. Invent. Math. **60**, 105-134 (1980)
7. Friedlander, E.M.: Étale K-theory II: Connections with algebraic K-theory. Ann. Scient. Éc. Norm. Sup. (in press 1982)
8. Friedlander, E.M.: Étale homotopy of simplicial schemes. Princeton University Press (in press 1982)
9. Fulton, W.: Riemann-Roch for singular varieties. Algebraic Geometry: Arcata 1974, Amer. Math. Soc. 1975, pp. 449-457
10. Jouanolou, J.P.: Une suite exacte de Mayer-Vietoris en K-théorie algébrique. Lecture Notes in Math. Vol. 341, pp. 293-316 Berlin-Heidelberg-New York: Springer 1973
11. Snaith, V.: Algebraic cobordism and K-theory. Memoir of the Amer. Math. Soc. **221**, (1979)
12. Snaith, V.: Algebraic K-theory and localized stable homotopy. (Preprint)
13. Soulé, C.: K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale. Invent. Math. **55**, 251-295 (1979)
14. Thomason, R.W.: Algebraic K-theory and étale cohomology. (Preprint)
15. Thomason, R.W.: Riemann-Roch for algebraic vs. topological K-theory. (In press 1982)