Review: Expected Utility

ECON 40565
Fall 2007

Intermediate Micro

- Workhorse model of intermediate micro
  - Utility maximization problem
  - Consumers Max U(x,y) subject to the budget constraint, $I=P_x x + P_y y$
- Problem is made easier by the fact that we assume all parameters are known
  - Consumers know prices and income, simple mathematical problem for person

- Many cases, some of the parameters are unknown
- Uncertainty about prices? Income? State of the world? Will you have a job?
- This section, will review utility theory under uncertainty
  - Hopefully, most of this will be a review of ECON306

- Will emphasize the special role of insurance in a generic sense
  - Why insurance is 'good'
  - How much insurance should people purchase
- Insurance markets may generate incentives that reduce the welfare gains of insurance
  - Moral hazard
  - Adverse selection

Definitions

- Probability - likelihood and event will occur
  - P_i be the probability event i happens
  - $0 \leq P_i \leq 1$
  - Suppose there are n potential events
  - $P_1+P_2+P_3+\ldots P_n=1$
- Probabilities can be 'subjective' or 'objective', depending on the model

- Expected value –
  - Weighted average of possibilities, weight is probability
  - Sum of the possibilities times probabilities
- $x=\{x_1,x_2\ldots x_n\}$
- $P=\{P_1,P_2,\ldots P_n\}$
- $E(x) = P_1X_1 + P_2X_2 + P_3X_3 + \ldots . P_nX_n$

- Roll of a die, all sides have (1/6) prob. What is expected roll?
- E(x) = 1(1/6) + 2(1/6) + … 6(1/6) = 3.5
- Suppose you have: 25% chance of an A, 50% B, 20% C, 4% D and 1% F
- E[quality points] = 4(.25) + 3(.5) + 2(.2) + 1(.04) + 0(.01) = 2.94

Expected utility

- Suppose income is random. Two potential values (Y_1 or Y_2)
- Probabilities are either P_1 or $P_2=1-P_1$
- When incomes are realized, consumer will experience a particular level of income and hence utility
- Looking at the problem beforehand, a person has a particular 'expected utility'
- May have to choose over different lotteries
- How do they choose?

Assumptions about utility with uncertainty

- One element (income or wealth)
- U = U(Y)

- Marginal utility is positive
  - $U' = dU/dY > 0$

- Standard assumption, risk averse, U``<0
- (will relax this in the future)

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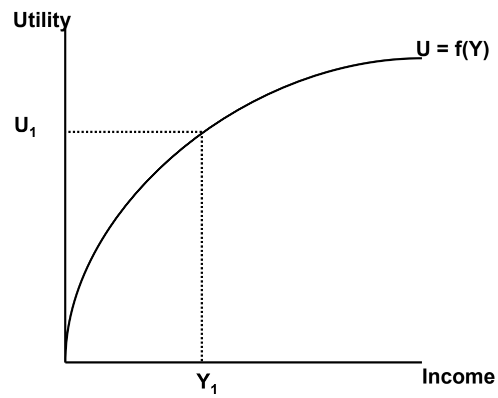

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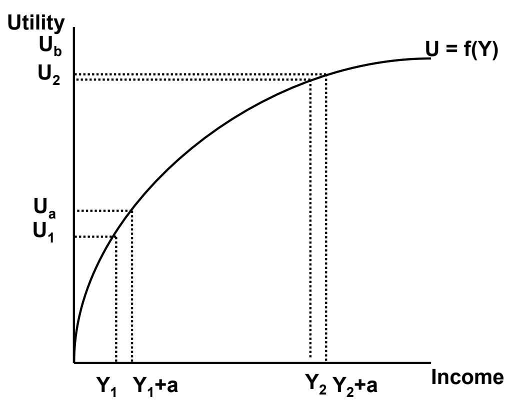

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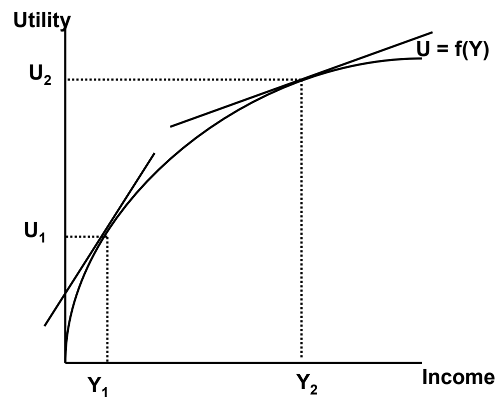

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Von Neumann-Morganstern Utility

- N states of the world, with incomes defined as Y_1 Y_2 ….Y_n
- The probabilities for each of these states is P_1 P_2…P_n
- A valid utility function is the expected utility of the gamble
- $E(U) = P_1U(Y_1) + P_2U(Y_2) \dots + P_nU(Y_n)$

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- E(U) is the sum of the possibilities times probabilities
- Example:
  - 40% chance of earning $2500/month
  - 60% change of $1600/month
  - $E(U) = P_1U(Y_1) + P_2U(Y_2)$
  - $U(Y) = Y^{0.5}$
  - $E(U) = 0.4(2500)^{0.5} + 0.6(1600)^{0.5}$
    $= 0.4(50) + 0.6(40) = 44$

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- Note that expected utility in this case is very different from expected income
  - $E(Y) = 0.4(2500) + 0.6(1600) = 1960$
- Expected utility allows people to compare gambles
- We assume people prefer situation 1 to 2 if $E(U_1) > E(U_2)$
- People maximize expected utility

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- Expected income is not what we compare
- Two situations can have the same expected value but very different expected utilities
- Job A: certain income of $50K
- Job B: 50% chance of $10K and 50% chance of $100K
- Expected income is the same ($50K), one is much more certain

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- Job 1
  - 40% chance of $2500
  - 60% chance of $1600
  - $E(Y) = 0.4*2500 + .6*1600 = \$1960$
  - $E(U_1) = (0.4)(2500)^{0.5} + (0.6)(1600)^{0.5} = 44$

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- Job 2
  - 25% chance of $5000
  - 75% chance of $1000
  - $E(Y) = .25(5000) + .75(1000) = \$2000$
  - $E(U_2) = 0.25(5000)^{0.5} + 0.75(1000)^{0.5} = 41.4$
- Notice that $E(Y_1) < E(Y_2)$
- But $E(U_1) > E(U_2)$

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How to represent graphically

- Probability P_1 of having Y_1
- $(1-P_1)$ of having Y_2
- U_1 and U_2 are utility that one would receive if they received Y_1 and Y_2 respectively
- $E(Y) = P_1Y_1 + (1-P_1)Y_2 = Y_3$
- U_3 is utility they would receive if they had income Y_3 with certainty

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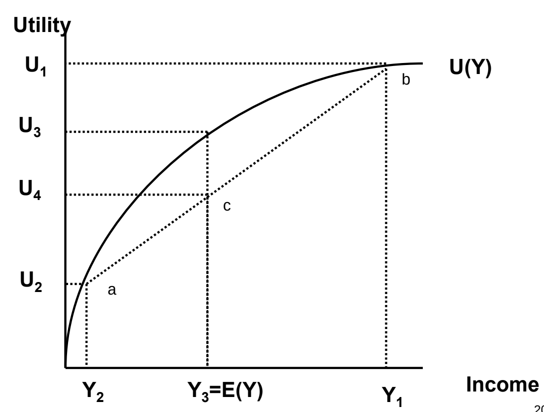

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- Notice that E(U) is a weighted average of utilities in the good and bad states of the world
- $E(U) = P_1U(Y_1) + (1-P_1)U(Y_2)$
- The weights sum to 1 (the probabilities)
- Draw a line from points (a,b)
- Represent all the possible 'weighted averages' of $U(Y_1)$ and $U(Y_2)$
- What is the one represented by this gamble?

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- Draw vertical line from E(Y) to the line segment. This is E(U)
- U_4 is Expected utility
- $U_4 = E(U) = P_1U(Y_1) + (1-P_1)U(Y_2)$

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- Suppose offered two jobs
  - Job A: Has chance of a high (Y1) and low (Y2) wages
  - Job B: Has chance of high (Y3) and low (Y4) wages
  - Expected income from both jobs is the same
  - P_a and P_b are the probabilities of getting the high wage situation

$$P_aY_1 + (1-P_a)Y_2 = P_bY_3 + (1-P_b)Y_4 = E(Y)$$

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Numeric Example

- Job A
  - 20% chance of $150,000
  - 80% chance of $20,000
  - E(Y) = 0.2(150K) + 0.8(20K) = $46K
- Job B
  - 60% chance of $50K
  - 40% chance of $40K
  - E(Y) = 0.6(50K) + 0.4(40K) = $46K

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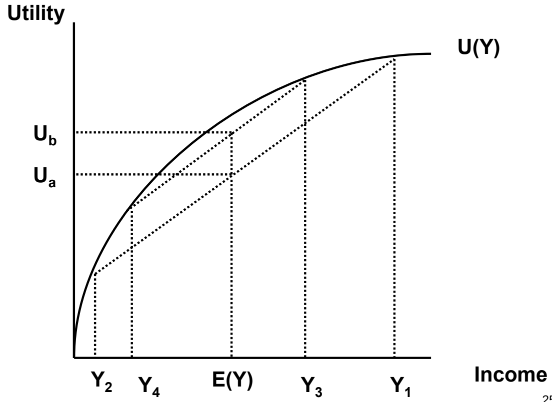

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- Notice that Job A and B have the same expected income
- Job A is riskier – bigger downside for Job A
- Prefer Job B (Why? Will answer in a moment)

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- The prior example about the two jobs is instructive. Two jobs, same expected income, very different expected utility
- People prefer the job with the lower risk, even though they have the same expected income
- People prefer to 'shed' risk – to get rid of it.
- How much are they willing to pay to shed risk?

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Example

- Suppose have $200,000 home (wealth).
- Small chance that a fire will damage you house. If does, will generate $75,000 in loss (L)
- U(W) = ln(W)
- Prob of a loss is 0.02 or 2%
- Wealth in "good" state = W
- Wealth in bad state = W-L

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- E(W) = (1-P) W + (1-P)(W-L)
- E(W) = 0.98(200,000) + 0.02(125,000) = $198,500

- E(U) = (1-P) ln(W) + P ln(W-L)
- E(U) = 0.98 ln(200K) + 0.02 ln(200K-75K) = 12.197

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- Suppose you can add a fire detection/prevention system to your house.
- This would reduce the chance of a bad event to 0 but it would cost you $C to install
- What is the most you are willing to pay for the security system?
- E(U) in the current situation is 12.197
- Utility with the security system is U(W-C)
- Set U(W-C) equal to 12.197 and solve for C

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- ln(W-C) =12.197
- Recall that $e^{\ln(x)} = x$
- Raise both sides to the e
- $e^{\ln(W-C)} = W-C = e^{12.197} = 198,128$
- W – 198,128 = 200,000 – 198,128 = C = $1872

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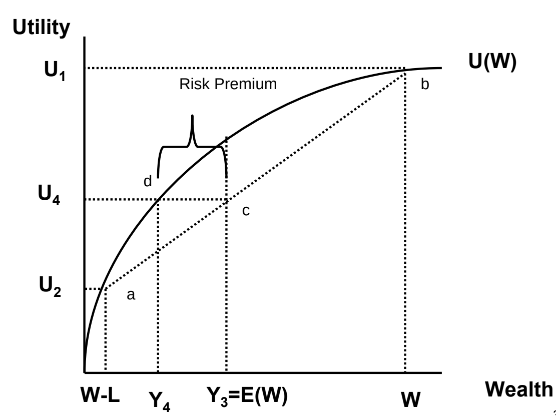

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- Will earn Y_1 with probability p_1
  - Generates utility U_1
- Will earn Y_2 with probability p_2=1-p_1
  - Generates utility U_2
- E(I) =p_1Y_1 + (1-p_1)Y_2 = Y_3
- Line (ab) is a weighted average of U_1 and U_2
- Note that expected utility is also a weighted average
- A line from E(Y) to the line (ab) give E(U) for given E(Y)

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- Take the expected income, E(Y). Draw a line to (ab). The height of this line is E(U).
- E(U) at E(Y) is U_4
- Suppose income is know with certainty at I_3. Notice that utility would be U_3, which is greater that U_4
- Look at Y_4. Note that the $Y_4<Y_3$=E(Y) but these two situations generate the same utility

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- The line segment (cd) is the “Risk Premium.” It is the amount a person is willing to pay to avoid the risky situation.
- If you offered a person the gamble of Y_3 or income Y_4, they would be indifferent.
- Therefore, people are willing to sacrifice cash to ‘shed’ risk.

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Some numbers

- Person has a job that has uncertain income
  - 50% chance of making $30K, U(30K) = 18
  - 50% chance of making $10K, U(10K) = 10
- E(I) = (0.5)($30K) + (0.05)($10K) = $20K
- E(U) = 0.5U(30K) + 0.5U(10K) = 14

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- **Expected utility. Weighted average of U(30) and U(10). E(U) = 14**
- **Notice that a gamble that gives expected income of $20K is equal in value to a certain income of only $16K**
- **This person dislikes risk.**
  - **Indifferent between certain income of $16 and uncertain income with expected value of $20**
  - **Utility of certain $20 is a lot higher than utility of uncertain income with expected value of $20**

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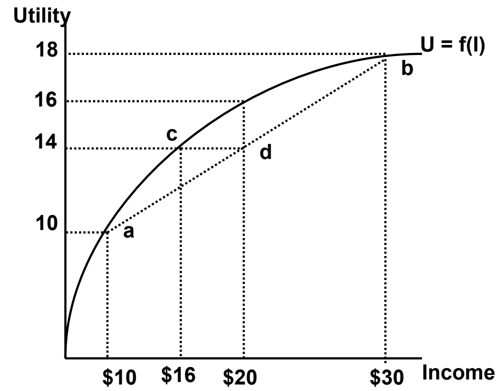

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- **Although both jobs provide the same expected income, the person would prefer the guaranteed $20K.**
- **Why? Because of our assumption about diminishing marginal utility**
  - **In the 'good' state of the world, the gain from $20K to $30K is not as valued as the 1st $10**
  - **In the 'bad' state, because the first $10K is valued more than the last $10K, you lose lots of utils.**

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- **Notice also that the person is indifferent between a job with $16K in certain income and $20,000 in uncertain**
- **They are willing to sacrifice up to $4000 in income to reduce risk, risk premium**

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Example

- $U = y^{0.5}$
- Job with certain income
  - $400 week
  - $U=400^{0.5}=20$
- Can take another job that
  - 40% chance of $900/week, U=30
  - 60% chance of $100/week, U=10
  - E(I) = 420, E(U) = 0.4(30) + 0.6(10) = 18

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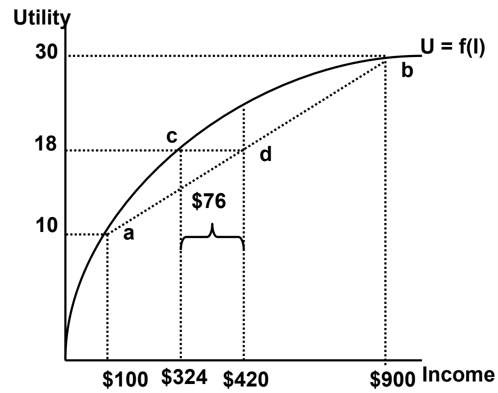

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- Notice that utility from certain income stream is higher even though expected income is lower
- What is the risk premium??
- What certain income would leave the person with a utility of 18? $U=Y^{0.5}$
- So if $18 = Y^{0.5}$, $18^2= Y =324$
- Person is willing to pay 400-324 = $76 to avoid moving to the risky job

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Allais Paradox

- Which gamble would you prefer
  - 1A: $1 million w/ certainty
  - 1B: (.89, $1 million), (0.01, $0), (0.1, $5 million)
- Which gamble would you prefer
  - 2A: (0.89, $0), (0.11, $1 million)
  - 2B: (0.9, $0), (0.10, $5 million)

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- 1^{st} gamble:
  - U(1) > 0.89U(1) + 0.01U(0) + 0.1U(5)
  - 0.11U(1) > 0.01U(0) + 0.1U(5)
- Now consider gamble 2
  - 0.9U(0) + 0.1U(5) > 0.89U(0) + 0.11U(1)
  - 0.01U(0) + 0.1U(5) > 0.11U(1)
- Choice of Lottery 1A and 2B is inconsistent with expected utility theory

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St Petersburg Paradox

- Fixed fee to enter a game
- Pot starts at $1. Flip a coin and if a head appears, the pot doubles. If tails appears, you win the pot and the game ends.
- So, if you get H, H, H T, you win $8
- What would you be willing to pay to 'play' this game?

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- Probabilities?
- Pr(h)=Pr(t) = 0.5
- All events are independent
- Pr(h on 2^{nd} | h on 1^{st}) = Pr(h on 1^{st})
- Recall definition of independence
- If A and B and independent events
  - Pr (A ∩ B) = Pr(A)Pr(B)

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- Note, Pr(first tail on kth toss) =
- Pr(h on 1^{st})Pr(h on 2^{nd} …)…Pr(t on kth)=
- (1/2)(1/2)….(1/2) = $(1/2)^k$
- What is the expected pot on the kth trial?
- 1 on 1^{st} or 2^0
- 2 on 2^{nd} or 2^1
- 4 on 3^{rd}, or 2^2
- So the payoff on the k^{th} is 2^{k-1}

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- What is the expected value of the gamble
- E = (1/2)$1 + (1/2)²$2 + (1/2)³$4 + (1/2)⁴$8

$$E=\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\left(2^{k-1}\right)=\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)=\infty$$

- The expected payout is infinite

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Round	Winnings	Probability
5th	$16	0.03125
10th	$512	0.000977
15th	$16,384	3.05E-5
20th	$524,288	9.54E-7
25th	$16,777,216	2.98E-8

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Risk Loving

- The desire to shed risk is due to the assumption of declining marginal utility of income
- Consider the next situation.
- The graph shows increasing marginal utility of income
- U\`($Y_1$) > U\`(Y_2) even though Y_1>Y_2

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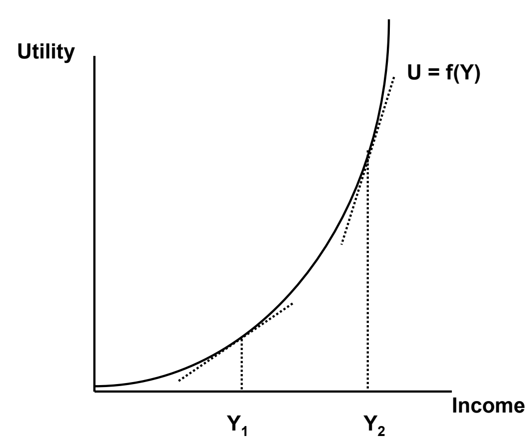

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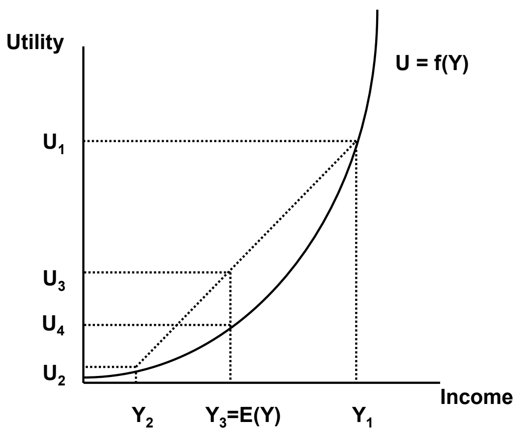

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- What does this imply about tolerance for risk?
- Notice that at $E(Y) = Y_3$, expected utility is U_3.
- Utility from a certain stream of income at Y_3 would generate U_4. Note that $U_3 > U_4$
- This person prefers an uncertain stream of Y_3 instead of a certain stream of Y_3
- This person is 'risk loving'. Again, the result is driven by the assumption are U``

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Risk Neutral

- If utility function is linear, the marginal utility of income is the same for all values of income
  - U`>0
  - U``=0
- The uncertain income E(Y) and the certain income Y_3 generate the same utility
- This person is considered risk neutral
- We usually make the assumption firms are risk neutral

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Example

- 25% chance of $100
- 75% chance of $1000
- E[Y] = 0.25(100) + 0.75(1000) = $775
- U = Y
- Compare to certain stream of $775

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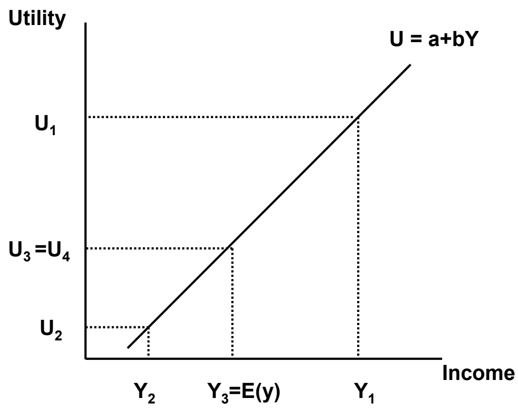

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Benefits of insurance

- Assume declining marginal utility
- Person dislikes risk
  - Is willing to receive lower certain income rather than higher expected income
- Firms can capitalize on the dislike for risk by helping people shed risk via insurance

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Simple insurance example

- Suppose income is know (Y_1) but random shocks can reduce income
  - House or car is damaged
  - Can pay $ to repair, return you to a state of normal
- L is the loss if the bad event happens
- Probability of loss is P_1
- Expected utility without insurance is
- $E(U) = (1-P_1)U(Y_1) + P_1U(Y_1-L)$

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- Suppose you can buy insurance that costs you PREM. The insurance pay you to compensate for the loss L.
  - In good state, income is
    - Y-Prem
  - In bad state, paid PREM, lose L but receive PAYMENT, therefore, income is
    - Y-Prem-L+Payment
  - For now, lets assume PAYMENT=L, so
  - Income in the bad state is also
    - Y-Prem

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- Notice that insurance has made income certain. You will always have income of Y-PREM
- What is the most this person will pay for insurance?
- The expected loss is p_1L
- Expected income is E(Y)
- The expected utility is U_2
- People would always be willing to pay a premium that equaled the expected loss

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- But they are also willing to pay a premium to shed risk (line cd)
- The maximum amount they are willing to pay is expected loss + risk premium

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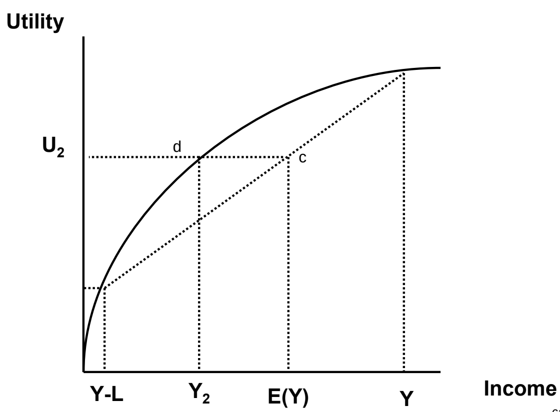

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- Suppose income is $50K, and there is a 5% chance of having a car accident that will generate $15,000 in loss
- Expected loss is .05(15K) = $750
- U = ln(y)
- Some properties of logs
  - Y=ln(x) then e^y = exp(y) = x
  - Y=ln(x^a) = a ln(x)
  - Y=ln(xz) = ln(x) + ln(z)

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- E(U) = P ln(Y-L) + (1-P)ln(Y)
- E(U) = 0.05 ln(35,000) + 0.95 ln(50,000)
- E(U) = 10.8

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- What is the most someone will pay for insurance?
- People would purchase insurance so long as utility with certainty is at least 10.8 (expected utility without insurance)
- U_a =U(Y – Prem) ≥ 10.8
- Ln(Y-PREM) ≥10.8
- Y-PREM = exp(10.8)
- PREM =Y-exp(10.8) = 50,000 – 49,021 =979

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- Recall that the expected loss is $750 but this person is willing to pay more than the expected loss to avoid the risk
- Pay $750 (expected loss), plus the risk premium ($979-$750) = 229

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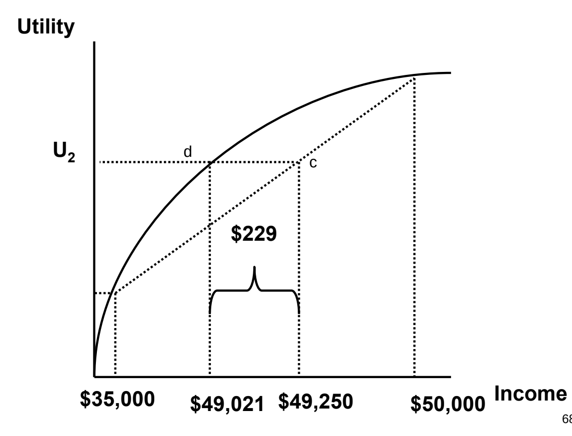

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Supply of Insurance

- Suppose there are a lot of people with the same situation as in the previous slide
- Each of these people have a probability of loss P and when a loss occurs, they have L expenses
- A firm could collect money from as many people as possible in advance. If bad event happens, they pay back a specified amount.

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- Firms are risk neutral, so they are interested in expected profits
- Expected profits = revenues – costs
  - Revenues are known
  - Some of the costs are random (e.g., exactly how many claims you will pay)

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- Think of the profits made on sales to one person
- A person buys a policy that will pay them q dollars ($q \leq L$) back if the event occurs
- To buy this insurance, person will pay "a" dollars per dollar of coverage
- Cost per policy is fixed t

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- Revenues = aq
  - a is the price per dollar of coverage
- Costs =pq +t
  - For every dollar of coverage (q) expect to pay this p percent of time
- $E(\pi) = aq - pq - t$
- Let assume a perfectly competitive market, so in the long run $\pi = 0$
- What should the firm charge per dollar of coverage?
- $E(\pi) = aq - pq - t = 0$

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- a = p + (t/q)
- The cost per dollar of coverage is proportion to risk
- t/q is the loading factor. Portion of price to cover administrative costs
- Make it simple, suppose t=0.
  - a = p
  - If the probability of loss is 0.05, will change 5 cents per $1.00 of coverage

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- In this situation, if a person buys a policy to insure L dollars, the 'actuarially fair' premium will be LP
- An actuarially fair premium is one where the premium equals the expected loss
- In the real world, no premiums are 'actuarially fair' because prices include administrative costs called 'loading factors'

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How much insurance will people purchase?

- With insurance
  - Pay a premium that is subtracted from income
  - If bad state happens, lose L but get back the amount of insurance q
  - They pay p+(t/q) per dollar of coverage. Have q dollars of coverage to pay a premium of pq+t in total
- Utility in good state
  - U = U[Y – pq - t]

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- Utility in bad state
  - U[Y- L + q – pq - t]
- E(u) = (1-p)U[Y – pq – t] + pU[Y-L+q-pq-t]
- Simplify, let t=0 (no loading costs)
- E(u) = (1-p)U[Y – pq] + pU[Y-L+q-pq]
- Maximize utility by picking optimal q
- dE(u)/dq = 0

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- E(u) = (1-p)U[Y – pq] + pU[Y-L+q-pq]
- dE(u)/dq = (1-p) U`(y-pq)(-p)
- + pU`(Y-L+q-pq)(1-p) = 0
- p(1-p)U`(Y-L+q-pq) = (1-p)pU`(Y-pq)
- (1-p)p cancel on each side

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- U`(Y-L+q-pq) = U`(Y-pq)
- Optimal insurance is one that sets marginal utilities in the bad and good states equal
- Y-L+q-pq = Y-pq
- Y's cancel, pq's cancel,
- q=L
- If people can buy insurance that is 'fair' they will fully insure loses.

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Insurance w/ loading costs

- Insurance is not actuarially fair and insurance does have loading costs
- Can show (but more difficult) that with loading costs, people will now under-insure, that is, will insure for less than the loss L
- Intution? For every dollar of expected loss you cover, will cost more than a $1
- Only get back $1 in coverage if the bad state of the world happens

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- Recall:
  - q is the amount of insurance purchased
  - Without loading costs, cost per dollar of coverage is p
  - Now, for simplicity, assume that price per dollar of coverage is pK where K>1 (loading costs)
- Buy q $ worth of coverage
- Pay qpK in premiums

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- E(u) = (1-p)U[Y – pqk] + pU[Y-L+q-pqk]

- dE(u)/dq = (1-p) U`(y-pqk)(-pk)
- + pU`(Y-L+q-pqk)(1-pk) = 0

- p(1-pk)U`(Y-L+q-pq) = (1-p)pkU`(Y-pq)

- p cancel on each side

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- (1-pk)U`(Y-L+q-pq) = (1-p)kU`(Y-pq)
- (a)(b) = (c)(d)
- Since k > 1, can show that
- (1-pk) < (1-p)k
- Since (a) < (c), must be the case that
- (b) > (d)
- U`(Y-L+q-pq) > U`(Y-pq)
- Since U'(y1) > U'(y2), must be that y1 < y2

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- (Y-L+q-pqk) < (Y-pqk)
- Y and –pqk cancel
- -L + q < 0
- Which means that q < L
- When price is not 'fair' will not fully insure

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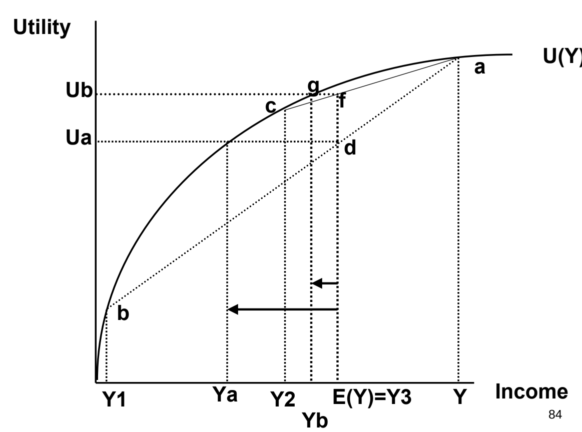

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Demand for Insurance

- Both people have income of Y
- Each person has a potential health shock
  - The shock will leave person 1 w/ expenses of E1 and will leave income at Y1=Y-E1
  - The shock will leave person 2 w/ expenses of E2 and will leave income at Y2=Y-E2
- Suppose that
  - E1>E2, Y1<Y2

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- Probabilities the health shock will occur are P1 and P2
- Expected Income of person 1
  - $E(Y)_1$ = (1-P1)Y + P1*(Y-E1)
  - $E(Y)_2$ = (1-P2)Y + P2*(Y-E2)
  - Suppose that $E(Y)_1 = E(Y)_2$ = Y3

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- In this case
  - Shock 1 is a low probability/high cost shock
  - Shock 2 is a high probability/low cost shock
- Example
  - Y=$60,000
  - Shock 1 is 1% probability of $50,000 expense
  - Shock 2 is a 50% chance of $1000 expense
  - E(Y) = $59500

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- Expected utility locus
  - Line ab for person 1
  - Line ac for person 2
- Expected utility is
  - Ua in case 1
  - Ub in case 2
- Certainty premium –
  - Line (de) for person 1, Difference Y3 – Ya
  - Line (fg) for person 2, Difference Y3 - Yb

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Implications

- Do not insure small risks/high probability events
  - If you know with certainty that a costs will happen, or, costs are low when a bad event occurs, then do not insure
  - Example: teeth cleanings. You know they happen twice a year, why pay the loading cost on an event that will happen?

- Insure catastrophic events
  - Large but rare risks
- As we will see, man of the insurance contracts we see do not fit these characteristics – they pay for small predictable expenses and leave exposed catastrophic events

Some adjustments to this model

- The model assumes that poor health has a monetary cost and that is all.
  - When experience a bad health shock, it costs you L to recover and you are returned to new
- Many situations where
  - health shocks generate large expenses
  - And the expenses may not return you to normal
  - AIDS, stroke, diabetes, etc.

- In these cases, the health shock has fundamentally changed life.
- We can deal with this situation in the expected utility model with adjustment in the utility function
- "State dependent" utility
  - U(y) utility in healthy state
  - V(y) utility in unhealthy state

- **Typical assumption**
  - **U(Y) >V(Y)**
    - **For any given income level, get higher utility in the healthy state**
  - **U`(Y) > V`(Y)**
    - **For any given income level, marginal utility of the next dollar is higher in the healthy state**

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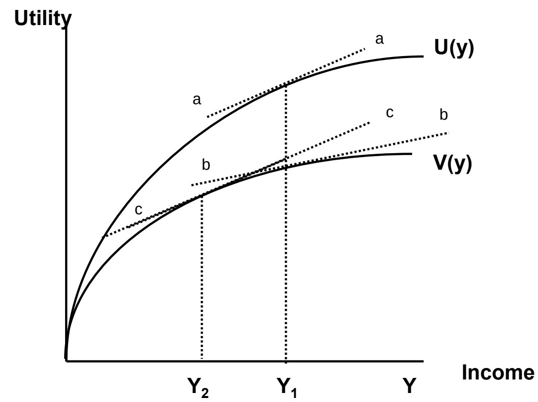

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Note that:

- **At Y_1,**
  - **$U(Y_1) > V(Y_1)$**
  - **$U`(Y_1) > V`(Y_1)$**
  - **Slope of line aa > slope of line bb**
- **Notice that slope line aa = slope of line cc**
  - **$U`(Y_1) = V`(Y_2)$**

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What does this do to optimal insurance

- **E(u) = (1-p)U[Y – pq – t] + pV[Y-L+q-pq-t]**
- **Again, lets set t=0 to make things easy**
- **E(u) = (1-p)U[Y – pq] + pV[Y-L+q-pq]**

- **dE(u)/dq = (1-p)(-p)U`[Y-pq]**
  +p(1-p)V`[Y-l+q+pq] = 0
- **U`[Y-pq] = V`[Y-l+q-pq]**

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- Just like in previous case, we equalize marginal utility across the good and bad states of the world
- Recall that
  - U`(y) > V`(y)
  - U`($y_1$) = V`(y_2) if y_1>y_2
- Since U`[Y-pq] = V`[Y-l+q-pq]
- In order to equalize marginal utilities of income, must be the case that

  [Y-pq] > [Y-l+q+pq]

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- Income in healthy state > income in unhealthy state
- Do not fully insure losses. Why?
  - With insurance, you take $ from the good state of the world (where MU of income is high) and transfer $ to the bad state of the world (where MU is low)
  - Do not want good money to chance bad

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