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Lake Geometry: Implications for Production and Sediment Accretion Rates

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A geometric analysis distinguishes between the size and shape components of lake morphometry. Implications of shape for nutrient recycling are investigated using models that relate two ratios: (1) sediment surface area in contact with the epilimnion to epilmnetic volume, and (2) mean depth to maximum depth (the depth ratio). The epilimnion's sediment surface area to volume ratio declines with depth ratio, in lakes with thermoclines shallower than about 1·2 times the mean depth. As depth ratio decreases, therefore, potential nutrient recycling from the sediment surface, productivity, and sediment accretion rates are predicted to increase. This prediction is borne out by a significant netative correlation between productivity and depth ratio. The theoretical relationship of depth ratio to productivity and sediment accretion rate helps explain some limnological differences among lake districts. Glacially-formed lakes often have low depth ratios, and should therefore be more productive and fill with sediment more rapidly than otherwise similar lakes with high depth ratios, which often originate from volcanic or tectonic events.

1. Introduction

Limnologists have historically recognized three kinds of factors that influence lake typology, productivity, and succession: climatic factors, which are characteristic of a lake's biographic region; edaphic factors, which depend on the geochemistry of the watershed; and morphometric factors, which are determined by the shape of the lake's basin. The relative importance of the three kinds of factors is debatable, and probably quite variable among lakes. However, Rawson (1939, p. 12), Deevey (1940, p. 738), and Mortimer (1942, p. 191) are nearly unanimous in their acknowledgement of the importance of all three classes of factors. More recently, limnologists have tended to emphasize control of productivity by nutrient loading (an edaphic factor), because models that predict the consequences of anthropogenic nutrient inputs have been notably successful (Moss, 1980). While nutrient loading is significantly correlated with lacustrine primary

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production, a substantial amount of variance in the data remains unexplained (Schindler, 1978), suggesting that additional factors are involved.

Correlations between morphometric and biotic characteristics of lakes were first recognized nearly a century ago, according to Patalas' (1980) historical account. Some comparative limnologists have attempted theoretical or empirical corrections for morphometric effects on productivity (Hayes, 1957; Kerekes, 1975, 1977). Fisheries biologists have related fish yield to lake mean depth (Ryder, 1982). Recently, Fee (1979) has found that the ratio of sediment surface area in the epilimnion to epilimnetic volume is strongly correlated with lake productivity. Morphometry influences productivity by affecting the rate of nutrient recycling from sediments to the water by processes such as diffusion, resuspension, bioturbation, and decay of macrophytes (Wetzel, 1979; Kitchell *et al.*, 1979; Carpenter, 1980; de Groot, 1981).

Evidence varies on the relative importance of edaphic and morphometric factors in determining the trophic status of lakes. For example, Moss (1980, p. 222), Pennington (1981), and Spence (1982) emphasize edaphic factors, while Manny, Wetzel & Bailey (1978). Whitehead & Crisman (1978), Wetzel (1979), and Carpenter (1981) point out the importance of morphometry. The apparently variable role of morphometry demands closer analysis, but is difficult to study rigorously because the shapes of lakes are so complex.

Most lake productivity models view the lake as a homogeneous mixed reactor and therefore cannot directly consider morphometric effects. In contrast, this paper takes an explicitly three-dimensional approach, using geometric models of lakes. While the analytic geometry of these models is well known, the models have not previously been applied to questions of nutrient recycling from sediment or primary production. My analyses relate a dimensionless shape parameter to primary production, and suggest a heuristically useful morphometric classification of lake basins.

2. Lake Basin Shape Models

Two distinct classes of models have been proposed for the shapes of lakes: quadric surfaces and sinusoids (Neumann, 1959; Junge, 1966). The ratio of average depth ($\bar{z}$) to maximum depth (z_m), henceforth referred to as the depth ratio, takes on a particular range for each class of shapes. Distributions of area and volume with depth are fully determined by the depth ratio for both classes of shapes. All models approximate the morphometry of lakes with single central depressions and elliptical surfaces. Profiles of model basins appear in Fig. 1.

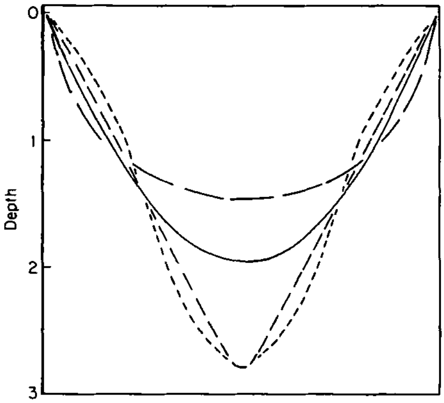

FIG. 1. Cross sections through the deepest points of models of lakes with identical surface areas, mean depths, and volumes. Vertical scale is in multiples of mean depth. ——, paraboloid (depth ratio = 0·5); — — —, ellipsoid (depth ratio = 0·66), - - -, hyperboloid (depth ratio = 0·35); and -----, sinusoid (depth ratio = 0·35).

Quadric surfaces include the hyperboloid (depth ratio between 1/3 and 1/2) and ellipsoid (depth ratio between 1/2 and 2/3). The paraboloid is an intermediate shape with depth ratio exactly 1/2. The proportion of the lake's surface area that encompasses water depths $\leq z$ is

$$A(u) = pu^2 + (1-p)u \tag{1}$$

where u is relative depth, z/z_m. The value of p is determined by the depth ratio:

$$p = 6(\bar{z}/z_m) - 3 \tag{2}$$

The proportion of the lake's volume that lies above depth z is

$$V(u) = [6u - 3(1-p)u^2 - 2pu^3]/(3+p) \tag{3}$$

Junge (1966) gives a full mathematical treatment of the quadric surfaces.

Sinusoids have depth ratios between 0·297 and 0·5, overlapping the range of depth ratios for hyperboloids. Area and volume distributions of the paraboloid are identical to those of the sinusoid with depth ratio 1/2 (Junge, 1966). However, sinusoids and quadric surfaces are distinctly different for depth ratios less than 1/2. The sinusoid's area distribution is given by Junge (1966):

$$A(u) = 1 - \left\{\frac{\cos^{-1}[1 - r(1-u)]}{\cos^{-1}(1-r)}\right\}^2. \tag{4}$$

The volume distribution is

$$V(u)=g[1-r(1-u)]/g(1-r) \tag{5}$$

where the function g is

$$g(x)=x(\cos^{-1}x)^2+2(1-x)-2(1-x^2)^{1/2}(\cos^{-1}x) \tag{6}$$

and x is a dummy variable (Lehman, 1975). The value of r is determined by the depth ratio (Junge, 1966):

$$\frac{\bar{z}}{z_m}=\frac{2}{r}\left\{\frac{(2r-r^2)^{1/2}}{\cos^{-1}(1-r)}-\frac{r}{[\cos^{-1}(1-r)]^2}-\frac{1-r}{r}\right\}. \tag{7}$$

For any particular depth ratio, equation (7) is solved numerically for r. A conversion table is provided here for the convenience of future users of

TABLE 1

Values of the sinusoid parameter r *determined for selected depth ratios by the regula falsi algorithm (Stark,* 1970) *to a tolerance of* 10^{-6}

Depth ratio	r	Depth ratio	r
0·30	1·99990	0·41	1·67370
0·31	1·99763	0·42	158803
0·32	1·99201	0·43	1·48587
0·33	1·98254	0·44	1·36447
0·34	1·96862	0·45	1·22061
0·35	1·94957	0·46	1·05044
0·36	1·92462	0·47	0·84940
0·37	1·89285	0·48	0·61194
0·40	1·74519	0·49	0·33147

this shape model (Table 1). Approximate values of r accurate to two digits may be obtained for $0{\cdot}30 \leq \bar{z}/z_m \leq 0{\cdot}49$ from the polynomial

$$r=-15{\cdot}98+202{\cdot}5\left(\frac{\bar{z}}{z_m}\right)-867{\cdot}9\left(\frac{\bar{z}}{z_m}\right)^2+1689{\cdot}6\left(\frac{\bar{z}}{z_m}\right)^3-1272{\cdot}0\left(\frac{\bar{z}}{z_m}\right)^4. \tag{8}$$

About 87% of natural lakes in a literature sample have depth ratios within the range accounted for by the models (Table 2). Exceptions fall into two categories. Lakes with very low depth ratios ($\bar{z}/z_m<0{\cdot}297$) contain deep but narrow pits. Such pits have little influence on $\bar{z}$, but greatly increase z_m. Lakes with very high depth ratios ($\bar{z}/z_m>0{\cdot}667$) have relatively

TABLE 2

Numbers of lakes within the ranges of $\bar{z}:z_m$ *for various geometric shapes, from selected literature sources. The total number of lakes in each sample may be less than the row sum because the ranges of* $\bar{z}:z_m$ *for the sinusoid and hyperboloid overlap. Ranges of* $\bar{z}:z_m$ *for the shapes are: sinusoid,* 0·297–0·499; *hyperboloid,* 0·333–0·499; *paraboloid,* 0·500; *and ellipsoid,* 0·501–0·667

	$\bar{z}:z_m$							
Region	<0·297	0·297–0·499	0·333–0·499	0·500	0·501–0·667	>0·677	Total	Source
Worldwide	10	59	55	2	32	4	107	Neumann (1959)
Washington, U.S.A.	1	19	17	1	21	4	46	Lehman (1975)
E.L.A., Canada	0	16	15	1	2	0	19	Fee (1979)
Saskatchewan, Canada	5	6	4	0	2	0	13	Rawson (1960)
English Lake District	1	10	10	0	5	0	16	Gorham *et al.* (1974)
Wisconsin	4	33	32	0	3	2	42	Juday (1914)

steep sides and flat bottoms. Such lakes can sometimes be approximated by steep-sided frustrums (Lehman, 1975).

A more rigorous assessment of the shape models would compare predicted and observed distributions of area or volume. Junge (1966), using data from Balkan ponds and backwaters, found satisfactory agreement between observed volume distributions and volume distributions of quadric surfaces. Lehman (1975) examined 41 lakes with depth ratios in the range accounted for by the models (Table 2), and found satisfactory fits of observed and calculated volume distributions in 29 cases. In sum, the shape models adequately approximate the shapes of the basins of a substantial fraction of the world's lakes.

3. Morphometric Comparison of Shape Models

Here the morphometries of model lake basins identical in size but different in shape are compared. The hypothetical lakes have unit surface area and total volume, and therefore have unit average depth since by definition

$$\bar{z} = V(z_m)/A(z_m) \tag{9}$$

Since the shapes differ in depth ratio, the modeled basins differ in z_m.

Morphometric curves of all possible basin models are bounded by those for shapes with extreme depth ratios (Fig. 2). Curves for quadric surfaces are intermediate to those of the extreme ellipsoid ($\bar{z}/z_m = 2/3$) and the extreme hyperboloid ($\bar{z}/z_m = 1/3$). Curves for sinusoids are bounded by those of the extreme sinusoid ($\bar{z}/z_m = 0{\cdot}297$) and the paraboloid ($\bar{z}/z_m = 1/2$).

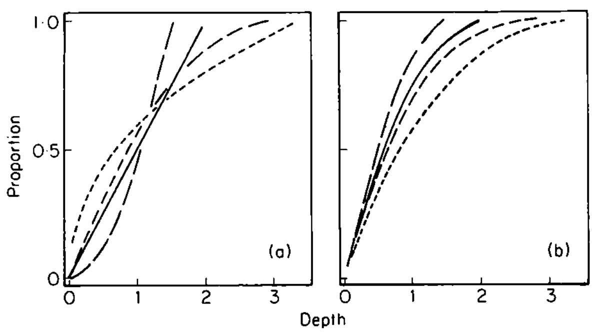

FIG. 2 Proportions of surface area of the sediment-water interface (a) and volume (b) above the depth contour versus depth (in multiples of $\bar{z}$). ——, paraboloid; — — —, extreme ellipsoid; - - -, extreme hyperboloid; and ----- denote extreme sinusoid.

The volumes of extreme shapes follow a consistent ranking at all common depths: extreme ellipsoid > paraboloid > extreme hyperboloid > extreme sinusoid (Fig. 2). The relationship of the area distributions is more complex. For depths above 1·2 $\bar{z}$, the area ranking is the reverse of the volume ranking, i.e. extreme sinusoid > extreme hyperboloid > paraboloid > extreme ellipsoid.

The effects of nutrients recycled from sediments depend on the extent to which they are diluted in the epilimnion. Therefore, the ratio of area of the sediment in contact with the epilimnion to volume of the epilimnion is a crucial factor in lake metabolism (Fee, 1979). In the present notation, this ratio is

$$S = A(z_T)/V(z_T) \tag{10}$$

where z_T is the depth to the thermocline and the functions A and V are specified for either quadric surfaces or sinusoids. Curves of s versus depth to the thermocline show that the extreme shapes follow a consistnent ranking for thermoclines above 1·2 $\bar{z}$ (Fig. 3): extreme sinusoid > extreme

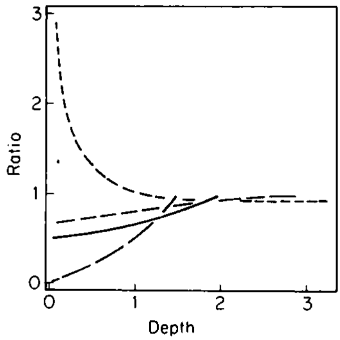

FIG. 3. Ratio of area of the sediment-water interface to volume of the epilimnion (S, in units of $\bar{z}$) as a function of depth to the thermocline (in units of $\bar{z}$). Key to shapes as in Fig. 2.

hyperboloid > paraboloid > extreme ellipsoid. The ranking of the first three shapes, which have depth ratios less than 1/2, holds for thermoclines as deep as 1·75 $\bar{z}$. These rankings suggest that epilimnetic area: volume ratio declines with increasing depth ratio. Such a relationship is confirmed by plots of s versus $\bar{z}/z_m$ for quadric surfaces and sinusoids (Fig. 4).

Sinusoids and hyperboloids of identical depth ratio, average depth, maximum depth, volume, and surface area can have quite different curves

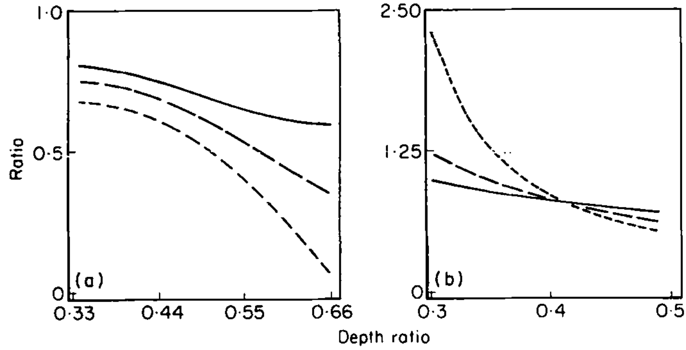

FIG. 4. Ratio of the area of the sediment-water interface to volume of the epilimnion (S, in units of $1/\bar{z}$) as a function of depth ratio for quadric surfaces (a) and sinusoids (b). In each graph, curves are shown for three depths to the thermocline (in units of $\bar{z}$): 1·1 (——), 0·7 (— — —), and 0·3 (-----).

of S versus depth to the thermocline (Fig. 5). the sinusoid and hyperboloid become more similar as depth ratio increases because both shapes approach the paraboloid as the depth ratio approaches 1/2.

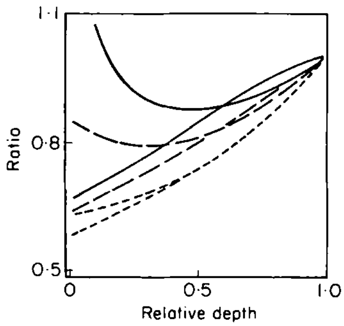

FIG. 5. Ratio of the area of the sediment-water interface to volume of the epilimnion (S, in units of $1/\bar{z}$) as a function of relative depth u for hyperboloids and sinusoids with depth ratios of 0·35 (——), 0.40 (— — —), and 0·45 (-----). In each pair of curves, the sinusoid lies above the hyperboloid at the smallest relative depths.

4. Implications of Shape for Primary Production

The ratio of sediment-water interface area to water volume in the epilimnion (S) declines as depth ratio increases. Therefore, nutrient recycling from sediments, and hence productivity, should decline as the depth ratio increases. This prediction is restricted to stratified lakes with thermoclines shallower than 1·2 $\bar{z}$, or lakes with thermoclines shallower than 1·75 $\bar{z}$ if the depth ratios are less than 1/2. A statistical test of the prediction requires data for a group of lakes similar in size, belonging to one family of shapes (i.e. quadric surfaces or sinusoids), and subject to similar nutrient inputs and climatic influences.

Fee's (1979) data are suitable for testing effects of shape on productivity because he used a single, well-validated technique on a group of lakes that varied in morphometry but were otherwise similar: they occurred in the same geologic formation, experienced the same climatic regime, and had similar watershed vegetation and nutrient loadings. The stratified unmanipulated lakes were used in the following analysis. These lakes range in depth ratio from 0·33 to 0·50. All but one have thermoclines shallower than their mean depths; the exception (Lake 221) has the thermocline at 1.5 $\bar{z}$. Lake 228 was omitted from the analysis because it is a highly influential outlying data point, beng much larger (5 to 34 times deeper and 30 to 840 times greater in area) than the other lakes.

The range of depth ratios indicates that the sample could be a mixture of hyperboloids and sinusoids. This possibility was examined by comparing observed curves of $A(z)/V(z)$ versus z with expected curves for hyperboloids and sinusoids having the same depth ratios as the lakes (Table 3). Of the seven lakes for which morphometric data could be found, five fit

TABLE 3

Square root of the mean squared deviation between paired curves of A(z)/V(z) *versus depth. For each lake, observed curves are compared with both the hyperboloid and sinusoid of the same depth ratio, and the hyperboloid and sinusoid are compared. Data from* Brunskill & Schindler (1971; Lakes 114, 239, 240, and 305); Schindler, Ruszczynski & Fee (1980; Lake 302S); Schindler *et al.* (1980; Lake 223); and Hesslein, Broecker & Schindler (1980; Lake 224)

		Square root of mean square deviation		
Lake	Depth ratio	Hyperboloid versus observed	Sinusoid versus observed	Hyperboloid versus sinusoid
114	0·340	0·211	0·325	0·116
223	0·500	0·050	0·050	0
224	0·423	0·018	0·034	0·033
239	0·345	0·062	0·154	0·169
240	0·466	0·060	0·063	0·007
302S	0·481	0·111	0·115	0·006
305	0·462	0·037	0·032	0·007

the hyperboloid more closely than the sinusoid. Lake 223 fit both models equally well, because hyperboloids and sinusoids with depth ratios of 0·5 are identical. Lake 305 fits the sinusoid slightly more closely than the hyperboloid. However, the depth ratio of this lake is high and the discrepancy between the two models is therefore small. Of the remaining three lakes, one (Lake 383) has a high depth ratio of 0·48, and therefore must deviate nearly the same amount from each model. The remaining two lakes (120 and 221) both have depth ratios of 0·4. In sum, morphometric deviations from the quadric surfaces seem unlikely to affect the following analysis. Any such deviation adds to the noise in the data and makes detection of the predicted effect less likely; in this sense, the following result is conservative.

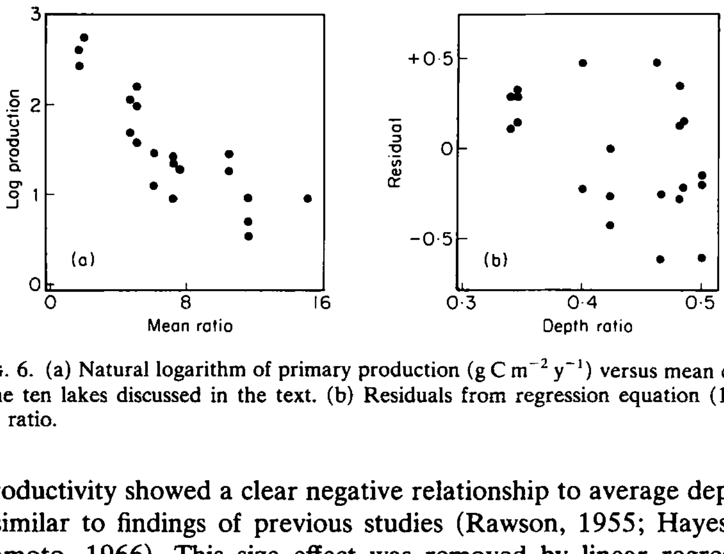

FIG. 6. (a) Natural logarithm of primary production (g C m^{-2} y^{-1}) versus mean depth (m) for the ten lakes discussed in the text. (b) Residuals from regression equation (11) versus depth ratio.

Productivity showed a clear negative relationship to average depth (Fig. 6), similar to findings of previous studies (Rawson, 1955; Hayes, 1957; Sakamoto, 1966). This size effect was removed by linear regression of production (P) on $\bar{z}$. the regression equation

$$\ln P = -0{\cdot}137\,\bar{z} + 2{\cdot}55 \qquad (11)$$

has a significantly negative slope ($F_{1,21} = 48{\cdot}11$, $p < 0{\cdot}0001$) and explains 69·6% of the variance. Consistent with the prediction based on the shape models, the residuals from this regression line correlate negatively with the depth ratio ($r = -0{\cdot}676$, $p = 0{\cdot}0279$).

Production shows the expected relationship to the morphometric variables. A higher percentage of the variance can be explained by explicitly incorporating thermocline depth in the morphometric predictors (Fee, 1979). Although I am aware of no data set more appropriate than this one, these data may underestimate the importance of morphometry for two reasons. First, the range of depth ratios is relatively small and, if extended, could strengthen the apparent relationship between depth ratio and productivity. Second, these lakes are relatively unproductive. Some highly productive lakes exhibit considerable nutient recycling from sediments or littoral zones (Cooke *et al.*, 1977; Prentki *et al.*, 1979; Larsen, Schults & Maleug, 1981; Jacoby *et al.*, 1982).

5. Implications for Sediment Accretion Rates

Sediment accretion rates are fundamental data in studies of lake aging. Lehman (1975) has analyzed effects of basin shape on spatial distribution of sedimenting materials. The following analysis suggests that lake shape directly affects the amount of material added to the sediment annually.

Sediment accretion rates are usually obtained from increments of compacted sediment at a lake's center, and thus correspond to estimates of $-\mathrm{d}z_m/\mathrm{d}t$. A simple equation relating $-\mathrm{d}z_m/\mathrm{d}t$ to productivity and external inputs is

$$-\frac{dz_m}{\mathrm{d}t} = k_1\left\{\left[\frac{k_2}{V(z_T)}\right][k_3A(z_T)+L]+W\right\}. \qquad (12)$$

Nutrient recycling from epilimnetic sediments is determined by a constant k_3 times sediment surface area $A(z_T)$. These recycled nutrients, combined with external nutrient loading L, are diluted in epilimnetic water volume $V(z_T)$ and converted to primary production by constant k_2. Primary production plus inputs of sedimenting material from the watershed (W) are converted to increments of compacted sediment by the constant k_1. Reaarangement of equation (12) and substitution using equation (10) yields

$$-\frac{\mathrm{d}z_m}{k_1\mathrm{d}t} = k_2k_3S + \frac{k_2L}{V(z_T)} + W. \qquad (13)$$

Clearly the first term on the right of equation (13) increases as the depth ratio decreases (Fig. 4). The second term also increases as depth ratio decreases, because the volume above any given depth declines as the depth ratio declines (Fig. 2). Therefore, sediment accretion rate increases as depth ratio decreases.

Relatively high sediment accretion rates due to nutrient recycling from sediments are likely to be amplified as the lake fills, epilimnetic sediment surface area increases, and epilimnetic volume decreases (Carpenter, 1981).

Therefore, lakes with low depth ratios are most likely to show increases in productivity and sedimentation rates through time. This morphometric factor may explain some apparently dispartate paleolimnological observations that are often attributed to edaphic factors (cf. Moss, 1980). For example, Hutchinson & Cowgill (1970, p. 170) conclude that the productivity of Lago di Monterosi has not increased consistently since the basin was formed. In contrast, Whitehead & Crisman (1978, p. 476) discovered relatively steady increases in postglacial productivity of Berry Pond (superimposed on oscillations which were attributed to edaphic factors). Lago di Monterosi, a caldera lake, is an ellipsoid ($\bar{z}/z_m = 0{\cdot}66$; see morphometric data in Hutchinson, 1970). Berry Pond, a kettle lake, has a depth ratio of only 0·46. The contrasting trends in postglacial productivity found in these studies are consistent with the differences in the shapes of the basins.

6. Conclusions

Lake morphometry has two components; size, reflected in the mean depth $\bar{z}$, and shape, reflected in the dimensionless depth ratio $\bar{z}/z_m$. Shape models indicate that the depth ratio is negatively related to potential nutrient recycling from sediments in quadric surfaces with thermoclines shallower than $1{\cdot}2\ \bar{z}$, and in sinusoids with thermoclines shallower than $1{\cdot}75\ \bar{z}$. This inference is supported by published data which show a negative correlation between $\bar{z}/z_m$ and productivity corrected for effects of mean depth. These results suggest that lakes with low depth ratios fill with sediment more rapidly than lakes with high depth ratios.

The depth ratio provides a heuristically useful classification of lake shapes. Lakes with depth ratios below 1/2 are often kettle lakes, solution lakes, or other lake types with nearly conical basins (Hutchinson, 1957). Such lakes can often be approximated by sinusoids or hyperboloids. These lakes are expected to be more productive and accrete sediment more rapidly than otherwise similar lakes with higher depth ratios. Lake with depth ratios above 1/2 are often caldera, fjord, or graben lakes, which have steep sides and relatively flat bottoms (Hutchinson, 1957). Such lakes can be often approximated by ellipsoids or, in extreme cases, steep-sided frustrums. These lakes are expected to be less productive and accrete sediment relatively slowly, in comparison to otherwise similar lakes with lower depth ratios. Since depth ratios are similar among lakes of similar geologic origin, morphometric differences may contribute to metabolic and paleolimnological differences among the world's lake districts.

Quadric surfaces and sinusoids adequately approximate the shapes of a large fraction of the world's lakes. Such models facilitate study of lake morphometry by substituting mathematically tractable functions for complex lake shapes. However, many lakes do not adequately fit these models. Further work is needed to categorize and analyze shapes of such lakes.

These conclusions could not have emerged from a model that viewed lakes as homogeneous systems. Study of morphometric factors requires models that incorporate spatial heterogeneity. This paper underscores the importance of bathymetric data in comparative limnology. At a minimum, lake area, volume, and maximum depth are needed to study relationships between morphometry and metabolism. More detailed data, such as volume-depth and area-depth curves, are needed to compare actual lakes to idealized shape models. Further theoretical and empirical analyses may reveal properties of lake size and shape that relate more closely to lake metabolic rates than the mean depth and depth ratio used here.

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