

SOLUTIONS TO PRACTICE EXAM 2, MATH 10560

1. Which of the following expressions gives the partial fraction decomposition of the function

$$f(x) = \frac{x^2 - 2x + 6}{x^3(x-3)(x^2+4)}?$$

Solution: Since x is a linear factor of multiplicity 3, $(x-3)$ is a linear factor of multiplicity 1 and (x^2+4) is an irreducible quadratic factor of multiplicity 1, then

$$\frac{x^2 - 2x + 6}{x^3(x-3)(x^2+4)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{x-3} + \frac{Ex+F}{x^2+4}.$$

2. Use the trapezoidal rule with step size $\Delta x = 2$ to approximate the integral $\int_0^4 f(x)dx$.

Solution: Note

$$n = \frac{4-0}{2} = 2.$$

Then by the trapezoidal rule

$$\int_0^4 f(x)dx \approx \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + f(x_2)) = \frac{2}{2}(2 + 8 + 0) = 10.$$

3. Evaluate the following improper integral:

$$\int_e^\infty \frac{1}{x(\ln x)^2} dx.$$

Solution: Use the definition of improper integral and make the substitution $u = \ln x$ with $dx = xdu$. Then

$$\begin{aligned} \int_e^\infty \frac{1}{x(\ln x)^2} dx &= \lim_{t \rightarrow \infty} \int_e^t \frac{1}{x(\ln x)^2} dx = \lim_{t \rightarrow \infty} \int_1^{\ln t} \frac{1}{u^2} du \\ &= \lim_{t \rightarrow \infty} \left[-\frac{1}{u} \right]_1^{\ln t} = \lim_{t \rightarrow \infty} \left(-\frac{1}{\ln t} + 1 \right) = 1. \end{aligned}$$

4. Find $\int_{-2}^2 \frac{1}{x+1} dx$.

Solution: Function $\frac{1}{x+1}$ has an infinite discontinuity at the point $x = -1$. Therefore

$$\int_{-2}^2 \frac{1}{x+1} dx = \int_{-2}^{-1} \frac{1}{x+1} dx + \int_{-1}^2 \frac{1}{x+1} dx,$$

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where each of the integrals is improper. Compute the first integral as follows

$$\int_{-2}^{-1} \frac{1}{x+1} dx = \lim_{t \rightarrow -1} \int_{-2}^t \frac{1}{x+1} dx = \lim_{t \rightarrow -1} [\ln|x+1|]_{-2}^t = \lim_{t \rightarrow -1} \ln|t+1| - \ln 1 = -\infty.$$

Since $\int_{-2}^{-1} \frac{1}{x+1} dx$ diverges, then the initial integral diverges as well.

5. Which of the following is an expression for the area of the surface formed by rotating the curve $y = 5^x$ between $x = 0$ and $x = 2$ about the y -axis?

Solution: Distance from the axis of the revolution (y -axis) and the graph of the function $y = 5^x$ is x . Therefore

$$S = \int_a^b 2\pi x \sqrt{1 + (y')^2} dx = \int_0^2 2\pi x \sqrt{1 + (\ln 5)^2 \cdot 25^x} dx.$$

6. Find the centroid of the region bounded by $y = 1/x$, $y = 0$, $x = 1$, and $x = 2$.

Solution: Use

$$\bar{x} = \frac{1}{A} \int_a^b x f(x) dx,$$

$$\bar{y} = \frac{1}{2A} \int_a^b f^2(x) dx,$$

where A is the area of the given region. Therefore

$$\bar{x} = \frac{1}{\ln 2} \int_1^2 x \frac{1}{x} dx = \frac{1}{\ln 2} \int_1^2 1 dx = \frac{1}{\ln 2} x \Big|_1^2 = \frac{1}{\ln 2} (2 - 1) = \frac{1}{\ln 2},$$

$$\bar{y} = \frac{1}{2 \ln 2} \int_1^2 \frac{1}{x^2} dx = \frac{1}{2 \ln 2} \left[-\frac{1}{x} \right]_1^2 = \frac{1}{2 \ln 2} \left(-\frac{1}{2} + 1 \right) = \frac{1}{4 \ln 2}.$$

7. Use Euler's method with step size 0.5 to estimate $y(1)$ where $y(x)$ is the solution to the initial value problem

$$y' = y + 2xy, \quad y(0) = 1.$$

Solution: Note $\Delta x = 0.5$, $a = 0$, $b = 1$, $n = \frac{1-0}{0.5} = 2$. Therefore using $F(x, y) = y + 2xy$ for Euler's method

$$y(0) = y_0 = 1,$$

$$y(0.5) \approx y_1 = y_0 + F(x_0, y_0) \Delta x = y_0 + (y_0 + 2x_0 y_0) \Delta x = 1 + (1 + 0) 0.5 = 1.5,$$

$$y(1) \approx y_2 = y_1 + F(x_1, y_1) \Delta x = y_1 + (y_1 + 2x_1 y_1) \Delta x = 1.5 + (1.5 + 1.5) 0.5 = 3.$$

8. Find the solution to the initial value problem

$$y' = \frac{\sin x}{2y + 1}, \quad y(0) = 2.$$

Solution: Separate variables and then integrate

$$(2y + 1)y' = \sin x,$$

or

$$\int (2y + 1)dy = \int \sin x dx.$$

We get

$$y^2 + y = -\cos x + C.$$

Now use the initial value to find C as follows

$$2^2 + 2 = -1 + C.$$

Hence $C = 7$, and

$$y^2 + y = 7 - \cos x.$$

9. Find the integral

$$\int \frac{3x + 1}{x^3 + x^2} dx.$$

Solution: Use partial fraction decomposition

$$\frac{3x + 1}{x^3 + x^2} = \frac{3x + 1}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1} = \frac{Ax(x + 1) + B(x + 1) + Cx^2}{x^2(x + 1)}.$$

Therefore

$$3x + 1 = (A + C)x^2 + (A + B)x + B.$$

It follows that

$$\begin{aligned} A + C &= 0, & A + B &= 3, & B &= 1, \\ A &= 2, & B &= 1, & C &= -2, \end{aligned}$$

and

$$\int \frac{3x + 1}{x^3 + x^2} dx = \int \left(\frac{2}{x} + \frac{1}{x^2} - \frac{2}{x + 1} \right) dx = 2 \ln |x| - \frac{1}{x} - 2 \ln |x + 1| + C.$$

10. Calculate the integral

$$\int \frac{dx}{x + \sqrt[3]{x}}.$$

Solution: Make substitution $u = x^{1/3}$. Then $u^3 = x$ and with $dx = 3u^2 du$

$$\int \frac{dx}{x + \sqrt[3]{x}} = \int \frac{3u^2 du}{u(u^2 + 1)} = \int \frac{3udu}{u^2 + 1} = \frac{3}{2} \ln(u^2 + 1) = \frac{3}{2} \ln(x^{2/3} + 1) + C.$$

11. Calculate the arc length of the curve if $y = \frac{x^2}{4} - \ln(\sqrt{x})$, where $2 \leq x \leq 4$.

Solution: Recall

$$L = \int_a^b \sqrt{1 + (y')^2} dx.$$

Note

$$y' = \frac{x}{2} - \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2} \left(x - \frac{1}{x} \right).$$

Thus

$$\begin{aligned} 1 + (y')^2 &= 1 + \frac{1}{4}\left(x - \frac{1}{x}\right)^2 = 1 + \frac{1}{4}\left(x^2 - 2x\frac{1}{x} + \frac{1}{x^2}\right) = 1 + \frac{1}{4}\left(x^2 - 2 + \frac{1}{x^2}\right) \\ &= 1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}\left(x^2 + 2x\frac{1}{x} + \frac{1}{x^2}\right) = \frac{1}{4}\left(x + \frac{1}{x}\right)^2. \end{aligned}$$

Therefore

$$L = \int_2^4 \sqrt{1/4(x + 1/x)^2} dx = \int_2^4 \frac{1}{2}\left(x + \frac{1}{x}\right) dx = \frac{1}{2} \left[\frac{x^2}{2} + \ln x \right]_2^4 = 3 + \frac{1}{2} \ln 2.$$

12. *Solve the initial value problem*

$$\begin{aligned} xy' + xy + y &= e^{-x} \\ y(1) &= \frac{2}{e}. \end{aligned}$$

Solution: This is a linear differential equation. Since it can be reduced to the form

$$y' + \left(1 + \frac{1}{x}\right)y = \frac{e^{-x}}{x},$$

an integrating factor is

$$I(x) = e^{\int (1 + \frac{1}{x}) dx} = e^{x + \ln x} = xe^x.$$

Multiply both sides of the differential equation by $I(x)$ to get

$$xe^x y' + y(x+1)e^x = 1,$$

and hence

$$(xe^x y)' = 1.$$

Integrate both sides to obtain

$$xe^x y = x + C,$$

or

$$y = e^{-x} \left(1 + \frac{C}{x}\right).$$

Using the initial value, we have

$$y(1) = \frac{2}{e} = \frac{1}{e} (1 + C), \quad C = 1.$$

Hence

$$y = e^{-x} \left(1 + \frac{1}{x}\right).$$