

SOLUTIONS TO PRACTICE EXAM 1, MATH 10560

1. Simplify the following expression for x

$$x = \log_3 81 + \log_3 \frac{1}{9}.$$

Solution:

$$x = \log_3 81 + \log_3 \frac{1}{9} = \log_3 \frac{81}{9} = \log_3 9 = 2.$$

2. The function $f(x) = x^3 + 3x + e^{2x}$ is one-to-one. Compute $(f^{-1})'(1)$.

Solution:

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

$f^{-1}(1) = 0$ and $f'(x) = 3x^2 + 3 + 2e^{2x}$. Hence $f'(f^{-1}(1)) = f'(0) = 5$. Therefore $(f^{-1})'(1) = \frac{1}{5}$.

3. Differentiate the function

$$f(x) = \frac{(x^2 - 1)^4}{\sqrt{x^2 + 1}}.$$

Solution: Use logarithmic differentiation.

$$\ln f = 4 \ln(x^2 - 1) - \frac{1}{2} \ln(x^2 + 1)$$

$$\frac{f'}{f} = \frac{8x}{x^2 - 1} - \frac{x}{x^2 + 1}$$

$$f'(x) = \frac{x(x^2 - 1)^4}{\sqrt{x^2 + 1}} \left(\frac{8}{x^2 - 1} - \frac{1}{x^2 + 1} \right).$$

4. Compute the integral

$$\int_{2e}^{2e^2} \frac{1}{x(\ln \frac{x}{2})^2} dx.$$

Solution: Make the substitution $u = \ln \frac{x}{2}$ with $dx = x du$.

$$\int_{2e}^{2e^2} \frac{1}{x(\ln \frac{x}{2})^2} dx = \int_1^2 \frac{1}{u^2} du = \left[-\frac{1}{u} \right]_1^2 = \frac{1}{2}.$$

5. Compute the limit

$$\lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^{2x} - e^{-2x}}.$$

Solution:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^{2x} - e^{-2x}} &= \lim_{x \rightarrow \infty} \frac{e^x(1 - e^{-2x})}{e^{2x}(1 - e^{-4x})} \\ &= \lim_{x \rightarrow \infty} \frac{1 - e^{-2x}}{e^x(1 - e^{-4x})} = 0.\end{aligned}$$

6. Find $f'(x)$ if

$$f(x) = x^{\ln x}.$$

Solution: One method is to use logarithmic differentiation.

$$\ln y = \ln(x^{\ln x}) = (\ln x)(\ln x) = (\ln x)^2.$$

$$\frac{y'}{y} = \frac{2 \ln x}{x}.$$

Therefore $f'(x) = y' = 2(\ln x)x^{(\ln x)-1}$.

7. Calculate the following integral

$$\int_0^1 \frac{\arctan x}{1+x^2} dx.$$

Solution: Make the substitution $u = \arctan x$ with $dx = (1+x^2)du$.

$$\int_0^1 \frac{\arctan x}{1+x^2} dx = \int_0^{\frac{\pi}{4}} u du = \left[\frac{u^2}{2} \right]_0^{\frac{\pi}{4}} = \frac{\pi^2}{32}.$$

8. Evaluate the integral

$$\int_0^{\pi/2} \sin^3(x) \cos^5(x) dx.$$

Solution: Use the identity $1 - \cos^2(x) = \sin^2(x)$.

$$\begin{aligned}\int_0^{\pi/2} \sin^3(x) \cos^5(x) dx &= \int_0^{\pi/2} (1 - \cos^2(x)) \sin(x) \cos^5(x) dx \\ &= - \int_1^0 u^5 - u^7 du \quad (u = \cos(x), du = -\sin(x) du) \\ &= \int_0^1 u^5 - u^7 du \\ &= \left[\frac{u^6}{6} - \frac{u^8}{8} \right]_0^1 \\ &= \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.\end{aligned}$$

9. Evaluate the limit

$$\lim_{x \rightarrow 0} (\cosh(x))^{1/x^2}.$$

Solution: The limit has indeterminate form 1^∞ . Let $L = \lim_{x \rightarrow 0} (\cosh(x))^{1/x^2}$.

$$\begin{aligned} \ln L &= \lim_{x \rightarrow 0} \ln \left((\cosh(x))^{1/x^2} \right) \\ &= \lim_{x \rightarrow 0} \frac{\ln(\cosh(x))}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{\tanh(x)}{2x} \quad (\text{l'Hospital's rule}) \\ &= \lim_{x \rightarrow 0} \frac{\operatorname{sech}^2(x)}{2} \quad (\text{l'Hospital's rule}) \\ &= \frac{1}{2}. \end{aligned}$$

Therefore $L = e^{\frac{1}{2}}$.

10. Evaluate the integral

$$\int x^2 \cos(2x) dx.$$

Solution:

$$\begin{aligned} &\int x^2 \cos(2x) dx \\ &= \frac{1}{2} x^2 \sin(2x) - \int x \sin(2x) dx \quad (\text{integration by parts}) \\ &= \frac{1}{2} x^2 \sin(2x) - \left[-\frac{1}{2} x \cos(2x) - \int -\frac{1}{2} \cos(2x) dx \right] \quad (\text{integration by parts}) \\ &= \frac{1}{2} x^2 \sin(2x) + \frac{1}{2} x \cos(2x) - \frac{1}{4} \sin(2x) + C \end{aligned}$$

11. Evaluate

$$\int \frac{1}{3} x^3 \sqrt{9 - x^2} dx.$$

Solution: Two approaches work: trigonometric substitution with $x = 3 \sin \theta$ and u substitution with $u = 9 - x^2$. The method of trigonometric substitution is outlined here,

although the latter method may be somewhat easier.

$$\begin{aligned}
 \int \frac{1}{3}x^3\sqrt{9-x^2} dx &= \int 81 \sin^3 \theta \cos^2 \theta d\theta \quad (x = 3 \sin \theta, dx = 3 \cos \theta d\theta) \\
 &= \int 81(1 - \cos^2 \theta) \sin \theta \cos^2 \theta d\theta \quad (u = \cos \theta, du = -\sin \theta d\theta) \\
 &= \int 81(u^4 - u^2) du \\
 &= \frac{81 \cos^5 \theta}{5} - 27 \cos^3 \theta + C \quad (\cos \theta = \frac{1}{3}\sqrt{9-x^2}) \\
 &= \frac{(9-x^2)^{\frac{5}{2}}}{15} - (9-x^2)^{\frac{3}{2}} + C.
 \end{aligned}$$

12. Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Thus $C'(t) = kC(t)$, where k is a constant. The initial concentration of the drug is 4 mg/ml. After 5 hours, the concentration is 3 mg/ml.

(a) Give a formula for the concentration of the drug at time t .

(b) How much drug will there be in 10 hours?

(c) How long will it take for the concentration to drop to 0.5 mg/ml?

Solution: (a)

$$\begin{aligned}
 C(t) &= C(0)e^{kt} = 4e^{kt} \\
 C(5) &= 3 = 4e^{k5} \quad (\text{solve for } k) \\
 k &= \frac{1}{5} \ln \left(\frac{3}{4} \right) \quad (\text{substitute into } C(t)) \\
 C(t) &= 4 \left(\frac{3}{4} \right)^{\frac{1}{5}t}.
 \end{aligned}$$

(b)

$$C(10) = 4 \left(\frac{3}{4} \right)^2 = \frac{9}{4}.$$

(c)

$$\begin{aligned}
 C(t) &= 4 \left(\frac{3}{4} \right)^{\frac{1}{5}t} = \frac{1}{2} \quad (\text{solve for } t) \\
 t &= -5 \log_{3/4}(8) = \frac{-5 \ln 8}{\ln 3 - \ln 4}.
 \end{aligned}$$