

The Effects of Statistically Dependent Values
on Equilibrium Strategies of Bilateral k -Double Auctions

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Abstract: This paper describes how introducing statistical dependency among trader values changes the equilibrium bidding strategies in bilateral k -double auctions and uses the special case of affiliation to illustrate the range of equilibrium responses to a change in the value distribution. Consistent with standard intuition, a change from independent to strictly affiliated valuations can result in high-value buyers and low-value sellers bidding closer to their actual values while low-value buyers and high-value sellers distort their bids farther from their true values. However, there also exist equilibria for which either type of trader responds to a change in the distribution of values in the opposite direction. Whether a given trader bids more aggressively or less aggressively can also be non-monotonic with respect to the trader's value.

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1. Introduction.

Economists have long studied the role of statistically dependent buyer values in auction models with a primary emphasis on affiliated values.¹ In double auction models, the effect of dependent trader values has received less attention in large part because of the lack of results on the existence of positive-trade equilibria. With a sufficiently large number of buyers and sellers, existence and efficiency results have been developed for double auction models with correlated values by Reny and Perry (2006), Fudenberg, Mobius, and Szeidl (2007), and Cripps and Swinkels (2006). For a very general set of single and double auctions with dependent, private values, Jackson and Swinkels (2005) prove existence of equilibria in which trade occurs with positive probability as long as there exist at least two buyers or at least two sellers. A drawback of these papers is that they are not able to study the relationship between the distribution of trader valuations and equilibrium bidding strategies.

For the case of one buyer and one seller, Kadan (2007) characterizes the set of positive-trade, strictly-increasing equilibria of k -double auctions with $k \in [0,1]$ for a subset of affiliated buyer and seller valuation distributions.² He presents several examples to suggest how changes in the statistical dependency between the traders' valuations influences equilibrium bidding strategies. The purpose of this paper is to provide a general analysis of the differences in equilibrium bidding strategies induced by a change in the distribution of trader valuations. The impact of affiliation on equilibrium bidding strategies is included as a special case. It will turn out that Kadan's examples are not representative of the range of equilibrium responses to a change in the distribution of trader valuations even for the affiliation case.

For bilateral k -double auctions with independent, private values, Chatterjee and Samuelson (1983) (CS) and Satterthwaite and Williams (1989) (SW) derived a pair of differential equations which captures each trader's trade-off between changing her bid to improve her gains if trade occurs or to improve the probability she trades. They show that any equilibrium must satisfy these differential equations at all trader values for which the traders' strategies are locally C^1 and each trader's interim probability of trade is strictly positive. For a large set of independent value distributions, CS and SW

¹For a nice summary of this literature, see Krishna (Chapter 6, 2002). De Castro (2007) provides a serious critique of the assumption of affiliation in auction models.

²When $k \in \{0,1\}$, Kadan proves existence of a positive-trade, strictly-increasing equilibrium for a more general set of valuation distributions than he requires to prove existence when $k \in (0,1)$.

also prove that the differential equations are sufficient to characterize all strictly increasing equilibria over the set of values for which trade occurs with positive probability. If the traders' valuations are dependently distributed, a natural generalization of the CS/SW differential equations exists which again any equilibrium must satisfy at all values for which the traders' strategies are C^1 and for which each trader has a strictly positive interim probability of trade. For a subset of affiliated value distributions, Kadan proves that these generalized differential equations are also sufficient to characterize all strictly increasing equilibria over the set of values for which trade occurs with positive probability. The main insights of the current paper come from understanding how changes in the distribution of trader values affects the traders' marginal trade-offs in these generalized differential equations.

Let $F(c, v)$ denote a joint probability distribution over trader values where $c \in [0, 1]$ denotes the seller's value and $v \in [0, 1]$ denotes the buyer's value. Define C^* to be the set of vectors $(\hat{c}, \hat{b}, \hat{v})$ such that $0 < \hat{c} < \hat{b} < \hat{v} < 1$ where a seller with value \hat{c} bids \hat{b} and a buyer with value \hat{v} bids \hat{b} . If the traders' values are independently distributed, then for any $k \in (0, 1)$, SW show, that for each $(\hat{c}, \hat{b}, \hat{v})$, there exists a strictly increasing equilibrium defined by the solution to the CS/SW differential equations. Now perturb the value distribution so that the joint support of c and v does not change but the buyer and seller values are no longer independent. Kadan's system of differential equations will determine how the traders' bidding strategies must change near each initial condition vector if the new strategies remain C^1 . This set of initial condition vectors that retain local differentiability can be quite large. For the set of affiliated distributions Kadan considers, the set of initial condition vectors for which there exists a strictly increasing equilibrium is precisely C^* .

Consider what happens when the traders' values change from being independently distributed to being strictly affiliated.³ Kadan (2007, pp. 497 and 506) suggests that affiliation creates an incentive for the traders to adopt steeper strategies. The traders who are most likely to trade, high-value buyers and low-value sellers, bid less aggressively (they bid closer to their values), while the traders least likely to trade, low-value buyers and high-value sellers, bid more aggressively (they submit bids farther from their values). Such changes in the seller's equilibrium strategy near \hat{c} are a response by the seller to changes in the beliefs of the buyer with value \hat{v} that imply affiliation has *increased* the probability of the traders submitting tie bids *conditional* on the seller submitting a bid no greater than the buyer's bid. The buyer

³Since independently distributed valuations are also affiliated, I use the term "strict affiliation" to refer to non-independent, affiliated valuations. See Bikhchandani and Riley (1991) for a formal definition.

with value \hat{v} will still bid \hat{b} in the new equilibrium with affiliated values because, on the margin, the new seller strategy implies the \hat{v} buyer faces a *larger* conditional probability of no trade resulting from a bid below \hat{b} . Similarly, such changes in the buyer's equilibrium strategy near \hat{v} are a response by the seller with value \hat{c} that imply affiliation has *increased* the probability of ties bids *conditional* on the buyer submitting a bid at least as large as the seller's bid. Consistent with this intuition, all of Kadan's examples show that affiliation makes both buyer and seller strategies steeper. However, the examples fail to recognize that across the set of initial condition vectors neither conditional probability need increase. Even with a change in the value distribution from independence to the well-behaved form of statistical dependence implied by strict affiliation, the inframarginal probabilities that trade occurs, $F(\hat{c}|\hat{v})$ and $1 - F(\hat{v}|\hat{c})$, can increase or decrease; the weight a trader puts on her marginal gains from trade, $f(\hat{c}|\hat{v})$ and $f(\hat{v}|\hat{c})$, can increase or decrease; and most importantly, the inverse hazard rates, $F(\hat{c}|\hat{v})/f(\hat{c}|\hat{v})$ and $(1 - F(\hat{v}|\hat{c}))/f(\hat{v}|\hat{c})$, can increase or decrease.⁴ The forthcoming analysis will demonstrate that across C^* each of the four possible combinations of a steeper or a flatter buyer strategy and of a steeper or a flatter seller strategy can occur near some initial condition vector $(\hat{c}, \hat{b}, \hat{v})$.

The effect of changes in the value distribution is even more complex than would be suggested by a comparison of inverse hazard rates at each initial condition vector because it creates both local and global effects. The local effects arise if there exists a locally C^1 equilibrium through the same initial condition vector under the old and the new distribution. However, local changes in an equilibrium need not persist. For a set of initial condition vectors of positive measure, a change in the value distribution can induce global changes that are different from the local changes near the initial condition vector. There may be trader values at which a trader's strategy become steeper and other values at which the same trader's strategy becomes flatter. For example, one can observe a non-monotonic pattern in which, as a trader's valuation increases, her strategy switches from being more aggressive to less and then back to being more aggressive

Section 2 presents a standard bilateral k -double auction model with dependent, private trader values and presents Kadan's equilibrium characterization. Section 3 shows how to apply this characterization to identify the range of local differences in equilibria induced by a change in the valuation distribution. Section 4 investigates the global effects. The lack of persistence of some of the local differences is shown to be a general property. Section 5 offers concluding remarks.

⁴The inverse hazard rates are the inverses of the probabilities of tie bids conditional on the seller's bid not exceeding the buyer's bid..

2. The Model.

For comparison purposes, the notation in this paper is essentially the same as that used by Kadan (2007). A buyer and a seller will negotiate the trade of an indivisible object using a k -double auction. This auction requires each trader to simultaneously submit a bid to a third party. Let b denote the buyer's bid and let s denote the seller's bid. The object is traded if $b \geq s$ at the price $kb + (1-k)s$ for $k \in [0,1]$. If $b < s$, no trade occurs and no money changes hands.

Each trader knows her value for the object but does not know the other trader's value. Let $c \in I$ denote the seller's value and let $v \in I$ denote the buyer's value. Unless otherwise stated, assume without loss of generality that $I = [0,1]$. The traders have common ex ante beliefs concerning the distribution of values. $F(c,v)$ denotes the joint distribution and $f(c,v)$ denotes the joint density. Similar to k -double auctions with independent, private values, the inverse hazard rate expressions, $R(c|v) = F(c|v)/f(c|v)$ and $T(v|c) = (F(v|c) - 1)/f(v|c)$, are important in defining equilibria of dependent, private value k -double auctions.

The following assumptions are used throughout the paper.

A1. For all $(c,v) \in I \times I$, $f(c,v)$ is strictly positive and C^1 .

A2. For all $(c,v) \in I \times I$, $R_c(c|v) > 0$ and $T_v(v|c) > 0$.⁵

A3. Affiliation: For all $c_1, c_2, v_1, v_2 \in I$ such that $c_1 < c_2$ and $v_1 < v_2$, $f(c_1, v_1)f(c_2, v_2) \geq f(c_1, v_2)f(c_2, v_1)$

A4. Bounded Association: For all $(c,v) \in I \times I$, $v f_v(c|v)/f(c|v) > -1$ and $(1 - c) f_c(v|c)/f(v|c) < 1$.

Assumption A1 ensures that the support of the conditional densities is always equal to the support of the unconditional densities while the boundedness condition guarantees that all likelihood ratios are well-defined. Assumption A2 is equivalent to assuming that $F(c|v)$ is log-concave in c for all v and that $F(v|c)$ is log-concave in v for all c and is sufficient to guarantee that each trader's interim probability of ending up with the good is increasing in her value. This assumption is stronger than the corresponding assumption in Kadan (2007) of $R_c > -1$ and $T_v > -1$. The stronger assumption simplifies some of the analysis without limiting the range of qualitative effects a change in the distribution of trader values has on equilibrium bidding strategies. Assumption A3 implies the traders' values are affiliated random variables. Assumption A4 is used by Kadan (2007) to limit the extent to which a change in a trader's value changes her beliefs about the other trader's value. Bounded Association is satisfied if $f(v|c)$ is not too elastic with respect to c and $f(c|v)$ is not too elastic with respect to v . That is, Bounded Association limits the difference in belief updating between two different value sellers and between two different

⁵Letter subscripts denote partial differentiation.

value buyers. While this property is important in Kadan's existence proofs, it does not play a role in the results of this paper aside from making it possible to invoke Kadan's existence results. Let \mathcal{F} denote the set of all joint distributions satisfying Assumptions A1-A4.

Let $S:I \rightarrow I$ and $B:I \rightarrow I$ denote the seller and buyer strategies. Given a strategy profile (S,B) and a realization of values, c and v , trader payoffs when $B(v) \geq S(c)$ are

$$v - kB(v) - (1-k)S(c)$$

for the buyer and

$$kB(v) + (1-k)S(c) - c$$

for the seller; otherwise each trader's payoff is zero. (S,B) is a Bayesian equilibrium if

$$S(c) \in \operatorname{argmax}_s \int (kB(v) + (1-k)s - c) 1_{\{s \leq B(v)\}} dF(v|c)$$

and

$$B(v) \in \operatorname{argmax}_b \int (v - kb - (1-k)S(c)) 1_{\{b \geq S(c)\}} dF(c|v).$$

As in Satterthwaite and Williams (1989) and Kadan (2007), I restrict attention to equilibria with the following properties:

B1: $S(\cdot)$ and $B(\cdot)$ are continuous and strictly increasing

B2: For all $c \in I$, $c \leq S(c) \leq 1$.

B3: For all $v \in I$, $0 \leq B(v) \leq v$.

B4: For $c \geq B(1)$, $S(c) = c$.

B5: For $v \leq S(0)$, $B(v) = v$.

B6: $S(\cdot)$ is C^1 on $[0, B(1)]$.

B7: $B(\cdot)$ is C^1 on $[S(0), 1]$.

An equilibrium strategy profile is regular if conditions B1-B7 are satisfied.

The following theorem is due to Kadan (2007).

Theorem 1. Suppose $k \in (0, 1)$ and $F \in \mathcal{F}$.

(i) There exist a continuum of regular equilibria of the k -double auction.

(ii) (S,B) is a regular equilibrium of the k -double auction if, and only if, for $c \leq B(1)$ and $v \geq S(0)$,

$$kS'(c)R(c|B^{-1}(S(c))) + S(c) = B^{-1}(S(c)) \quad (1)$$

and

$$(1-k)B'(v)T(v|S^{-1}(B(v))) + B(v) = S^{-1}(B(v)). \quad (2)$$

Theorem 1 is reported to establish the existence of regular equilibria for an interesting and economically relevant set of dependent value distributions. As noted in the introduction, (1) and (2) are necessary conditions for any locally C^1 equilibrium given any distribution satisfying A1 and A2. The special but important cases of $k=0$ or $k=1$ are discussed below.

Given the role of $R(c|v)$ and $T(v|c)$ in defining equilibria, it will be important to understand how these inverse hazard rate functions change when statistical dependence is introduced. Let F^0 and F^1 be distributions satisfying A1 and A2. Associate with each distribution F^i the inverse hazard rate functions $R^i(c|v)$ and $T^i(v|c)$. In general, the signs of $R^1(c|v) - R^0(c|v)$ and $T^1(v|c) - T^0(v|c)$ can change in arbitrary ways as c and v change. If F^0 and F^1 have the same marginal distributions, then $E_v E_c R^1(c|v) = E_v E_c R^0(c|v)$ and $E_v E_c T^1(v|c) = E_v E_c T^0(v|c)$. This means if, for some c and v , $R^1(c|v) - R^0(c|v) \neq 0$ or $T^1(v|c) - T^0(v|c) \neq 0$, then there must exist other values of c and v for which each difference changes sign. If c and v are independently distributed under F^0 and strictly affiliated under F^1 , then the inverse hazard rate changes must be weakly monotonic since, from Milgrom and Weber (1982), $R_v^1(c|v) < R_v^0(c|v) \equiv 0$ and $T_c^1(v|c) < T_c^0(v|c) \equiv 0$. These inequalities imply $T^1(v|c) - T^0(v|c)$ must follow one of three patterns:

- (1) given v , if $T^1(v|0) < T^0(v|0)$, then $T^1(v|c) < T^0(v|c)$ for all $c > 0$;
- (2) given v , if $T^1(v|1) > T^0(v|1)$, then $T^1(v|c) > T^0(v|c)$ for all $c < 1$; or
- (3) given v , there exists c^* such that for all $c < c^*$, $T^1(v|c) > T^0(v|c)$, and for all $c > c^*$, $T^1(v|c) < T^0(v|c)$.

(Similar patterns for $R(c|v)$ also exist.) Section 3 will show that it is the ability of the inverse hazard rate differences to change sign, such as in pattern (3) with affiliation, that generates the range of equilibrium responses to a change in the distribution of trader valuations.

3. Local Equilibrium Changes.

Using the technique developed in SW, define $c(b) = S^{-1}(b)$ and define $v(b) = B^{-1}(b)$. Then, (1) and (2) are equivalent to

$$\dot{c}(b) = kR(c(b)|v(b))/(v(b) - b) \quad (3)$$

and

$$\dot{v}(b) = -(1-k)T(v(b)|c(b))/(b - c(b)). \quad (4)$$

Combined with the tautology $\dot{b} = 1$, (3) and (4) define a vector field. A regular equilibrium corresponds

to the solution defined by this vector field through a point $(\hat{c}, \hat{b}, \hat{v})$ such that $0 < \hat{c} < \hat{b} < \hat{v} < 1$. At this point, $S(\hat{c}) = B(\hat{v}) = \hat{b}$. Let $\underline{b} = S(0)$ and let $\bar{b} = B(1)$. Then Assumptions B4 and B5 imply $c(b) = 1$ for all $b > \bar{b}$ and $v(b) = 1$ for all $b < \underline{b}$. Trade occurs with positive probability only at bids between \underline{b} and \bar{b} . Any equilibrium that is locally C^1 at \hat{b} with initial condition $(\hat{c}, \hat{b}, \hat{v})$ must satisfy (3) and (4) in a neighborhood about \hat{b} .

For F^0 and F^1 with inverse hazard rate functions (R^0, T^0) and (R^1, T^1) , let $(c^i(b), v^i(b))$ denote an equilibrium for $i \in \{0, 1\}$ that is locally C^1 about \hat{b} where $c^i(\hat{b}) = \hat{c}$ and $v^i(\hat{b}) = \hat{v}$. That is, both equilibria have the same initial condition. The existence of distributions F^0 and F^1 for which such equilibria exist when at least one distribution is dependent is guaranteed by Theorem 1. Then,

$$\dot{c}^1(b) - \dot{c}^0(b) = \frac{kR^1(c^1(b)|v^1(b))}{v^1(b) - b} - \frac{kR^0(c^0(b)|v^0(b))}{v^0(b) - b} \quad (5)$$

and

$$\dot{v}^1(b) - \dot{v}^0(b) = \frac{(1-k)T^0(v^0(b)|c^0(b))}{b - c^0(b)} - \frac{(1-k)T^1(v^1(b)|c^1(b))}{b - c^1(b)}. \quad (6)$$

Eqs. (5) and (6) can be used to describe changes in equilibrium strategies in a neighborhood of \hat{b} .

Since $c^0(\hat{b}) = c^1(\hat{b}) = \hat{c} < \hat{b}$ and $v^0(\hat{b}) = v^1(\hat{b}) = \hat{v} > \hat{b}$, (5) and (6) imply

$$\dot{c}^1(\hat{b}) - \dot{c}^0(\hat{b}) = k[R^1(\hat{c}|\hat{v}) - R^0(\hat{c}|\hat{v})]/(\hat{v} - \hat{b}) > 0 \text{ if, and only if, } R^1(\hat{c}|\hat{v}) > R^0(\hat{c}|\hat{v}) \quad (7)$$

and

$$\dot{v}^1(\hat{b}) - \dot{v}^0(\hat{b}) = (1-k)[T^0(\hat{v}|\hat{c}) - T^1(\hat{v}|\hat{c})]/(\hat{b} - \hat{c}) > 0 \text{ if, and only if, } T^1(\hat{v}|\hat{c}) < T^0(\hat{v}|\hat{c}). \quad (8)$$

Theorem 2. Fix $k \in (0, 1)$ and let F^0 and F^1 satisfy A1 and A2. Suppose for each $i \in \{0, 1\}$ that $(c^i(\cdot), v^i(\cdot))$ is an equilibrium of the k -double auction under F^i with initial condition $(\hat{c}, \hat{b}, \hat{v})$ that is locally C^1 about \hat{b} . Then relative to F^0

- a) the seller's equilibrium strategy under F^1 will be flatter at \hat{b} if, and only if, the probability of ties bids conditional on trade occurring at (\hat{c}, \hat{v}) decreases; and
- b) the buyer's equilibrium strategy under F^1 will be flatter at \hat{b} if, and only if, the probability of tie bids conditional on trade occurring at (\hat{c}, \hat{v}) increases.

Theorem 2 follows directly from (7) and (8) and it allows us to determine the changes in any locally C^1 equilibrium in a neighborhood of \hat{b} by looking at the changes in the inverse hazard rates. It is based on the following intuition. $1/R(\hat{c}|\hat{v})$ equals the probability of tie bids conditional on trade occurring

($s < \hat{b}$). If this probability falls (R increases), the \hat{v} buyer has an incentive to shade his bid more (bid less than \hat{b}). To counter this effect so that \hat{b} remains the equilibrium bid for traders with values \hat{c} and \hat{v} , the seller's strategy must be flatter so that the buyer's probability of trade associated with bids below \hat{b} falls faster thus making such bid reductions by the buyer undesirable. A flatter seller strategy in turn implies a steeper inverse seller strategy, $c(b)$. Thus, Theorem 2 allows one to determine which traders with values near \hat{c} and \hat{v} will reduce the difference between their bids and their values (bid less aggressively) in response to a change in the value distribution and which will do the opposite (bid more aggressively). When $\hat{c}^1(\hat{b}) < \hat{c}^0(\hat{b})$, a seller with a value above \hat{c} will bid more aggressively while a seller with a value below \hat{c} will bid less aggressively. When $\hat{v}^1(\hat{b}) < \hat{v}^0(\hat{b})$, a buyer with a value above \hat{v} will bid less aggressively and a buyer with a value below \hat{v} will bid more aggressively. These were the changes observed in Kadan's examples. However, it is possible to observe four possible sign changes in the inverse hazard rate differences due to a change in the value distribution and hence it is possible to observe the following four possible changes in equilibrium strategies:

- (I) $\hat{c}^1(\hat{b}) < \hat{c}^0(\hat{b})$ and $\hat{v}^1(\hat{b}) > \hat{v}^0(\hat{b})$,
- (II) $\hat{c}^1(\hat{b}) < \hat{c}^0(\hat{b})$ and $\hat{v}^1(\hat{b}) < \hat{v}^0(\hat{b})$,
- (III) $\hat{c}^1(\hat{b}) > \hat{c}^0(\hat{b})$ and $\hat{v}^1(\hat{b}) < \hat{v}^0(\hat{b})$, and
- (IV) $\hat{c}^1(\hat{b}) > \hat{c}^0(\hat{b})$ and $\hat{v}^1(\hat{b}) > \hat{v}^0(\hat{b})$.

Case II corresponds to the examples in Kadan (2007) that are consistent with the intuition described in the introduction whereby affiliation makes equilibrium strategies steeper. The following example shows that all four cases can arise even for distributions within the set \mathcal{F} .

Example. Farlie-Gumbel-Morgenstern distributions

Let $G_1(c)$ and $G_2(v)$ denote marginal distributions. For $\alpha \in (-1, 1)$, the FGM copula is defined by the joint distributions

$$F(c, v, \alpha) = G_1(c)G_2(v)(1 + \alpha(1 - G_1(c))(1 - G_2(v))). \quad (9)$$

For any $\alpha \in (-1, 1)$, the marginal distributions of $F(\cdot, \cdot, \alpha)$ are $G_1(\cdot)$ and $G_2(\cdot)$ (with densities $g_1(\cdot)$ and $g_2(\cdot)$), $f(c|v, \alpha) = g_1(c)[1 + \alpha(1 - 2G_1(c))(1 - 2G_2(v))]$, and $F(c|v, \alpha) = G_1(c)[1 + \alpha(1 - G_1(c))(1 - 2G_2(v))]$. The trader values are independently distributed when $\alpha = 0$ and they are strictly affiliated when $\alpha > 0$. If the marginals are uniform, Kadan (2007) proves the joint distribution will satisfy Bounded Association if $\alpha < 1/3$.

Kosmopoulou and Williams (1998) used the FGM distributions with uniform marginals to study the effect of affiliation on the existence of efficient bilateral bargaining mechanisms. Kadan (2007) used the more general family defined in (9) to prove that for the k -double auction, the regular-equilibrium

correspondence is lower hemi-continuous.

For FGM distributions, let c_m and v_m denote the median values for $G_1(\cdot)$ and $G_2(\cdot)$. For all $c > 0$,

$$\partial R(c|v, \alpha) / \partial \alpha > 0 \text{ if, and only if, } v < v_m \quad (10)$$

and for all $v < 1$,

$$\partial T(v|c, \alpha) / \partial \alpha > 0 \text{ if, and only if, } c < c_m. \quad (11)$$

Let α_0 and α_1 denote values of the affiliation parameter, α , that satisfy Bounded Association and $0 \leq \alpha_0 < \alpha_1$. Let $F^0(c, v) \equiv F(c, v, \alpha_0)$ and $F^1(c, v) \equiv F(c, v, \alpha_1)$ as defined by (9) and let $(c^0(\cdot), v^0(\cdot))$ and $(c^1(\cdot), v^1(\cdot))$ denote the regular k -double auction equilibrium under each distribution with initial condition $(\hat{c}, \hat{b}, \hat{v})$. By Theorem 1, such equilibria exists for all $(\hat{c}, \hat{b}, \hat{v}) \in C^*$. Then (10) and (11) imply that for each $j \in \{I, II, III\}$ there exists an initial condition $(\hat{c}, \hat{b}, \hat{v})$ such that the change in the equilibrium from $(c^0(\cdot), v^0(\cdot))$ to $(c^1(\cdot), v^1(\cdot))$ in a neighborhood about \hat{b} is described by case j . Initial conditions consistent with case IV exist only if $c_m < v_m$. Specifically,

- (I) $\hat{c} > c_m$ and $\hat{v} > v_m$ implies $\dot{c}^1(\hat{b}) < \dot{c}^0(\hat{b})$ and $\dot{v}^1(\hat{b}) > \dot{v}^0(\hat{b})$,
- (II) $\hat{c} < c_m$ and $\hat{v} > v_m$ implies $\dot{c}^1(\hat{b}) < \dot{c}^0(\hat{b})$ and $\dot{v}^1(\hat{b}) < \dot{v}^0(\hat{b})$,
- (III) $\hat{c} < c_m$ and $\hat{v} < v_m$ implies $\dot{c}^1(\hat{b}) > \dot{c}^0(\hat{b})$ and $\dot{v}^1(\hat{b}) < \dot{v}^0(\hat{b})$, and
- (IV) $\hat{c} > c_m$ and $\hat{v} < v_m$ implies $\dot{c}^1(\hat{b}) > \dot{c}^0(\hat{b})$ and $\dot{v}^1(\hat{b}) > \dot{v}^0(\hat{b})$.⁶

Figures 1a-1d illustrate the effect of affiliation in Cases (I)-(IV) when buyer and seller values are distributed over $[0,1]$ with uniform marginal distributions. The top two lines in each figure are equilibrium seller strategies and the bottom two lines are equilibrium buyer strategies for $k=1/2$. The solid lines are equilibrium strategies for $\alpha=0$ and the dashed lines are equilibrium strategies for $\alpha=.95$. Technically, $\alpha=.95$ does not satisfy the Bounded Association property, A4, so the dashed lines may not

⁶For the special cases when $k \in \{0,1\}$, there exists a unique regular equilibrium in which one trader plays her weakly dominant strategy of truthful bidding. This uniqueness changes the way the inverse hazard rates effect equilibrium bidding. In the buyer's bid double auction ($k=1$) that Kadan (2007) analyzes $\dot{c}^0(\hat{b}) = \dot{c}^0(\hat{b}) \equiv 1$ so only two cases can exist: $\dot{v}^1(\hat{b}) < \dot{v}^0(\hat{b})$ and $\dot{v}^1(\hat{b}) > \dot{v}^0(\hat{b})$. The buyer's equilibrium strategy is defined by $v(\hat{b}) = \hat{b} + R(\hat{b}|v(\hat{b}))$ instead of (4) which relies on T . However, some aspects of the analysis when $k \in (0,1)$ can be applied to these double auctions as well. For example, if F^0 and F^1 have the same marginals, then \hat{v} must equal v_m and (10) implies affiliation must make the buyer's strategy steeper at v_m . This analysis allows one to extend Kadan's Proposition 1, which assumes uniform marginals, to FGM distributions with non-uniform marginals that satisfy A1, A2, and some additional technical conditions.

correspond to actual equilibrium strategies. This large value of α was used to create enough separation between the strategies for visualization purposes only. Similar calculations for values of α less than $1/3$, which satisfy A4, have been done. They reveal identical qualitative differences between equilibria as seen in the figures except that the magnitude of the differences make some comparisons harder to visualize in a single graph. The initial conditions for Figures 1a-1c were chosen to coincide with the linear equilibrium first identified by CS when $\alpha=0$.

Figure 1a corresponds to Case I. Strict affiliation makes the seller's strategy steeper at $c=5/8$ but it makes the buyer's strategy flatter at $v=7/8$. (Checking the calculations that support this figure verifies that the buyer's strategy under strict affiliation cuts below the strategy under $\alpha=0$ at $v=7/8$. Similar verifications have been performed for all the other figures.) Increasing α causes the buyer with a value just below $7/8$ and the seller with a value just below $5/8$ to bid less aggressively and it causes the buyer with a value just above $7/8$ and the seller with a value just above $5/8$ to bid more aggressively. Figure 1b is analogous to Figure 2 in Kadan (2007) and corresponds to Case II. Strict affiliation results in traders who trade with high probabilities (low-value sellers and high-value buyers) bidding less aggressively and traders who trade with low probabilities bidding more aggressively. Figure 1c corresponds to Case III. Increasing α causes the buyer with a value just above $3/8$ and the seller with a value just above $1/8$ to bid less aggressively and it causes the buyer with a value just below $3/8$ and the seller with a value just below $1/8$ to bid more aggressively. To generate a Case IV example, Figure 1d is drawn assuming the marginal distribution of seller values is still uniform while the marginal distribution of buyer values is now $G_2(v) = v^2$. Strict affiliation makes both the buyer and the seller strategies flatter near a bid of $.6$. Increasing α causes the buyer with a value just below $.68$ and the seller with a value just above $.54$ to bid less aggressively and it causes the buyer with a value just above $.68$ and the seller with a value just below $.54$ to bid more aggressively.

4. Global Equilibrium Changes.

The figures for cases I, III, and IV show that local changes in bidding strategies due to a change in the value distribution need not persist over the entire range of bids. This lack of persistence in local changes can be traced to the fact that in (5) and (6) changes in the seller's strategy can generate moderating effects on the buyer's strategy and vice versa. For instance, case I implies that $c^1 - c^0$ is initially decreasing and $v^1 - v^0$ is initially increasing for $b > \hat{b}$. As $c^1 - c^0$ decreases, (6) implies that $\dot{v}^1 - \dot{v}^0$ also decreases thus moderating the difference between v^1 and v^0 as b increases above \hat{b} .

Cases I and III equilibria must always exhibit a weak change in sign for either $c^1(b) - c^0(b)$ or $v^1(b) - v^0(b)$. Figure 2 illustrates this point for case I. The solid lines again represent the seller and

buyer strategies under distribution F^0 and the dashed lines correspond to the seller and buyer strategies under F^1 if the local properties near \hat{b} were to extend for all larger bids at which trade occurs. Note that for cases I and III, $(c^1(b) - c^0(b))(v^1(b) - v^0(b)) < 0$ for bids in a neighborhood about \hat{b} . That is, for case I (case III), both the seller with a value just above \hat{c} and the buyer with a value just above \hat{v} bid more (less) aggressively under F^1 . As Figure 2 shows, if both traders bid more aggressively for all $b \in (\hat{b}, \max\{\bar{b}^0, \bar{b}^1\})$, then $c^1(b) < c^0(b)$ implies $\bar{b}^1 < \bar{b}^0$ while $v^1(b) > v^0(b)$ implies $\bar{b}^1 > \bar{b}^0$. (Recall that $c^i(\bar{b}^i) = \bar{b}^i$ and $v^i(\bar{b}^i) = 1$.) Thus, case I cannot persist globally in the sense that either $c^1(b) - c^0(b)$ or $v^1(b) - v^0(b)$ must weakly change sign for some $b > \hat{b}$. An analogous argument applied to case III equilibria reaches a similar contradiction about \underline{b}^0 and \underline{b}^1 . Cases II and IV equilibria do not yield this type of contradiction. Theorem 3 formalizes this argument which shows that the global effects in cases I and III work against the local effects.

Theorem 3. Fix $k \in (0, 1)$. Suppose for each $i \in \{0, 1\}$ that $(c^i(\cdot), v^i(\cdot))$ is an equilibrium of the k -double auction under F^i with initial condition $(\hat{c}, \hat{b}, \hat{v}) \in \mathcal{C}^*$. Assume that either case I or case III describes the local behavior of the equilibria about \hat{b} so that near \hat{b} both the seller and the buyer bid strictly more aggressively or both bid strictly less aggressively under F^1 . There must exist at least one bid not equal to \hat{b} at which one trader bids more aggressively and the other bids weakly less aggressively under F^1 or one bids less aggressively and the other bids weakly more aggressively under F^1 .

In the case of FGM distributions, the possibility that $\bar{b}^0 = \bar{b}^1$ can be ruled out thereby establishing that any local change in equilibrium bidding consistent with cases I or III is temporary. To do so assume there exists equilibria $(c^i(\cdot), v^i(\cdot))$ for $i \in \{0, 1\}$ with initial condition $(\hat{c}, \hat{b}, \hat{v})$ that implies case I at \hat{b} , assume for all $b > \hat{b}$ that $c^1(b) \leq c^0(b)$ and $v^1(b) \geq v^0(b)$ (weak case I persistence), and assume $\bar{b}^0 \geq \bar{b}^1$. With $\bar{b}^0 \geq \bar{b}^1$,

$$\int_{b=\hat{b}}^{\bar{b}^0} [c^1(b) - c^0(b)] db = c^1(\bar{b}^0) - c^0(\bar{b}^0) - c^1(\hat{b}) + c^0(\hat{b}) = \bar{b}^0 - \bar{b}^0 - \hat{b} + \hat{b} = 0. \quad (12)$$

But (3) implies

$$\begin{aligned}
\int_{b=\hat{b}}^{\bar{b}^0} [\dot{c}^1(b) - \dot{c}^0(b)]db &= k \int_{b=\hat{b}}^{\bar{b}^1} [R^1(c^1(b)|v^1(b))/(v^1(b) - b) - R^0(c^0(b)|v^0(b))/(v^0(b) - b)]db \\
&\quad + \int_{b=\bar{b}^1}^{\bar{b}^0} [1 - \dot{c}^0(b)]db \\
&< k \int_{b=\hat{b}}^{\bar{b}^1} [(R^1(c^1(b)|v^1(b)) - R^0(c^0(b)|v^1(b)))/(v^1(b) - b)]db \\
&< k \int_{b=\hat{b}}^{\bar{b}^1} [(R^1(c^1(b)|v^1(b)) - R^1(c^0(b)|v^1(b)))/(v^1(b) - b)]db < 0
\end{aligned} \tag{13}$$

which contradicts (12).

The first inequality arises for three reasons. First the integral in line 2 of (13) equals $c^0(\bar{b}^1) - \bar{b}^1$ which is negative. Second, the assumption of weak case I persistence implies that $v^1(b) \geq v^0(b)$ for all $b \geq \hat{b}$ and the assumption of a (local) case I equilibria ensures that $v^1(b) > v^0(b)$ on a set of positive Lebesgue measure for bids greater than \hat{b} . Thus, replacing v^0 with v^1 in the denominator of \dot{c}^0 strictly increases the integral on the right-hand side of line 1 of (13). Third, strict affiliation implies $R_v < 0$. Hence, replacing v^0 with v^1 in R^0 also strictly increases the same integral.

To establish the second inequality in (13) recall that, with FGM distributions, strict affiliation implies $R^1(c|v) < R^0(c|v)$ for all $c > 0$ and for all $v > v_m$ and that case I implies $\hat{v} > v_m$. The fact that case I persistence implies $c^1(b) \leq c^0(b)$, with strict inequality in a neighborhood of \hat{b} and the fact that A2 implies $R_c > 0$ yields the final inequality.

In other words, if a case I equilibrium is globally persistent then \bar{b}^1 must be strictly larger than \bar{b}^0 (the measure of high-values sellers who do not trade must decrease) but if $\bar{b}^1 > \bar{b}^0$ then at some bid, b , $c^1(b)$ must be larger than $c^0(b)$ and the local properties of case I equilibria will no longer persist. Thus, the non-monotonic effects of affiliation on the equilibrium strategies as seen in Figure 1a are common to all FGM distributions and more generally to any pair of distributions which admit regular, case I equilibria and which for all $c > 0$ there exists v^* such that $R^1(c|v) - R^0(c|v)$ is positive for all $v < v^*$ and negative for all $v > v^*$. A similar argument that assumes instead that $\underline{b}^0 \leq \underline{b}^1$ applied to case III equilibria reaches an analogous contradiction.

Persistent case IV equilibria cannot be ruled out using the above arguments because $R^1(c|v) > R^0(c|v)$ only for $v \in (\hat{v}, v_m)$. Sufficiently large bids will imply $v > v_m$ and hence $R^1(c|v) < R^0(c|v)$. The changing sign of $R^1 - R^0$ for seller values above \hat{v} indicates that strict affiliation creates countervailing global effects for case IV equilibria and suggests that one may need to use finer

details of case IV equilibria in order to establish results about the global persistence of local case IV properties. Two possibly relevant implications of globally persistent case IV equilibria are $\bar{b}^1 < \bar{b}^0$ and $\underline{b}^1 > \underline{b}^0$. These inequalities mean that strict affiliation could increase the range of buyer values and seller values that never trade in a case IV equilibrium. In contrast, globally persistent case II equilibria, which we know exist, imply that strict affiliation results in a smaller range of buyer and seller values that never trade. I conjecture that, because a globally persistent case IV equilibrium implies a reduction in the range of buyer and seller values associated with a positive probability of trade, strict affiliation cannot generate such equilibria.

To support this conjecture consider that a necessary condition for a regular equilibrium is $\dot{c}(\bar{b}^i) \geq 1$ (see SW). Given A2, this inequality places a lower bound on values of \bar{b}^i consistent with a regular equilibrium. Suppose then that F^0 and F^1 are distributions which, consistent with FGM distributions, satisfy $R^1(c|1) < R^0(c|1)$ for all $c > 0$ and suppose there are regular equilibria (c^0, v^0) and (c^1, v^1) with the same case IV initial condition such that $\bar{b}^1 < \bar{b}^0$ and $\dot{c}^0(\bar{b}^0) = 1$. Because $c^i(\bar{b}^i) = \bar{b}^i$ and $v^i(\bar{b}^i) = 1$,

$$1 = \dot{c}^0(\bar{b}^0) = kR^0(\bar{b}^0|1)/(1 - \bar{b}^0) > kR^0(\bar{b}^1|1)/(1 - \bar{b}^1) > kR^1(\bar{b}^1|1)/(1 - \bar{b}^1) = \dot{c}^1(\bar{b}^1). \quad (14)$$

Thus, globally persistent case IV equilibria require $\dot{c}(\bar{b}^0)$ sufficiently larger than 1. However, larger values of $\dot{c}(\bar{b}^0)$ and hence larger values of \bar{b}^0 are more likely to be associated with case I or III equilibria.

5. Conclusion.

This paper shows how to exploit the multiplicity of equilibria that generally exist for bilateral k -double auctions with $k \in (0, 1)$ to study the effect of changes in the distribution of valuations. As a special case, the analysis identifies a wider range of equilibrium responses to strict affiliation than has been suggested by the intuition emphasized in Kadan (2007). A change in the distribution of trader values from independence to dependence can result in the buyer adopting a steeper or a flatter strategy and the seller adopting a steeper or a flatter strategy over some range of values and this full range of equilibrium responses arises even when the statistical dependency between the traders' values changes the inverse hazard rates in a well-behaved monotonic fashion implied by strict affiliation. An analysis of the global effects of a change in the value distribution shows that not all of the possible local equilibrium changes can persist locally. This lack of persistence results in changes in equilibrium strategies that are not monotonic in valuations.

References

- Bikhchandani, S. and J. Riley, 1991, Equilibria in Open Common Value Auctions. *Journal of Economic Theory* 53:101-130.
- Chatterjee, K. and W. Samuelson, 1983, Bargaining Under Incomplete Information. *Operations Research* 31: 835-51.
- Cripps, M. and J. Swinkels, 2006, Efficiency of Large Double Auctions. *Econometrica* 74:47-92.
- de Castro, L. 2007. Affiliation, Equilibrium Existence, and the Revenue Ranking of Auctions. mimeo.
- Fudenberg, D., M. Mobius, and A. Szeidl, 2007, Existence of Equilibrium in Large Double Auctions. *Journal of Economic Theory* 133:550-567.
- Jackson, M. and J. Swinkels. 2005. Existence of Equilibria in Single and Double Private Value Auctions. *Econometrica* 73:93-139.
- Kadan, O., 2007, Equilibrium in the Two Player, k -Double Auction with Affiliated Private Values. *Journal of Economic Theory* 135: 495-513.
- Kosmopoulou, G. and S. Williams, 1998, The Robustness of the Independent Private Value Model in Bayesian Mechanism Design. *Economic Theory* 12:393-421.
- Krishna, V., 2002, *Auction Theory*. Academic Press.
- Milgrom, P. and R. Weber, 1982, A Theory of Auctions and Competitive Bidding. *Econometrica* 50:1089-1122.
- Reny, P. and M. Perry, 2006, Toward a Strategic Foundation for Rational Expectations Equilibrium. *Econometrica* 74:1231-1269.
- Rustichini, A., M. Satterthwaite, and S. Williams, 1994, Convergence to Efficiency in a Simple Market with Incomplete Information. *Econometrica* 62:1041-1064.
- Satterthwaite, M. and S. Williams, 1989, Bilateral Trade with the Sealed Bid k -Double Auction: Existence and Efficiency. *Journal of Economic Theory* 48:107-133.

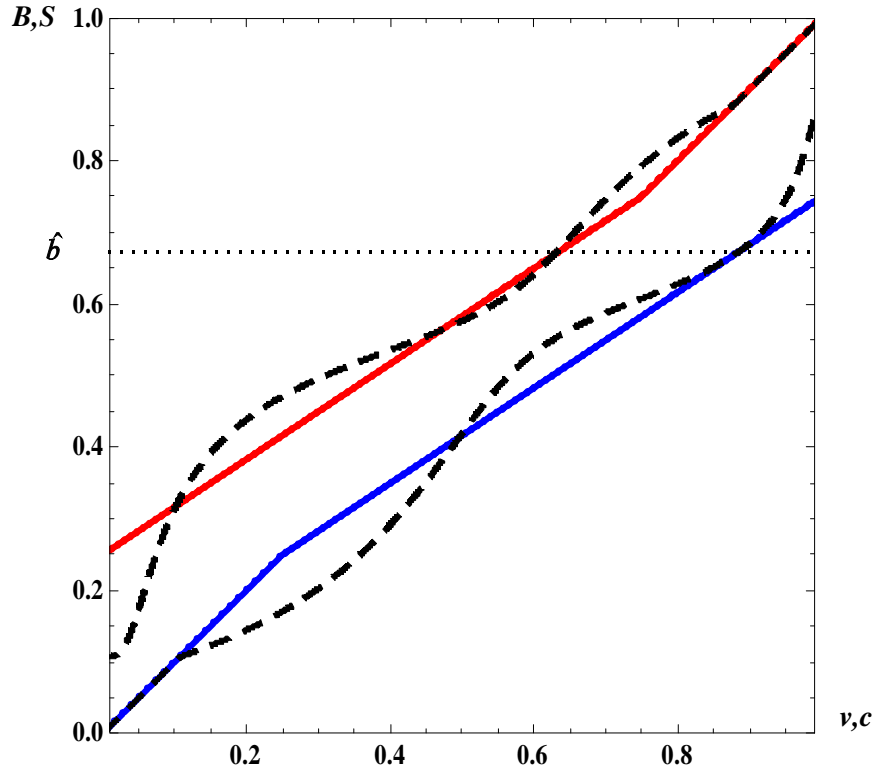


Figure 1a: $(\hat{c}, \hat{b}, \hat{v}) = (5/8, 2/3, 7/8)$ (Case I)

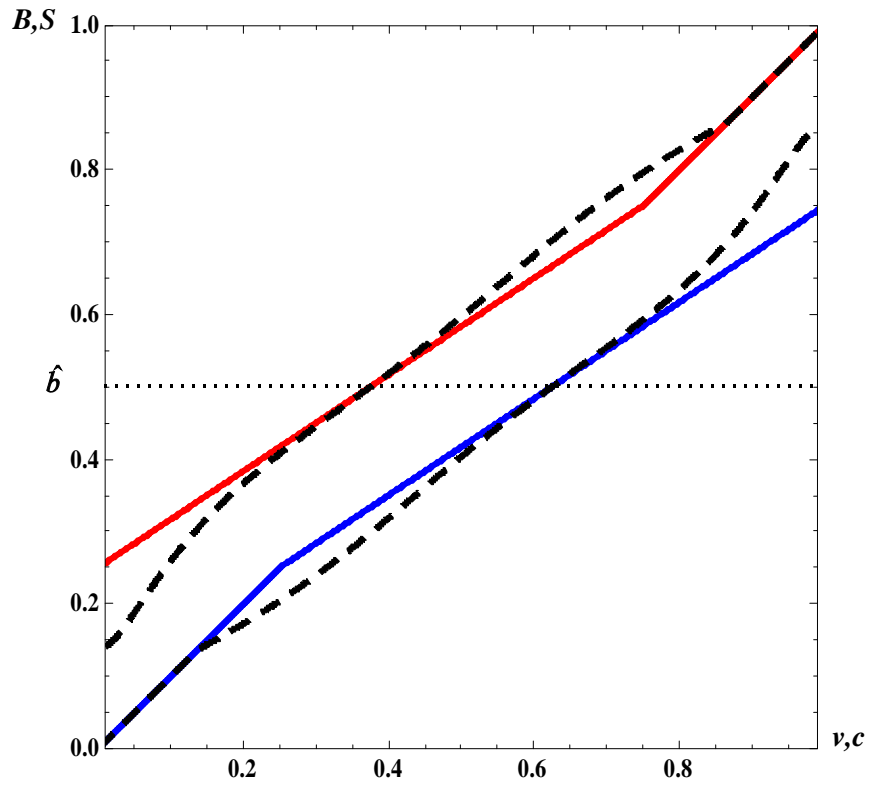
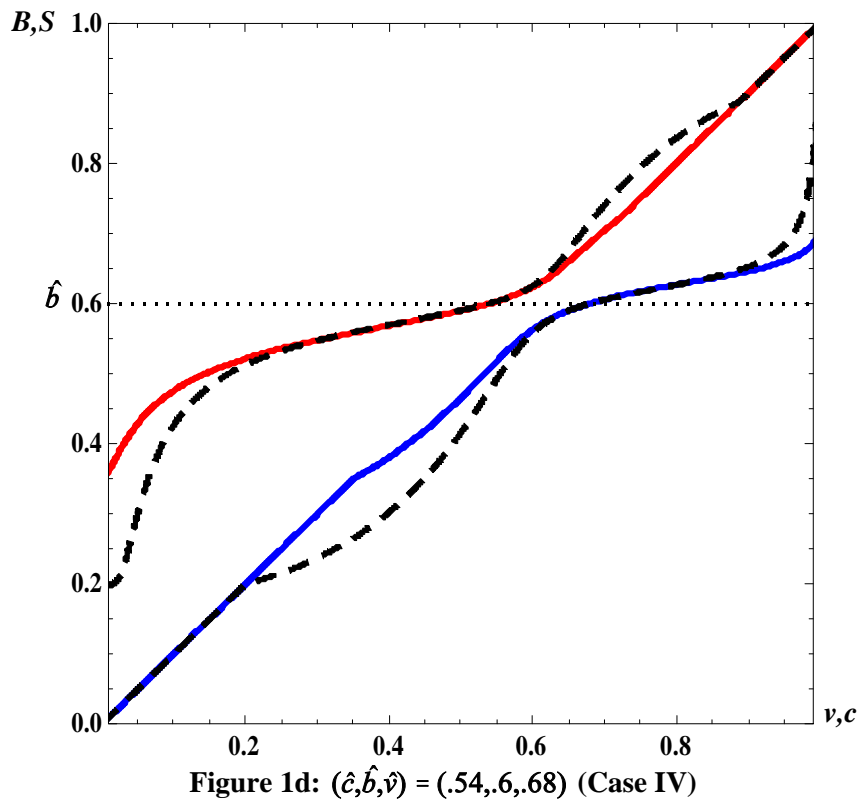
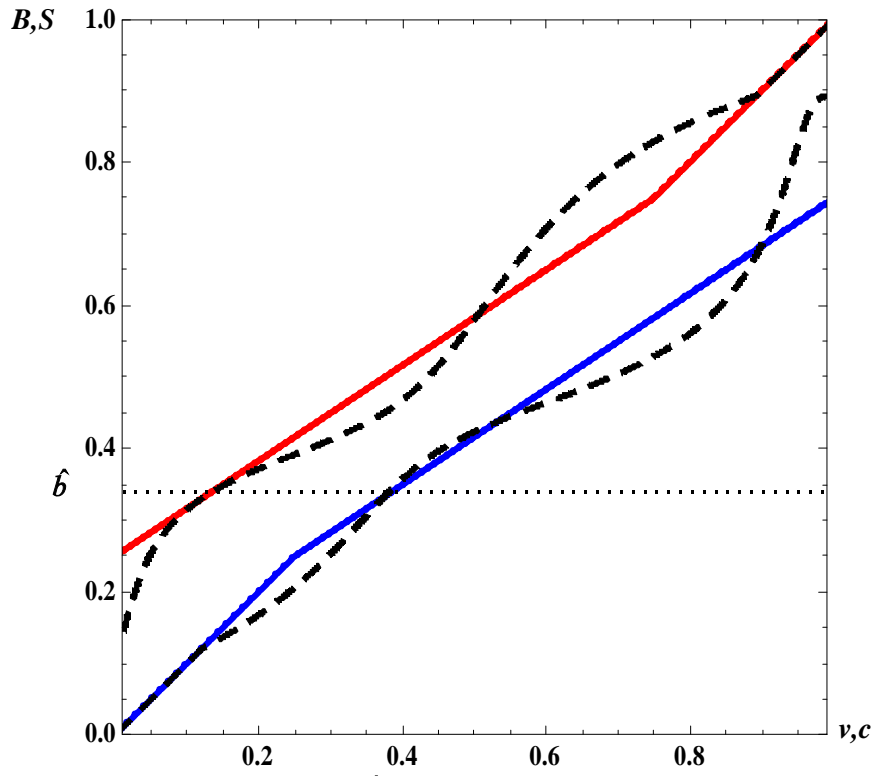


Figure 1b: $(\hat{c}, \hat{b}, \hat{v}) = (3/8, 1/2, 5/8)$ (Case II)



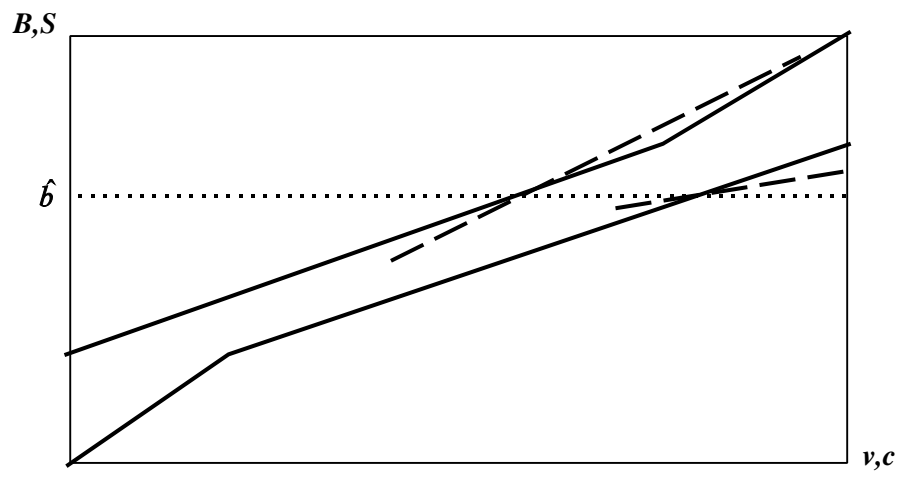


Figure 2: An Illustration of a Globally Persistent Case I Equilibrium