Asset Pricing with Long Run Risk and Stochastic Differential Utility: An Analytic Approach*

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December 5, 2009

Abstract:

This work studies the continuous time version of the long run risk model of Bansal and Yaron (2004). In this model the lifetime utility is recursive in the sense of Duffie and Epstein (1992a, 1992b) and it solves a nonlinear differential equation of second order with appropriate initial conditions. It is proved that this solution is analytic and its power series representation near the stationary mean has radius of convergence bigger than seven standard deviations of the long run risk variable. Consequently, the lifetime utility can be computed quickly and accurately using a higher order Taylor polynomial approximation. With the solution to the lifetime utility being analytic, one shows that the price-dividend function solves a second order linear differential equation with analytic coefficients depending on the lifetime utility. Therefore, it can be represented by a power series with a radius of convergence at least one third of that for the lifetime utility. Then, the long run risk model is compared with that of the external habit model of Campbell and Cochrane (1999) with respect to the time varying properties of stock returns. Finally, an alternative method from Hansen and Scheinkman (2009) is used to find the asymptotic rate of return on stocks for the long run risk model.

Journal of Economic Literature Classification Numbers: G12, G13, C63, D51

Key Words: Analyticity, Asset Pricing, Long Run Risk, Stochastic Differential Utility

*We would like to thank Adam Speight for helpful discussions.

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1 Introduction

The long run risk model of Bansal and Yaron (BY 2004) and the external habit model of Campbell and Cochrane (CC 1999) were developed to explain the statistical properties of stock returns. Bansal, Gallant, and Tauchen (2007, Table 3) are able to match key observed unconditional moments such as the equity premium for both models using quadratic approximations of the discrete time version of these models.[1] Subsequently, the debate over which model best represents the stylized facts of stock returns has centered on time varying properties of consumption growth and stock returns, such as whether the price-dividend ratio helps to predict consumption growth or stock returns.[2]

Here, we argue that both models require higher order polynomial approximations to accurately represent the solution to these models given the observed variation in consumption or dividend growth. In addition, the high order polynomial approximations influence the time varying properties of stock returns. Chen, Cosimano and Himonas (CCH 2009) proved that the price-dividend function for the continuous time version of the CC model is an analytic function of CC's state variable – the surplus consumption ratio. As a result, the price dividend function in the CC model is quickly and accurately computed by a high order Taylor polynomial approximation. This approximation is displayed in Figure 13 from CCH 2009.[3] The price-dividend function is used to find the expected return on stocks, solid line in Figure 14, and its standard deviation, the dotted line in Figure 14. Both, the expected return on stocks and its standard deviation decline as the surplus consumption ratio falls, since the price-dividend function is a concave function of the surplus consumption ratio. The concav-

[1] We show in CCH (2008a) that the discrete time model of CC does not have a solution in the space of price-dividend functions which have a bounded growth rate. However, for the parameters estimated by Bansal, Gallant and Tauchen (2007) the CC model does have such a solution. The main reason is that their estimates imply a short term interest rate of about 2%, while CC use a 1% rate of return. We will see below that the short term interest rate in the long run risk model also needs to be higher.

[2] See the debate between Beeler and Campbell (2009), and Bansal, Kiku, and Yaron (2009). Both papers use low order polynomial approximations to represent the solution to the discrete time version of these models.

[3] The parameter values are those chosen by CC in their original paper, Campbell and Cochrane (1999), rather than the parameters estimated by Bansal, Gallant, and Tauchen (2007). We do not know which parameters best represent the data since Bansal *et al.* base the choice of parameters on a quadratic approximation.

ity (down) of the price dividend function helps one to explain why the CC model predicts lower returns on stocks in the future when the current price-dividend ratio is high. Higher consumption growth leads to a persistent increase in the surplus consumption ratio. Furthermore, this increase in the surplus consumption ratio leads to a higher price-dividend ratio which persist for a long period of time. At the same time the expected return on stocks and its standard deviation are low so that the future stock returns will be lower, as long as the high surplus consumption ratio persists. Thus, the high order polynomial approximation is necessary to accurately represent the solution to the CC model given the fluctuation in consumption growth observed in the data. In addition, this approximation is necessary to capture the time varying properties of stock returns in that a high current price-dividend ratio helps to predict lower stock returns in the future.

BY argue that consumption and dividend growth are bombarded by shocks which persist for a long period of time.[4] As a result, the uncertainty in the common long term growth variable has a significant impact on the pricing of various financial assets. Subsequently, Bansal, Dittmar and Lundblad (2005) investigated the impact of this long run risk on the cross-section of one-period stock returns, while Hansen, Heaton, and Li (2008) examine its implications for pricing long-term cash flow risk. These authors demonstrate the importance of Epstein and Zin's (1989) recursive utility for the representative investor, so that the investor's attitude toward risk and intertemporal substitution of consumption may be separated. These authors use lower order polynomial approximations (linear or quadratic) to represent the solution to the long run risk model. In our work here we show that higher order approximations are needed so as to accurately compute the solution to this model. In fact, this is one of the main contributions of this paper.[5]

[4]For the BY parameters the half life of a shock to the long run risk variable is 38 months.

[5]The discrete time BY model also requires a high order polynomial approximation, since the integral equation is more complicated than the Mehra-Prescott (1985,2003) model (see CCH (2008c)). Calin, *et al.* (2005) and CCH (2008b) showed that the Mehra-Prescott model requires a ninth order Taylor polynomial approximation to accurately represent the solution. Continuous time is used rather than discrete time since the computer program is faster. The increase in spread results from the recursive procedure to solve for the coefficients of the approximation in continuous time as opposed to the simultaneous equation system which has to be solved to find the coefficients in discrete time.

High order polynomial approximation influence the choice of the parameters one would use in the long run risk model. The solution to the long run risk model in Figure 9 is convex (concave up) in the long run risk variable when we use the parameters advocated by BY in their original article in 2004. This price-dividend function does not capture some of the time varying properties of stock returns. The convexity of the price-dividend function yields an increasing expected return on stocks, the solid line in Figure 16 and its standard deviation, the dotted line in Figure 16, so that both will be higher when the long run risk variable is high. In this case a higher consumption growth leads to a persistent increase in the long run risk variable so that the price-dividend ratio is high for a long period of time. At the same time the expected return on stocks and its standard deviation are higher. Consequently, the higher price-dividend ratio predicts that stock returns will be higher in the future which is inconsistent with the time varying properties of stock returns. As a result, the parameters, which will deliver the appropriate time varying properties of stock returns, are definitely different than those chosen by Bansal and Yaron using a low order polynomial approximation of their model. Thus, having a procedure that can quickly and accurately solve the long run risk and the external habit model of asset prices, will inform the debate about the ability of stock return data to discriminate between these models.

A continuous time version of BY's model of consumption and dividend growth, which was updated by Bansal, Kiku, and Yaron (BKY 2007), is used to solve the long run risk model.[6] To capture the empirical evidence of BY that the volatility of consumption growth is negatively related to the price-dividend ratio, it is assumed that the variance of consumption growth is a negatively sloped logistic function of the long run risk variable. This property follows from the equilibrium price-dividend function being positively related to the long run risk variable. The parameters of this variance function are chosen such that the slope of a parametric plot of the standard deviation of consumption growth against the price-dividend ratio matches this empirical evidence from BY (see their Table III).

[6]These parameter's are also used in Bansal, Kiku, and Yaron (BKY 2009). Beeler and Campbell (2009) use these parameters in their BKY case.

In continuous time the investor's recursive utility is the solution of a backward stochastic differential equation which was developed by Duffie and Epstein (1992a, 1992b), and Duffie and Lions (1992). All the long-run risk models use the Kreps-Porteus (1978) functional form for the aggregator of future preferences in the recursive utility model. Following these works the lifetime utility of the investor subject to the stochastic behavior of consumption and dividend growth is the solution to a second order nonlinear ordinary differential equation (ODE). The analytic method developed in CCH (2009) is also used to demonstrate that this nonlinear ODE has an unique analytic solution so that the lifetime utility of the investor is given by a power series in the long run risk variable within some interval of convergence.[7] The method in CCH (2009) to identify the radius of convergence for the power series representation of this lifetime utility function cannot be used here since the ODE is nonlinear. However, one can find a dominant power series with a known radius of convergence, which bounds the power series of the lifetime utility function. As a result, an estimate of the relative error between the polynomial approximation of the lifetime utility and its true solution is determined. For the parameter values in BKY this relative error is less than 0.09% of the investor's lifetime utility. This estimate holds as long as the long run risk variable is bounded by the radius of convergence, which is 7 times its standard deviations, when a 100^{th} order Taylor polynomial approximation is used. In addition, a comparison between a 90^{th} and 100^{th} order Taylor polynomial yields a relative error of less than one in a 25 million within the region of convergence. Also a 4^{th} order Taylor polynomial approximation can yield an relative error as high as 55%. Thus, a higher order Taylor polynomial is necessary to adequately represent the solution to BY model over the wide range of the long run risk variable which is implied by consumption data.

Duffie and Epstein (1992b) show how the lifetime utility of a representative agent can be used to determine the state-pricing process, so that the price of any financial asset can be

[7] Duffie and Lions (1992) develop conditions to assure there is a unique solution to this equation for various values of the parameters of the aggregator function. Fisher and Gilles (1998) rely on the Cauchy-Kovalevksy theorem to prove analyticity of the solution, but they do not identify the radius of convergence or error in the solution.

determined once the stochastic process for the payoff from the security is known. This result allows one to derive the linear ODE that yields the equilibrium price-dividend function of the long run risk variable for the continuous time version of the BY model. As in CCH (2009) this ODE is formulated as an initial value problem which determines the unique analytic price-dividend function that matches the price-dividend ratio, the expected return on stocks, and the volatility of the equity premium at the stationary point of the long run risk variable. However, the risk free interest rate is about 2% too high. Following CCH (2009) the radius of convergence for the price-dividend function is at least the smallest radius of convergence of the coefficients and the forcing term of the ODE for the price-dividend function. These coefficients and forcing term are dependent on the lifetime utility function such that the smallest radius of convergence is equal to one third the radius of convergence of the lifetime utility function. Consequently, the error analysis for the price-dividend function is not as precise as that of the lifetime utility. Examination of the numerical solution to the price-dividend function shows that the difference between a 90^{th} and 100^{th} order Taylor polynomial approximation of the price-dividend function is less than one in 3 million for the long run risk variable within the interval of convergence of the lifetime utility function.

As pointed out in CCH (2009), the analytic solutions to the ODEs for the lifetime utility and price-dividend functions are unique when the initial conditions for the ODEs are given. It is shown that the first initial condition for the lifetime utility and price-dividend functions can be set to be consistent with the Feynman-Kac formula to each function at the risk neutral stationary mean of the long run risk variable. It is also shown that the elasticity of the lifetime utility and price-dividend functions can be related to the instantaneous expected equity premium and its standard deviation. Thus, the second initial conditions can be set such that the expected equity premium and its standard deviation are equal to financial market data at the stationary mean of the long run risk variable. Finally, the risk free interest rate at the stationary mean of the long run risk variable can be used to set the discount factor for the representative individual or the price-dividend at the risk neutral stationary mean of the long run risk variable. This trade-off between setting the price-dividend or the discount factor means that the risk free interest rate is too high by about 2% when one matches the historic average price-dividend ratio. Thus, the risk free interest puzzle of Weil (1989) is still present in the long run risk model but it is not as severe as in the Mehra and Prescott (1985, 2003) model of stock returns.

The analytic solution to the BY model is compared with the solution of the CC model from CCH (2009). The main difference is that the price-dividend ratio is convex in the state (long run risk) variable in the BY model using the BKY parameters, while the price-dividend ratio is concave in the state variable (surplus consumption ratio) in the CC model using CC (1999) parameters. Both state variables are designed to capture the cyclical and long term trends in the economy, so that an expansion occurs when the state variable is high. In addition, the increase in the state variable is designed to persist for a period of time. Consequently, an increase in the state variable has predictable effects on stock returns. In particular, an increase in the state variable leads to a higher price-dividend ratio, but eventually the state variable reverts back towards its mean, so that the price-dividend ratio is predicted to fall in the future. The implications of this mechanism for the time varying properties of stock returns is dependent on the concavity (convexity) of the price-dividend function. When the price-dividend ratio is concave (convex) in the state variable, the expected equity premium and its standard deviation will be lower (higher) in an expansion. As a result the time series properties of stock returns are fundamentally different between the BY and CC models. Thus, these higher order polynomial approximations will significantly influence the choice of parameters such that each model best represents the statistical properties of stock returns.

Finally, the long term risk properties of the BY model are examined and compared with the analysis of Hansen and Scheinkman (2009). Hansen and Scheinkman develop an operator method which identifies the long term rate of return for a financial asset, which they call the asymptotic rate of return, rather than the typical focus on short term moments. They identify the asymptotic rate of return by solving a particular eigenvalue problem for a differential equation. The solution to this eigenvalue problem must be consistent with the risk neutral stationary distribution of the state variable. In this paper the asymptotic rate of return is found as a by-product of the solution to the lifetime utility function for the representative investor and standard asset pricing formulas. The representative investor's lifetime utility determines the state price process, so that the Feynman-Kac probabilistic (risk neutral) solution to the price-dividend ratio is equivalent to the solution of the ODE modeling the price-dividend function, see Duffie (2001). This probabilistic solution determines the stochastic process for the long run risk variable such that the risk neutral formula for the price-dividend is true. If one examines the discount factor for the risk neutral price-dividend ratio at the stationary mean of the risk neutral mean of the stochastic process, then one obtains the asymptotic rate of return for the price-dividend function.[8] As in Hansen, Heaton and Li (2008) this asymptotic rate of return consists of 1.) the risk free interest rate at the stationary mean of the long run risk variable, 2.) the mean growth rate of dividends at the same point, and 3.) an adjustment for risk. This adjustment for risk is dependent on a.) the correlation between dividends and the long run risk variable, b.) the elasticity of the lifetime utility with respect to the long run risk variable, and c.) the effect of lifetime utility on the state price process. All these terms are evaluated at the stationary mean of the long run risk variable. In addition, this new procedure provides all the characteristics of the rate of return on equity rather than just the asymptotic properties.

The paper is structured as follows. The next section of the paper explains how the analytic method is used to solve the long run risk model of BY. Section 3 states and proves all the above properties of the long run risk model. Section 4 undertakes a simulation of the long run risk model using the analytic method. In addition, the property of the BY model using the BKY parameters is contrasted with the CC model using the CC (1999) parameters. Finally, we examine the long run rate of return implied by the solution to the BY model. Section 5 provides a summary of the results.

[8]The long run rate of return for the long term risk free asset of Cochrane (2008) is also found using this method.

2 Asset Pricing Model

This work is based on the BY model in which C consumption, D dividends, and x the long run risk variable follow the stochastic processes

$$d(\ln C) = (x + \bar{x})\, dt + \sigma(x)\, d\tilde{\omega}_1, \tag{1}$$

$$d(\ln D) = (\phi x + \bar{x})\, dt + \varphi_d \sigma(x)\, d\tilde{\omega}_2, \tag{2}$$

$$dx = -\kappa x\, dt + \varphi_e \sigma(x)\, d\tilde{\omega}_3. \tag{3}$$

Here, $\bar{x} > 0$ is the given stationary point for the consumption and dividend growth processes. The condition on the parameter $\phi > 1$ allows the dividend growth to be more sensitive to the long run risk variable, x. The stationary level of the long run risk variable is zero, while the parameter $\kappa > 0$ determines the persistence of this variable. The probability space $(\Omega, \mathcal{F}, P)$ is given together with a family $\{\mathcal{F}_t : t \in [0, \infty]\}$ of σ-algebras which is a filtration of the standard Brownian motion $d\omega = (d\tilde{\omega}_1, d\tilde{\omega}_2, d\tilde{\omega}_3)$. Assume that $d\tilde{\omega}_1 = d\omega_1$, $d\tilde{\omega}_2 = \rho_{cd}\, d\omega_1 + \sqrt{1 - \rho_{cd}^2}\, d\omega_2$ and $d\tilde{\omega}_3 = \rho_{xc}\, d\omega_1 + \sqrt{1 - \rho_{xc}^2}\, d\omega_3$ where $d\omega_1, d\omega_2$ and $d\omega_3$ are independent Brownian motions. As a result, the parameters ρ_{cd} and ρ_{xc} determine the instantaneous correlation between consumption growth, and dividend growth or the long run risk variable, respectively, while $\rho_{xc}\rho_{cd}$ is the correlation between dividend growth and the long run risk variable.[9] The parameters φ_d and φ_e determine the standard deviation of dividend growth and the long run risk variable, respectively.

BY argue that the consumption growth volatility changes over time. In particular, they find that the price-dividend ratio predicts future consumption growth volatility, σ, out to five years. In addition, a higher price-dividend ratio leads to lower consumption growth volatility. Based on this evidence they model the variance of consumption growth as a first order autoregressive process. The current work captures this empirical property by making variance a logistic function of the long run risk variable in which the variance falls when the

[9] BY allow for correlation between consumption an dividends, while Wachter (2002) and Bekaert, Engstrom, and Xing (2009) find evidence of correlation between consumption and the long run risk variable.

long run risk variable increases

$$\sigma^2(x) = \sigma_0^2 \frac{1+a}{a+e^{bx}}, \tag{4}$$

where a and b are positive parameters. We will show that the price-dividend ratio is an increasing function of the long run risk variable. As a result, consumption volatility falls when the price-dividend ratio is high in response to a high realization of the long run risk variable. In addition, the persistence of shocks to the long run risk variable means that the long run risk variable is high for a long period of time. Thus, our model of consumption volatility captures the empirical evidence that consumption volatility increases when the price-dividend ratio increases.

2.1 Lifetime Utility of Investor

The lifetime utility of the representative investor, V_t, follows the backward stochastic differential equation (SDE)

$$dV_t = -f(C,V)\,dt + \sigma_V(t)\,d\omega_t \tag{5}$$

given the terminal condition

$$V_T = \Gamma(C_T),$$

which is the terminal utility of the investor. The instantaneous mean is given by

$$f(C,V) = \frac{\beta}{\rho}\frac{C^\rho - [\alpha V]^{\rho/\alpha}}{[\alpha V]^{\rho/\alpha - 1}}.$$

The standard deviation for the lifetime utility $\sigma_V(t)$ is to be determined later (see footnote 10).

To interpret the SDE (5) consider the backward stochastic differential equation from Duffie and Epstein (1992a,b) and Duffie and Lions (1992).

$$dV_t = \left[-\bar{f}(C_t,V_t) - \frac{1}{2}\bar{A}(V_t)\|\sigma_V(t)\|^2\right] dt + \sigma_V(t)\,d\omega_t \text{ given } V_T = \Gamma(C_T), \tag{6}$$

where $\sigma_V(t)\,d\omega_t$ is the instantaneous standard deviation of lifetime utility and $\|\sigma_V(t)\|^2$ is the quadratic variation in the lifetime utility.[10] Following Kreps and Porteus (1978) it is assumed that

(1) the immediate benefit to the investor is $\bar{f}(C(t), V(t)) \equiv \frac{\beta}{\rho}\frac{C^\rho - V^\rho}{V^{\rho-1}}$, where $\psi = \frac{1}{1-\rho}$ is the investor's intertemporal rate of substitution.

(2) the investor's response to uncertainty is $\bar{A}(V(t)) \equiv -\frac{1-\alpha}{V}$, where $1-\alpha$ measures the investor's aversion to risk, and

(3) $V_T \equiv \Gamma(C_T)$ is the investor's utility in the terminal period T which is assumed to be $\frac{\xi^{\frac{\alpha}{\rho}}}{\alpha} C_T^\alpha$ for simplicity.

Duffie and Epstein (1992) define the integrable semimartingale for lifetime utility V as the stochastic process when it uniquely satisfies the integral equations

$$V_t^T = E\left[\int_t^T \left(\bar{f}(C_s, V_s) + \frac{1}{2}\bar{A}(V_s)\|\sigma_V(s)\|^2\right) ds|\mathcal{F}_t\right] \text{ a. s., } t \in [0, T]. \tag{7}$$

The representative investor is assumed to live forever so that following Duffie and Epstein (1992a, 1992b) define the infinite horizon lifetime utility process V_t as the pointwise limit, $\lim_{T\to\infty} V_t^T$. Following Duffie and Lions (1992) when the lifetime utility is well defined there is a measurable function such that

$$V_t = J(C_t, x_t), \quad t \geq 0. \tag{8}$$

In addition, Duffie and Epstein (1992a, 1992b) and Duffie and Lions (1992) use the change of variable

$$H(C) = \varphi\left(J(C)\right), \quad \text{where } \varphi(z) = \frac{z^\alpha}{\alpha}, \tag{9}$$

so that the equivalent aggregator is given by

$$f(C, V) = \frac{\beta}{\rho}\frac{C^\rho - [\alpha V]^{\rho/\alpha}}{[\alpha V]^{\rho/\alpha - 1}} \quad \text{and} \quad A(V) = 0. \tag{10}$$

[10] This standard deviation and quadratic variation are a by-product of Ito's lemma. When $V_t = V(C, x)$, then $\sigma_V(t)\, d\omega_t = CV_C\sigma(x)\, d\tilde{\omega}_1 + V_x\varphi_e\sigma(x)\, d\tilde{\omega}_3$. Here, V_j refers to the partial derivative of V with respect to the variable $j = C, x$. In this case the quadratic variation in lifetime utility is $\|\sigma_V(t)\|^2 = \left[(CV_C)^2\sigma^2(x) + 2CV_CV_x\varphi_e\sigma^2(x) + V_x^2\varphi_e^2\sigma^2(x)\right]$.

Consequently, the stochastic differential equation (6) becomes

$$dV_t = -f(C, V)\,dt + \sigma_V(t)\,d\omega_t \text{ given } V_T = \Gamma(C_T). \tag{11}$$

As a result, the lifetime utility function of the investor satisfies the PDE

$$E\left[dV(C, x)\right] = -f(C, V(C, x))\,dt, \tag{12}$$

where the differential of lifetime utility is found using Ito's Lemma.

Duffie and Epstein (1992b), Campbell and Viciera (2002), and Campbell, *et al.* (2004) demonstrate that the solution to this differential equation (12) is of the separable form

$$V(C, x) = C^{\alpha} v(x), \tag{13}$$

using a homogeneity property of the aggregator (10) and the linear homogeneity of the optimal consumption decision of the investor with respect to her wealth. This results in a highly nonlinear ordinary differential equation in $v(x)$. Following an insight from Fisher and Gilles (1998), Campbell and Viciera (2002), and Campbell, *et al.* (2004) this nonlinear differential equation can be simplified further by using the following change of variables

$$v(x) = \frac{1}{\alpha} g(x)^{\frac{\alpha}{\rho}}.\text{[11]} \tag{14}$$

Using these properties of the solution and Ito's lemma one proves that $g(x)$ solves the nonlinear ODE

$$\begin{aligned} 0 = \beta + \left[\rho x + \rho\bar{x} + \alpha\rho\frac{\sigma^2(x)}{2} - \beta\right] g(x) + \left[-\kappa x + \sigma^2(x)\varphi_e\rho_{xc}\alpha + \left(\frac{\alpha}{\rho} - 1\right)\frac{\varphi_e^2\sigma^2(x)}{2}\frac{g'(x)}{g(x)}\right] g'(x) \\ + \frac{\varphi_e^2\sigma^2(x)}{2} g''(x), \end{aligned} \tag{15}$$

with a given terminal condition

$$g(x_T) = \xi.$$

The details of the derivation of this ODE are provided in the appendix.

[11] $g(c(t), \psi(t))$ in Fisher and Gilles is given by $C^{\alpha}\frac{1}{\alpha}g(x)^{\frac{\alpha}{\rho}}$. Note they also use $\varphi(z) = \frac{z^{\alpha}-1}{\alpha}$ which does not effect our solution procedure.

2.2 The Probabilistic Form of Lifetime Utility

Following Karatzas and Shreve (1988), and Revuz and Yor (1991) as summarized by Duffie (2001, Appendix D and E), define the differential operator associated with (15) as

$$\mathcal{D}g(x) = \mu_g(x)g'(x) + \frac{\varphi_e^2\sigma^2(x)}{2}g''(x), \tag{16}$$

where

$$\mu_g(x) = \left[-\kappa x + \sigma^2(x)\varphi_e\rho_{xc}\alpha + \left(\frac{\alpha}{\rho} - 1\right)\frac{\varphi_e^2\sigma^2(x)}{2}\frac{g'(x)}{g(x)}\right]. \tag{17}$$

By Girsanov's Theorem (Duffie (2001, p.337-338)) the long run risk variable follows the stochastic differential equation (SDE)

$$dx = \mu_g(x)dt + \varphi_e\sigma(x)d\hat{\omega}_3, \tag{18}$$

where the twisted Brownian motion is

$$d\hat{\omega}_3 = d\tilde{\omega}_3 - \frac{\mu_g(x) + \kappa x}{\varphi_e\sigma(x)}dt. \tag{19}$$

The SDE (18) induces a new probability distribution for x in which the stationary mean of x has been changed from $x = 0$ to $x = x_g$ where x_g is the x such that $\mu_g(x_g) = 0$. Call x_g the risk neutral stationary mean of x under $g(x)$.

The Feynman-Kac formula for (15) is given by

$$g(x) = \lim_{T\to\infty} E_{x,g}\left[\int_t^T \beta_{t,s}^g(x_s)\beta ds + \beta_{t,T}^g(x_T)\xi\right] = \beta E_{x,g}\left[\int_t^\infty \beta_{t,s}^g(x_s)ds\right], \tag{20}$$

where

$$\beta_{t,s}^g(x) = \exp\left[-\int_t^s r_g\left(x_\tau\right)d\tau\right] = \exp\left[-(\beta - \rho\bar{x})(s-t) + \int_t^s \left[\rho x_\tau + \alpha\rho\frac{\sigma^2(x_\tau)}{2}\right]d\tau\right]. \tag{21}$$

$E_{x,g}$ denotes the expectation condition on x being the solution to the SDE (18) given initial condition $x_t = x$. Duffie (2001) refers to $g(x)$ as the probabilistic solution of the ODE. He also provides conditions on $\mu_g(x), \sigma_g(x), r_g(x)$ for a unique solution to the Cauchy problem (15) in the Feynman-Kac form (20). These conditions deal with Lipschitz and growth properties on these functions such that the integrals in (20) and (21) exist.

The main problem is whether the solution $g(x)$ exists. Since $\beta > 0$ the Monotone Convergence Theorem (Folland (1984, p.49)) assures that the limit in (20) exists as long as the transversality condition

$$\lim_{T\to\infty} E_x\left[\beta^g_{t,T}\right] = 0 \tag{22}$$

is satisfied. There are two properties of the Feyman-Kac solution which must be addressed so as to evaluate this limit. First, what is the property of $r_g(x)$ for $x \in \mathbb{R}$. Second, what is the probability distribution of x induced by the SDE (18). For the first property $\beta > \rho\bar{x}$ assures that the deterministic term in (21) converges to zero as $s \to \infty$. This condition places a restriction on the preference parameters of the representative investor, since $\bar{x}$ is the long term growth of consumption for the economy. In this case, the existence of

$$\lim_{T\to\infty} E_x\left[\exp\left[\int_t^T \left[\rho x_\tau + \alpha\rho\frac{\sigma^2(x_\tau)}{2}\right] d\tau\right]\right] \tag{23}$$

is sufficient for the transversality condition (22) to hold. The main problem is that the probability distribution of x induced by the SDE (18) is unknown, since it is dependent on the risk adjustment by the investor's with respect to the long run risk variable, $\left(\frac{\alpha}{\rho} - 1\right)\frac{\varphi_e^2\sigma^2(x)}{2}\frac{g'(x)}{g(x)}$. As a result, Monte Carlo methods suggested by Duffie (2001, Chapter 12) cannot be used directly to approximate a solution. Thus, solving for the lifetime utility of the investor is subject to a chicken and egg dilemma.

2.3 Approximating the Investor's Lifetime Utility

Our approach to solving for the investor's lifetime utility is to formulate the ODE (15) as an initial value problem, IVP, where $g(0) = g_0$ and $g'(0) = g_1$. $x_0 = 0$ is the stationary mean of the SDE (3).[12] The choice of $\frac{g_1}{g_0}$ will be related to the equity premium and its standard deviation such that one obtains the lifetime utility of the representative investor consistent with financial market data (see section 2.7). Given this value for $\frac{g_1}{g_0}$, g_0 is chosen to satisfy

[12] The risk neutral stationary mean of the SDE (18) cannot be used since it is not known until one knows the solution $g(x)$. Of course one can bootstrap on our solution, since our solution identifies accurate values for g(x) and g'(x) at the new stationary mean of the SDE (18).

the Feynman-Kac formula (20) at the risk neutral stationary mean, x_g, for the long run risk variable.

In the next section the analytic method of Cauchy-Kovalevsky is used to show that this nonlinear IVP has an analytic solution with a some radius of convergence, $r_g > 0$. An analytic solution near the point $x_0 = 0$ can be represented by its Taylor series, that is

$$g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(x_0)}{k!} x^k, |x| < r_g. \tag{24}$$

The method in CCH (2009) cannot be used to find the radius of convergence for the representative investor's lifetime utility, since the ODE (15) is nonlinear. As a result, an alternative method is developed for estimating the radius of convergence r_g which is at least as big as $7\varphi_e\sigma_0$ for BKY's parameters. In the next section, an algorithm is developed to approximate the analytic solution with a Taylor polynomial approximation

$$T_{g,n}(x) = \sum_{k=0}^{n} g_k (x - x_0)^k. \tag{25}$$

Here the g_n are determined by a recursive rule that satisfy ODE (15). The algorithm yields a lower bound for the radius of convergence r_g of the analytic solution (24). Also this method gives an estimate of the approximation errors

$$\max_{|x-x_0| \le r_g} |g(x) - T_{g,n}(x)| \text{ and } \max_{|x-x_0| \le \nu r_g} |g'(x) - T'_{g,n}(x)| \text{ for some } \nu < 1. \tag{26}$$

Thus, the mathematical properties of the solution is determined for $|x| < r_g$ so that an accurate approximation for the lifetime utility of the investor can be calculated within the interval of convergence.

Whether or not this solution to the IVP satisfies the Feynman-Kac formula (20) for the investor's problem (15) needs to be checked. The first issue is that the solution to the IVP is known for the long run risk variable, x, within the interval $[-r_g, r_g]$, while the probability distribution of x induced by the SDE (18) is defined on $\mathbb{R}$. Given $g(x)$ for $|x| < r_g$, an extension on $\mathbb{R}$ is chosen as follows:[13]

$$g(x) = \begin{cases} g(x_*) & \text{for } x \leq x_* = -r_g\,, \\ g(x^*) & \text{for } x \geq x^* = r_g \end{cases} \tag{27}$$

$g'(x)$ is defined at every point except x_* and x^*, however these values will not influence the value of (20). It is straightforward then to evaluate the drift (17) and the discount rate $r\,(x_\tau)$ in (21) to determine whether or not the conditions (Duffie (2001, p. 345)) for the uniqueness of the Feynman-Kac solution (20) are satisfied. The Lipschitz and growth conditions are satisfied for the volatility of consumption growth (4) since it was chosen to be bounded on $\mathbb{R}$.

2.4 State Price Process

Given the properties of the lifetime utility of the representative investor the state price process for the continuous time BY model can be found using Duffie and Epstein (1992b). Duffie and Epstein (1992b) demonstrate that the state price process for a representative investor satisfies the SDE

$$\frac{d\Lambda}{\Lambda} = f_V(C,V)\,dt + \frac{\mathcal{D}f_C(C,V)}{f_C(C,V)}, \tag{28}$$

where $\mathcal{D}f_C(C,V)$ is determined by Ito's Lemma. The state price vector at a given time t is found by integrating (28)

$$\Lambda_t(C_t,V_t) = \exp\left[\int_0^t f_V(C_s,V_s)\,ds\right] f_C(C_t,V_t)\frac{\Lambda_0}{f_C(C_0,V_0)}, \tag{29}$$

where it is assumed that $f_C(C_0,V_0) = \Lambda_0$.[14] As a result, risk premium for financial assets vary because of

(1) the marginal utility of current consumption $f_C(C,V) = \frac{\beta C^{\rho-1}}{(\alpha V)^{\rho/\alpha-1}}$ and

(2) the rate at which the future is discounted $f_V(C,V) = -\frac{\beta}{\rho}\left[\frac{(\rho-\alpha)C^\rho}{(\alpha V)^{\rho/\alpha}} + \alpha\right]$.

[13]Such an extension of the analytic solution is implicit in the approximate solution of Benzoni, Collin-Dufresne, and Goldstein (2005), since the Campbell and Shiller (1988) approach uses a first order Taylor polynomial without knowing the accuracy of such an approximation.

[14]Also see Duffie, Schroder, Skiadas (1997), Duffie and Skiadas (1994) for a discussion of using the state price process to value securties in the stochastic differential utility framwork.

When $\alpha = \rho$ the second effect disappears, so that in this constant relative risk averse case there is less fluctuation in the risk premium for financial assets.

By using the Kreps-Porteus functional form (10), the separation of variables (13), and the change of variable (14), the application of Ito's Lemma using the stochastic processes for consumption growth (1) and the long run risk variable (3) yields the stochastic process for the state price process.

$$\frac{d\Lambda}{\Lambda} = \mu_\Lambda(x)\,dt + \sigma(x)(\alpha-1)\,d\tilde{\omega}_1 + \frac{\sigma(x)\varphi_e(\alpha-\rho)}{\rho}\frac{g'(x)}{g(x)}\,d\tilde{\omega}_3, \tag{30}$$

where

$$\begin{aligned}\mu_\Lambda(x) = &- R^b(x) = (\rho-1)x + (\rho-1)\bar{x} - \beta + \frac{\sigma^2(x)}{2}(\alpha\rho - 2\alpha + 1)\\ &+ \rho_{xc}\varphi_e\sigma^2(x)\frac{\rho-\alpha}{\rho}\frac{g'(x)}{g(x)} + \frac{\varphi_e^2\sigma^2(x)}{2}\frac{\rho-\alpha}{\rho}\left(\frac{g'(x)}{g(x)}\right)^2.\end{aligned} \tag{31}$$

Here, $R^b(x)$ is the return on the risk free bond, since there would be no exposure to the fundamental shocks to consumption growth, $\sigma(x)d\tilde{\omega}_1$ or long term risk variable, $\sigma(x)\varphi_e d\tilde{\omega}_3$. The pricing of other financial assets is dictated by the correlation of the asset with fluctuations in consumption growth $\sigma(x)(\alpha-1)$ which is the risk exposure found in the constant relative risk aversion case, i.e., $\alpha = \rho$. The exposure to the standard deviation of the long run risk variable $\frac{(\alpha-\rho)}{\rho}\frac{g'}{g}$ is only relevant under recursive utility in which $\alpha \neq \rho$.[15] This new exposure to risk is the product of the elasticity of lifetime utility with respect to the long run risk variable, $\frac{1}{V(C,x)}\frac{\partial V}{\partial x} = \frac{\alpha}{\rho}\frac{g'(x)}{g(x)}$, and the effect of lifetime utility on the state price process $\frac{\partial \Lambda}{\partial V} = \frac{\alpha-\rho}{\alpha}$. Thus, the pricing of risk is dependent on knowing an accurate solution to the representative investor's lifetime utility $C^\alpha g(x)^{\frac{\alpha}{\rho}}$.

2.5 Pricing Equity under Stochastic Differential Utility and Long Run Risk

Duffie and Epstein (1992b) show that an asset promising dividends D has a price given by

$$P(t) = E_t\left[\int_t^\infty \frac{\Lambda_s}{\Lambda_t} D(s)\,ds\right] \quad \text{for } t \geq 0. \tag{32}$$

[15] See Hansen, Heaton and Li (2008) for a further discussion of this point.

In the long run risk model the dividends for a share of stock following (2) is given by

$$D(s) = D(t) \exp\left\{ \int_t^s \left[\phi x_\tau + \bar{x} + \frac{\varphi_d^2 \sigma^2(x_\tau)}{2} \right] d\tau + \varphi_d \int_t^s \sigma(x_\tau)\, d\tilde{\omega}_2(\tau) \right\}, \tag{33}$$

so that an individual stock or portfolio of stocks as in Bansal, Dittmar, Lundblad (2005), Bansal, Dittmar, Kiku (2007), Lettau and Wachter (2007), and Hansen, Heaton and Li (2008) can be captured by variation in the cash flow relative to the long run risk variable, ϕ, or the volatility of dividends φ_d. One can also account for various models of cointegration between consumption and dividend by restricting these parameters and placing an additional parameter on the stationary mean of dividend growth $\phi_1 \bar{x}$.

Duffie and Epstein (1992b) demonstrate the relation between the stock price process (32) and the differential equation

$$0 = \frac{D}{P} dt + E_t \left(\frac{d\Lambda}{\Lambda} + \frac{dP}{P} + \frac{d\Lambda}{\Lambda} \frac{dP}{P} \right) = \left(\frac{D}{P} - R^b \right) dt + E_t \left(\frac{dP}{P} + \frac{d\Lambda}{\Lambda} \frac{dP}{P} \right). \tag{34}$$

In the appendix this relation is used to derive the ODE for the price-dividend ratio $p \equiv \frac{P}{D}$ which is given by

$$\begin{aligned} 1 &+ \left\{ \phi x + \bar{x} + \frac{\sigma^2(x)}{2} \varphi_d^2 + \sigma^2(x) \varphi_d \rho_{cd} \left[\alpha - 1 + \frac{\varphi_e \rho_{xc} (\alpha - \rho)}{\rho} \frac{g'(x)}{g(x)} \right] - R^b(x) \right\} p(x) \\ &+ \left\{ \sigma^2(x) \varphi_e \rho_{xc} (\alpha - 1 + \varphi_d \rho_{cd}) - \kappa x + \left(\frac{\alpha}{\rho} - 1 \right) \frac{\varphi_e^2 \sigma^2(x)}{2} \frac{g'(x)}{g(x)} \right\} p'(x) \\ &+ \frac{\sigma^2(x)}{2} \varphi_e^2 p''(x) = 0 \end{aligned} \tag{35}$$

subject to $p(x(T)) = p_T$.

2.6 The Probabilistic Form of the Price-Dividend Ratio

The probabilistic solution to the price-dividend function can also be found as in the case of the lifetime utility function. The differential operator for the price-dividend ratio is

$$\mathcal{D} p(x) = \mu_p(x) p'(x) + \frac{\varphi_e^2 \sigma^2(x)}{2} p''(x), \tag{36}$$

where

$$\mu_p(x) = \left[-\kappa x + \sigma^2(x)\varphi_e\rho_{xc}(\alpha - 1 + \varphi_d\rho_{cd}) + \left(\frac{\alpha}{\rho} - 1\right)\frac{\varphi_e^2\sigma^2(x)}{2}\frac{g'(x)}{g(x)}\right]$$
$$= \mu_g(x) + \sigma^2(x)\varphi_e\rho_{xc}(\varphi_d\rho_{cd} - 1). \tag{37}$$

The extra effect on the instantaneous mean of the twisted long run risk variable, $\sigma^2(x)\varphi_e\rho_{xc}(\varphi_d\rho_{cd} - 1)$, comes from the correlation between dividend growth and the price-dividend ratio, and the correlation between consumption growth and the state price process.

The long run risk variable now follows the stochastic differential equation (SDE)

$$dx = \mu_p(x)dt + \varphi_e\sigma(x)d\hat{\omega}_3, \tag{38}$$

where the twisted Brownian motion is

$$d\hat{\omega}_3 = d\tilde{\omega}_3 - \frac{\mu_p(x) + \kappa x}{\varphi_e\sigma(x)}dt. \tag{39}$$

The Feynman-Kac solution to (35) is given by

$$p(x) = \lim_{T\to\infty} E_{x,p}\left[\int_t^T \beta_{t,s}^p ds + \beta_{t,T}^p p_T\right] = E_{x,p}\left[\int_t^\infty \beta_{t,s}^p ds\right], \tag{40}$$

where

$$\beta_{t,s}^p = \exp\left[-\int_t^s r_p(x_\tau)\, d\tau\right]$$
$$= \exp\left\{\int_t^s \left[\phi x_\tau + \bar{x} + \frac{\varphi_d^2\sigma^2(x_\tau)}{2} - R^b(x_\tau) + \sigma^2(x_\tau)\varphi_d\rho_{cd}\left(\alpha - 1 + \frac{\varphi_e\rho_{xc}(\alpha-\rho)}{\rho}\frac{g'(x_\tau)}{g(x_\tau)}\right)\right] d\tau\right\}. \tag{41}$$

$E_{x,p}$ denotes the expectation condition on x being the solution to the SDE (38) given initial condition x_t.

The price-dividend ratio (40) has two components to it. The first is the traditional Gordon growth model which bases the price-dividend ratio on the future growth rate of dividends $\phi x_\tau + \bar{x} + \frac{\varphi_d^2\sigma^2(x_\tau)}{2}$ relative to the future risk free interest rate $R^b(x_\tau)$. The second component is the future discount for risk $\sigma^2(x_\tau)\varphi_d\rho_{cd}\left[\alpha - 1 + \frac{\varphi_e\rho_{xc}(\alpha-\rho)}{\rho}\frac{g'(x_\tau)}{g(x_\tau)}\right]$ which is the correlation between dividend growth and the lifetime utility of the investor.

Given the solution to the lifetime utility function $g(x)$ one could use the Monte Carlo method suggested by Duffie (2001) to approximate the price-dividend function (40). However, the accuracy of the solution is not known. Instead the approach of CCH (2009) is used in the next section to approximate the price-dividend function with a Taylor polynomial approximation for a given radius of convergence, r_p. In this case (35) is viewed as an IVP whose solution is dependent on $p(0) = p_0$ and $p'(0) = p_1$. Following Theorem 3.1 in CCH (2009) the ODE (35) has a unique analytic solution $p(x)$ near the point $x_0 = 0$ with with radius of convergence, r_p, equal to at least the smallest radius of convergence of the coefficients and the forcing term. In this particular case the radius of convergence of the coefficients and forcing terms is $\frac{r_g}{3}$. Consequently, the radius of convergence of the lifetime utility determines where the price-dividend function is analytic. Thus, we can represent the solution to (35) by a power series

$$p(x) = \sum_{k=0}^{\infty} p_k x^k \text{ for } |x| < r_p, \tag{42}$$

in which the coefficients, p_k, are determined by a recursive rule following the ODE (35).

To complete the numerical solution to the long run risk model with stochastic differential utility the analytic solution is represented as a Taylor polynomial approximation

$$p_n(x) = \sum_{k=0}^{n} p_k x^k \quad \text{for } |x| < \nu r_p \text{ with } 0 < \nu < 1. \tag{43}$$

Finally, Corollary 3.2 of CCH (2009) is used to determine the error in the Taylor series remainder.

$$R_n(x) = p(x) - p_n(x) = \sum_{k=n+1}^{\infty} p_k x^k. \tag{44}$$

2.7 Initial Condition

The last step is to develop initial conditions. Here the initial conditions are related to financial market data and the Feynman-Kac equations for the lifetime utility (20) and the price-dividend ratio (40). Following the same procedure as in CCH (2009) the SDE for the equity premium is given by

$$d\left[R^e(x) - R^b(x)\right] = \left[E_t[R^e(x)] - R^b(x)\right] dt + \varphi_d \sigma(x) d\tilde{\omega}_2 + \varphi_e \sigma(x) \frac{p'(x)}{p(x)} d\tilde{\omega}_3. \tag{45}$$

The instantaneous expected equity premium is given by

$$\begin{aligned} E_t[R^e(x)] - R^b(x) =& - E_t\left[\frac{d\Lambda dp}{\Lambda p} + \frac{d\Lambda dD}{\Lambda D}\right] \\ =&\sigma^2(x)\left[\varphi_e \rho_{xc}(1-\alpha) + \frac{\varphi_e^2(\rho-\alpha)}{\rho}\frac{g'(x)}{g(x)}\right]\frac{p'(x)}{p(x)} \\ &+ \sigma^2(x)\varphi_d\rho_{cd}\left[1 - \alpha + \frac{\varphi_e\rho_{xc}(\rho-\alpha)}{\rho}\frac{g'(x)}{g(x)}\right]. \end{aligned} \tag{46}$$

The instantaneous variance for the equity premium is

$$\Sigma^2(x) = \sigma^2(x)\varphi_d^2 + 2\sigma^2(x)\varphi_e\varphi_d\rho_{xc}\rho_{cd}\frac{p'(x)}{p(x)} + \sigma^2(x)\varphi_e^2\left[\frac{p'(x)}{p(x)}\right]^2. \tag{47}$$

The equity premium and its volatility are functions of the elasticities of the lifetime utility $\frac{g'(x)}{g(x)}$ and the price-dividend ratio $\frac{p'(x)}{p(x)}$. To implement the analytic procedure one needs the value of these elasticities at the stationary mean of the long run risk variable $x = 0$. Following CCH (2009) historic observations can be used to determine estimates of $E_t[R^e(0)] - R^b(0)$ and $\Sigma^2(0)$ at the stationary mean $x = 0$. First, the volatility of stock returns $\Sigma^2(0)$ is used to set the elasticity of the price-dividend ratio at the stationary mean for the long run risk variable using (47).

$$\frac{p'(0)}{p(0)} = \frac{-\sigma_0\varphi_d\rho_{xc}\rho_{cd} + \sqrt{\sigma_0^2\varphi_d^2\rho_{xc}^2\rho_{cd}^2 + \Sigma_0^2 - \sigma_0^2\varphi_d^2}}{\sigma_0\varphi_e}. \tag{48}$$

Given the elasticity of the price-dividend ratio at this stationary mean, the elasticity of the lifetime utility can be found from (46).

$$\frac{g'(0)}{g(0)} = \frac{\rho\{R^b(0) - E_t[R^e(0)]\} - \sigma_0^2\rho(\alpha-1)(\varphi_e\rho_{xc}\frac{p'(0)}{p(0)} + \varphi_d\rho_{cd})}{\sigma_0^2\varphi_e(\alpha-\rho)(\varphi_e\frac{p'(0)}{p(0)} + \varphi_d\rho_{xc}\rho_{cd})}. \tag{49}$$

The next initial condition is a value for the lifetime utility function $g(0) = g_0$ at the stationary mean $x = 0$. This is accomplished by using the Feynman-Kac solution for the lifetime utility (20) to estimate $g(0)$. First, the risk neutral stationary mean, x_g, of the stochastic process (18) is found such that $\mu_g(x_g) = 0$. This takes two steps in the computer algorithm:

(1) Use the analytic method to approximate $g(x)$ with (25) given $g(0)$ from (20) when $x = 0$.[16] With this solution for lifetime utility the instantaneous mean of the risk neutral probability, x_g is given by the zero of (17), i.e., $\mu(x) = 0$.

(2) Approximate (20) at the risk neutral stationary point x_g so that $g(0)$ can be calculated using the Feynman-Kac equation (20). Set $g_0 = g(0)$. Further iterations on this algorithm did not change g_0, since the elasticity of the lifetime utility (49), and the mean of the risk neutral probability (17) are set independent of g_0.

The final initial condition is the price-dividend function $p(x)$ at the stationary point $x = 0$. Again the Feynman-Kac formula (40) is used to estimate $p(0)$. In this case only one step is necessary, since the elasticity of lifetime utility (48) is already known. As a result, the risk neutral stationary mean, x_p, for the price-dividend function can be found as the zero of the instantaneous mean for the risk neutral distribution for the price-dividend function (37), i.e., $\mu_p(x_p) = 0$. The initial value for the price-dividend function $p(0) = p_0$ is given by approximating (40) at the risk neutral stationary point x_p. Thus, all the initial conditions are available to apply the analytic method to solve the representative investor's lifetime utility and the price-dividend function.

[16] To illustrate the argument (21) is evaluated at the specific point so that $g(0) = 1.432968906$ can be calculated at this specific point. The same approximation is made for the price-dividend ratio. A more accurate value for g(0) can be found by using the Monte Carlo method suggested by Duffie (2001). If T is broken up into $100,000$ intervals and the integral over x in (20) is approximated by the trapezoidal rule than $g(0) = 1.433835331$. The Matlab program takes 437 seconds. This value does not change out to 16 digits when $70,000$ intervals were used. Only a few numbers in this paper change at the fourth digit so that this approximation does not material effect the results.

3 Results

In this section, all the claims made in the outline are proved starting with the analyticity of the lifetime utility function. For this it is worth recalling one of the most general, and powerful results in PDE theory.

Theorem 3.1. *[Cauchy-Kovalevsky] The IVP for the following m^{th} order nonlinear partial differential equation in $\mathbb{R}^{n+1}$*

$$\partial_t^m u = F\big(x, t, \{\partial_x^\alpha \partial_t^j u\}_{|\alpha|+j\leq m,\, j<m}\big) \tag{50}$$

$$\partial_t^j u(x,0) = u_j(x), \quad 0 \leq j \leq m-1, \quad x \in \mathbb{R}^n, t \in \mathbb{R}, \tag{51}$$

has a unique solution in the space of analytic functions near zero in $\mathbb{R}^{n+1}$, if all u_j are analytic near zero in $\mathbb{R}^n$, and F is analytic near $(0, 0, \{p_x^\alpha u_j(0)\}_{|\alpha|+j\leq m,\, j<m})$.

Here $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}$ is the multi-index with $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$.

The idea of the proof of this theorem is very simple. First, one can determine the coefficients of a formal power series solution

$$u(x,t) = \sum_{\alpha\in\mathbb{N}_0^n,\, j\geq 0} c_{\alpha,j} x^\alpha t^j, \tag{52}$$

uniquely by deriving the recurrence relation for the coefficients. Then, one finds a simpler "dominant" IVP which can be solved explicitly to show that its solution is analytic in a neighborhood of zero. This implies that the formal power series solution to the given IVP is also analytic in the same neighborhood of zero. The size of this neighborhood $(t_1, x_1, \cdots, x_n)$ is difficult to determine and has to be estimated for the particular case under consideration. In the Appendix this idea is implemented in the special case of second-order linear partial differential equations in $\mathbb{R}^2$ with analytic coefficient. Part of the motivation for this choice is that such equations arise in asset pricing theory. They are derived by using Ito's Lemma as in (15) and (35).

3.1 Solving the Investor's Lifetime Utility

The ODE for the lifetime utility of the representative investor (15) together with the initial conditions $g(0) = g_0$ and $g'(0) = g_1$, given by (49), is a special case of the IVP (50), so that it has a unique solution in the space of analytic functions near zero in $\mathbb{R}$. The main problem is showing that its radius of convergence is large enough such that $g(x)$ is analytic for reasonable values for the long run risk variable, x. In this subsection a recursive rule is found for determining the coefficients g_k. This rule is then used to find a dominant power series whose radius of convergence can be estimated from the data. Thus, all the ingredients necessary to find an accurate Taylor polynomial approximation (25) are established for the investor's lifetime utility.

One problem with regards to the ODE (15) is the small value of the variance of the long run risk variable $\phi_e \sigma^2(x)$ which multiplies the second derivative of the lifetime utility function, $g''(x)$. As a result, the recursive rule would involve division by a small number which would reduce the accuracy of the approximation. To avoid this problem one introduces the change of variable

$$x = \epsilon \varphi_e \sigma_0 s, \text{ where } \epsilon > 1.$$

Making this change of variable and substituting the logistic formula (4) for the variance of consumption growth,

$$\sigma^2(x) = \sigma_0^2 \frac{1+a}{a + e^{bx}},$$

yields

$$\begin{aligned} g(s)g''(s) =& A_0(g'(s))^2 + (B_2 s e^{Bs} + B_1 s + B_0) g(s) g'(s) + \\ & (C_3 s e^{Bs} + C_2 e^{Bs} + C_1 s + C_0) g^2(s) + (D_1 e^{Bs} + D_0) g(s) \\ & g(0) = g_0 \text{ and } g'(0) = \epsilon \sigma_0 \varphi_e g_1. \end{aligned} \tag{53}$$

The constants in (53) are as follows:

$$\begin{array}{ccc} A_0 = 1 - \alpha/\rho & B_0 = -2\epsilon\sigma_0\rho_{xc}\alpha & B_1 = 2\epsilon^2\kappa a/(1+a) \\ B_2 = 2\epsilon^2\kappa/(1+a) & C_0 = 2\epsilon^2(\beta - \rho\bar{x})a/(1+a) - \epsilon^2\sigma_0^2\alpha\rho & C_1 = -2\epsilon^3\sigma_0\varphi_e\rho a/(1+a) \\ C_2 = 2\epsilon^2(\beta - \rho\bar{x})/(1+a) & C_3 = -2\epsilon^3\sigma_0\varphi_e\rho/(1+a) & D_0 = -2\epsilon^2\beta a/(1+a) \\ D_1 = -2\epsilon^2\beta/(1+a) & B = b\epsilon\sigma_0\varphi_e & \end{array} \tag{54}$$

Following the procedures laid out in CCH (2009) the power series representation of the lifetime utility (24) and its derivatives are substituted into (53). After equating the coefficients of the terms of degree n one finds the following recursive rule

$$g_0 g_2 = A_0 g_1^2 + B_0 g_0 g_1 + (C_2 + C_0) g_0^2 + (D_1 + D_0) g_0, \tag{55}$$

and for all $n \geq 1$.

$$\begin{aligned} &(n+1)(n+2) g_0 g_{n+2} \\ &= -\sum_{k=0}^{n-1} (k+1)(k+2) g_{k+2} g_{n-k} + A_0 \sum_{k=0}^{n} (k+1)(n-k+1) g_{k+1} g_{n-k+1} \\ &+ B_2 \sum_{k=0}^{n-1} \sum_{r=0}^{k} (r+1) g_{r+1} g_{k-r} \frac{B^{n-k-1}}{(n-k-1)!} + B_1 \sum_{k=0}^{n-1} (k+1) g_{k+1} g_{n-k-1} \\ &+ B_0 \sum_{k=0}^{n} (k+1) g_{k+1} g_{n-k} + C_3 \sum_{k=0}^{n-1} \sum_{r=0}^{k} g_r g_{k-r} \frac{B^{n-k-1}}{(n-k-1)!} + C_2 \sum_{k=0}^{n} \sum_{r=0}^{k} g_r g_{k-r} \frac{B^{n-k}}{(n-k)!} \\ &+ C_1 \sum_{k=0}^{n-1} g_k g_{n-k-1} + C_0 \sum_{k=0}^{n} g_k g_{n-k} + D_1 \sum_{k=0}^{n} g_k \frac{B^{n-k}}{(n-k)!} + D_0 g_n \end{aligned} \tag{56}$$

To develop a dominant power series define $\tilde{g_n} = n^2 L(g_n/g_0)$ for all integers $n \geq 1$, where

$L \geq 1$ is a constant to be determined later. For all $n \geq 3$,

$$\begin{aligned}
\tilde{g}_{n+2} = \frac{n+2}{n+1} \Bigg\{ & -\frac{1}{L} \sum_{k=0}^{n-1} \frac{(k+1)\tilde{g}_{k+2}\tilde{g}_{n-k}}{(k+2)(n-k)^2} + \frac{A_0}{L} \sum_{k=0}^{n} \frac{\tilde{g}_{k+1}\tilde{g}_{n-k+1}}{(k+1)(n-k+1)} \\
& + \frac{B_2}{L} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} \frac{\tilde{g}_{r+1}\tilde{g}_{k-r}}{(r+1)(k-r)^2} \frac{B^{n-k-1}}{(n-k-1)!} + \frac{B_1}{L} \sum_{k=0}^{n-2} \frac{\tilde{g}_{k+1}\tilde{g}_{n-k-1}}{(k+1)(n-k-1)^2} \\
& + \frac{B_0}{L} \sum_{k=0}^{n-1} \frac{\tilde{g}_{k+1}\tilde{g}_{n-k}}{(k+1)(n-k)^2} + \frac{C_3}{L} \sum_{k=2}^{n-1} \sum_{r=1}^{k-1} \frac{\tilde{g}_r\tilde{g}_{k-r}}{r^2(k-r)^2} \frac{B^{n-k-1}}{(n-k-1)!} \\
& + \frac{C_2}{L} \sum_{k=2}^{n} \sum_{r=1}^{k-1} \frac{\tilde{g}_r\tilde{g}_{k-r}}{r^2(k-r)^2} \frac{B^{n-k}}{(n-k)!} + \frac{C_1}{L} \sum_{k=1}^{n-2} \frac{\tilde{g}_k\tilde{g}_{n-k-1}}{k^2(n-k-1)^2} + \frac{C_0}{L} \sum_{k=1}^{n-1} \frac{\tilde{g}_k\tilde{g}_{n-k}}{k^2(n-k)^2} \\
& + \frac{D_1}{g_0} \sum_{k=1}^{n} \frac{\tilde{g}_k}{k^2} \frac{B^{n-k}}{(n-k)!} + \frac{D_0}{g_0} \frac{\tilde{g}_n}{n^2} + B_2 \sum_{k=0}^{n-1} \frac{\tilde{g}_{k+1}}{k+1} \frac{B^{n-k-1}}{(n-k-1)!} + B_1 \frac{\tilde{g}_n}{n} + B_0 \frac{\tilde{g}_{n+1}}{n+1} \\
& + C_3 L \frac{B^{n-1}}{(n-1)!} + 2C_3 \sum_{k=1}^{n-1} \frac{\tilde{g}_k}{k^2} \frac{B^{n-k-1}}{(n-k-1)!} + C_2 L \frac{B^n}{n!} + 2C_2 \sum_{k=1}^{n} \frac{\tilde{g}_k}{k^2} \frac{B^{n-k}}{(n-k)!} \\
& + 2C_1 \frac{\tilde{g}_{n-1}}{(n-1)^2} + 2C_0 \frac{\tilde{g}_n}{n^2} + \frac{D_1 L}{g_0} \frac{B^n}{n!} \Bigg\}. \qquad (57)
\end{aligned}$$

There exist a real number $L \geq 1$ and an integer $N \geq 3$ such that for all $n \geq N$,

$$\begin{aligned}
\frac{n+2}{n+1} \Bigg\{ & \frac{1}{L} \sum_{k=0}^{n-1} \frac{k+1}{(k+2)(n-k)^2} + \frac{|A_0|}{L} \sum_{k=0}^{n} \frac{1}{(k+1)(n-k+1)} \\
& + \frac{|B_2|}{L} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} \frac{1}{(r+1)(k-r)^2} \frac{|B|^{n-k-1}}{(n-k-1)!} + \frac{|B_1|}{L} \sum_{k=0}^{n-2} \frac{1}{(k+1)(n-k-1)^2} \\
& + \frac{|B_0|}{L} \sum_{k=0}^{n-1} \frac{1}{(k+1)(n-k)^2} + \frac{|C_3|}{L} \sum_{k=2}^{n-1} \sum_{r=1}^{k-1} \frac{1}{r^2(k-r)^2} \frac{|B|^{n-k-1}}{(n-k-1)!} \\
& + \frac{|C_2|}{L} \sum_{k=2}^{n} \sum_{r=1}^{k-1} \frac{1}{r^2(k-r)^2} \frac{|B|^{n-k}}{(n-k)!} + \frac{|C_1|}{L} \sum_{k=1}^{n-2} \frac{1}{k^2(n-k-1)^2} + \frac{|C_0|}{L} \sum_{k=1}^{n-1} \frac{1}{k^2(n-k)^2} \\
& + \frac{|D_1|}{|g_0|} \sum_{k=1}^{n} \frac{1}{k^2} \frac{|B|^{n-k}}{(n-k)!} + \frac{|D_0|}{|g_0|} \frac{1}{n^2} + |B_2| \sum_{k=0}^{n-1} \frac{1}{k+1} \frac{|B|^{n-k-1}}{(n-k-1)!} + \frac{|B_1|}{n} + \frac{|B_0|}{n+1} \\
& + |C_3| L \frac{|B|^{n-1}}{(n-1)!} + 2|C_3| \sum_{k=1}^{n-1} \frac{1}{k^2} \frac{|B|^{n-k-1}}{(n-k-1)!} + |C_2| L \frac{|B|^n}{n!} + 2|C_2| \sum_{k=1}^{n} \frac{1}{k^2} \frac{|B|^{n-k}}{(n-k)!} \\
& + \frac{2|C_1|}{(n-1)^2} + \frac{2|C_0|}{n^2} + \frac{|D_1| L}{|g_0|} \frac{|B|^n}{n!} \Bigg\} \leq \mathcal{B}_\sigma(N, L) < 1, \qquad (58)
\end{aligned}$$

where

$$\begin{aligned}\mathcal{B}_\sigma(N,L) = \frac{N+2}{N+1}\Bigg[&\frac{\pi^2}{6L} + \frac{|A_0|}{L}U_{N+2} + \frac{|B_0|}{L}U_{N+1} + \frac{|B_1|+|C_0|}{L}U_N + \frac{|C_1|}{L}U_{N-1}\\ &+ \frac{|B_2|+|C_2|+|C_3|}{L}U_1 e^{|B|} + (|B_2| + 2|C_2| + |D_1/g_0|)U_N^{|B|} + 2|C_3|U_{N-1}^{|B|}\\ &+ (|C_2| + |D_1/g_0|)L\frac{|B|^N}{N!} + |C_3|L\frac{|B|^{N-1}}{(N-1)!}\\ &+\frac{2|C_0|+|D_0/g_0|}{N^2} + \frac{2|C_1|}{(N-1)^2} + \frac{|B_0|}{N+1} + \frac{|B_1|}{N}\Bigg]. \end{aligned} \tag{59}$$

The next two lemmas, whose proof is provided in the Appendix, are used in the error analysis.

Lemma 3.2. *If a, b, c, and d are integers such that $a \geq 0$, $b \geq 0$, $a+c > 0$, $b+d > 0$, and $c+d \geq 0$, then*

$$\sum_{k=a}^{n-b} \frac{1}{(k+c)(n-k+d)} \leq U_{n+c+d}, \tag{60}$$

where $U_k = 2[1 + \ln(k+1)]/k$ for $k = 1, 2, 3, \ldots$.

Lemma 3.3. *Let $B \geq 0$ be a real number. If b and d are integers such that $b \geq 0$, $d \geq 0$, and $b+d > 0$, then*

$$\sum_{k=b}^{n} \frac{1}{k+d} \cdot \frac{B^{n-k}}{(n-k)!} \leq U_{n+d}^B, \tag{61}$$

where $U_k^B = [(1 + 2Be^B) + 2Be^B \ln(k+1)]/k$ for $k = 1, 2, 3, \ldots$.

To obtain the bounds for the coefficients of the dominant power series one proceeds as follows. First, pick a real number $M_g \geq 1$ such that $|\tilde{g}_n| \leq M_g^n$ for $1 \leq n \leq N+1$. Then apply the following algorithm to construct a sequence $\{G_n\}$ of nonnegative real numbers.

1. Use the recurrence relation (56) and the initial values g_0, g_1 to calculate g_n, where $2 \leq n \leq N+1$.

2. Calculate $G_n = n^2 L|g_n/g_0|$ for $1 \leq n \leq N+1$.

3. Calculate the remaining terms G_{n+2}, with $n \geq N$, by the recurrence relation:

$$
\begin{aligned}
G_{n+2} = \frac{n+2}{n+1} \Bigg\{ & \frac{1}{L} \sum_{k=0}^{n-1} \frac{(k+1)G_{k+2}G_{n-k}}{(k+2)(n-k)^2} + \frac{|A_0|}{L} \sum_{k=0}^{n} \frac{G_{k+1}G_{n-k+1}}{(k+1)(n-k+1)} \\
& + \frac{|B_2|}{L} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} \frac{G_{r+1}G_{k-r}}{(r+1)(k-r)^2} \frac{|B|^{n-k-1}}{(n-k-1)!} + \frac{|B_1|}{L} \sum_{k=0}^{n-2} \frac{G_{k+1}G_{n-k-1}}{(k+1)(n-k-1)^2} \\
& + \frac{|B_0|}{L} \sum_{k=0}^{n-1} \frac{G_{k+1}G_{n-k}}{(k+1)(n-k)^2} + \frac{|C_3|}{L} \sum_{k=2}^{n-1} \sum_{r=1}^{k-1} \frac{G_r G_{k-r}}{r^2(k-r)^2} \frac{|B|^{n-k-1}}{(n-k-1)!} \\
& + \frac{|C_2|}{L} \sum_{k=2}^{n} \sum_{r=1}^{k-1} \frac{G_r G_{k-r}}{r^2(k-r)^2} \frac{|B|^{n-k}}{(n-k)!} + \frac{|C_1|}{L} \sum_{k=1}^{n-2} \frac{G_k G_{n-k-1}}{k^2(n-k-1)^2} + \frac{|C_0|}{L} \sum_{k=1}^{n-1} \frac{G_k G_{n-k}}{k^2(n-k)^2} \\
& + \frac{|D_1|}{|g_0|} \sum_{k=1}^{n} \frac{G_k}{k^2} \frac{|B|^{n-k}}{(n-k)!} + \frac{|D_0|}{|g_0|} \frac{G_n}{n^2} + |B_2| \sum_{k=0}^{n-1} \frac{G_{k+1}}{k+1} \frac{|B|^{n-k-1}}{(n-k-1)!} + |B_1| \frac{G_n}{n} + |B_0| \frac{G_{n+1}}{n+1} \\
& + |C_3| L \frac{|B|^{n-1}}{(n-1)!} + 2|C_3| \sum_{k=1}^{n-1} \frac{G_k}{k^2} \frac{|B|^{n-k-1}}{(n-k-1)!} + |C_2| L \frac{|B|^n}{n!} + 2|C_2| \sum_{k=1}^{n} \frac{G_k}{k^2} \frac{|B|^{n-k}}{(n-k)!} \\
& + 2|C_1| \frac{G_{n-1}}{(n-1)^2} + 2|C_0| \frac{G_n}{n^2} + \frac{|D_1| L}{|g_0|} \frac{|B|^n}{n!} \Bigg\}. \qquad (62)
\end{aligned}
$$

Next by mathematical induction, one can show that

$$
n^2 L |g_n/g_0| \le G_n \le M_g^n \quad \text{or} \quad |g_n| \le \frac{|g_0|}{L} \frac{M_g^n}{n^2} \quad \text{for all } n \ge 1. \qquad (63)
$$

Applying the root test for convergence, the lifetime utility can be represented as a convergent power series. Thus, the following is true.

Theorem 3.4. *Choose a real number $L \ge 1$ and an integer $N \ge 3$ such that $\mathcal{B}(N, L) < 1$ and set*

$$
M_g = \max\{1, \sqrt[n]{n^2 L |g_n/g_0|} \colon 1 \le n \le N+1\} \quad \textit{and} \quad r_g = \frac{\epsilon \sigma_0 \varphi_e}{M_g}. \qquad (64)
$$

The power series solution of the initial value problem (53)

$$
g(x) = \sum_{n=0}^{\infty} \frac{g_n}{(\epsilon \sigma_0 \varphi_e)^n} x^n \qquad (65)
$$

converges in the interval $-r_g \le x \le r_g$, where the g_n's are determined by the recurrence relation given in (56).

The estimate of the error in the Taylor polynomial approximation (26) follows immediately.

Corollary 3.5. *Let* $T_{g,n}(x) = \sum_{k=0}^{n}[g_k/(\epsilon\sigma_0\varphi_e)^k]x^k$ *be the* n^{th} *order Taylor polynomial of* $g(x)$*. For any positive number* $\nu < 1$*, we have*

$$\max_{|x|\le r_g} |g(x) - T_{g,n}(x)| \le \sum_{k=n+1}^{\infty} \frac{|g_0|}{L} \cdot \frac{1}{k^2} \le \frac{|g_0|}{L} \int_n^{\infty} \frac{d\lambda}{\lambda^2} \le \frac{|g_0|}{nL} \tag{66}$$

and

$$\max_{|x|\le \nu r_g} |g'(x) - T'_{g,n}(x)| \le \frac{M_g|g_0|}{\epsilon\varphi_e\sigma_0 L} \sum_{k=n}^{\infty} \frac{\nu^k}{k+1} \le \frac{M_g|g_0|}{\epsilon\varphi_e\sigma_0 L(n+1)} \frac{\nu^n}{1-\nu}. \tag{67}$$

3.2 Solving the Price-dividend Ratio

Having establish that the lifetime utility of the representative investor is an analytic function (65) which converges in the interval $-r_g \le x \le r_g$, the solution to the initial value problem for the price-dividend function (35) subject to the initial conditions $p(0) = p_0$ and $p'(0) = p_1$, see (48), can be addressed now. The results in Theorem 3.1 of CCH (2009) directly apply here since the ODE (35) is linear with analytic coefficients. Consequently, the price-dividend function (42) is also an analytic function of the long run risk variable. Recall that in this case the radius of convergence for $p(x)$ is at least as big as the smallest radius of convergence for the coefficients and the forcing term of the ODE (35). Below it is shown that the smallest radius of convergence of the coefficients and the forcing term is at least $r_p = \frac{r_g}{3}$. Thus, restricting x to the smaller interval of convergence $-\nu r_p < x < \nu r_p$, we can estimate the error of the Taylor polynomial approximation of $p(x)$.

Again to avoid division with small numbers we make the change of variables $x = \epsilon\varphi_e\sigma_0$

and obtain the following IVP.

$$
\begin{aligned}
g(s)p''(s) = {} & [(A_2 s e^{Bs} + A_1 s + A_0)g(s) + B_0 g'(s)]p'(s) \\
& + [(C_3 s e^{Bs} + C_2 e^{Bs} + C_1 s + C_0)g(s) + (D_1 e^{Bs} + D_0) \\
& \quad + (E_2 s e^{Bs} + E_1 s + E_0)g'(s) - g''(s)]p(s) \\
& + (F_1 e^{Bs} + F_0)g(s) \text{ subject to} \\
& p(0) = p_0 \text{ and } p'(0) = \left[-\sigma_0\varphi_d\rho_{xc}\rho_{cd} + \sqrt{\sigma_0^2\varphi_d^2\rho_{xc}^2\rho_{cd}^2 + \Sigma_0^2 - \sigma_0^2\varphi_d^2}\right]\epsilon p_0.
\end{aligned} \tag{68}
$$

The constants in the above ODE are given by

$$
\begin{array}{ccc}
A_0 = -2\epsilon\sigma_0\rho_{xc}(\alpha - 1 + \varphi_d\rho_{cd}), & A_1 = 2\epsilon^2\kappa a/(1+a), & A_2 = 2\epsilon^2\kappa/(1+a), \\
B_0 = 2(\rho - \alpha)/\rho, & \begin{array}{c} C_0 = 4\epsilon^2(\beta - \rho\bar{x})a/(1+a) - \epsilon^2\sigma_0^2 \times \\ [\varphi_d^2 + 2\varphi_d\rho_{cd}(\alpha - 1) + 2\alpha\rho - 2\alpha + 1], \end{array} & C_1 = -2\epsilon^3\sigma_0\varphi_e\rho a/(1+a), \\
C_2 = 4\epsilon^2(\beta - \rho\bar{x})/(1+a), & C_3 = -2\epsilon^3\sigma_0\varphi_e\rho/(1+a), & D_0 = -2\epsilon^2\beta a/(1+a), \\
D_1 = -2\epsilon^2\beta/(1+a), & \begin{array}{c} E_0 = -2\epsilon\sigma_0\rho_{xc}\left[\rho\alpha + \rho - \right. \\ \left. \alpha + \varphi_d\rho_{cd}(\alpha - \rho)\right]/\rho, \end{array} & E_1 = 2\epsilon^2\kappa a/(1+a), \\
E_2 = 2\epsilon^2\kappa/(1+a), & F_0 = -2\epsilon^2 a/(1+a), & F_1 = -2\epsilon^2/(1+a).
\end{array} \tag{69}
$$

Equation (68) is a second order linear ODE with analytic coefficients and forcing term and therefore the linear Cauchy-Kovalevsky Theorem in the form stated in Theorem 3.1 of CCH (2009) is applicable.

In this case the recurrence relation for the coefficients in the price-dividend function (42) is given by:

$$
g_0 p_2 = (A_0 g_0 + B_0 g_1)p_1 + [(C_2 + C_0)g_0 + D_1 + D_0 + E_0 g_1 - g_2]\, p_0 + (F_1 + F_0)g_0, \tag{70}
$$

and for $n \geq 1$

$$
\begin{aligned}
&(n+1)(n+2)g_0 p_{n+2} = -\sum_{k=0}^{n-1}(k+1)(k+2)g_{n-k}p_{k+2} \\
&+ A_2 \sum_{k=0}^{n-1}\sum_{r=0}^{k}(r+1)g_{k-r}p_{r+1}\frac{B^{n-k-1}}{(n-k-1)!} + A_1 \sum_{k=0}^{n-1}(k+1)g_{n-k-1}p_{k+1} + A_0 \sum_{k=0}^{n}(k+1)g_{n-k}p_{k+1} \\
&+ B_0 \sum_{k=0}^{n}(k+1)(n-k+1)g_{n-k+1}p_{k+1} + C_3 \sum_{k=0}^{n-1}\sum_{r=0}^{k} g_{k-r}p_r \frac{B^{n-k-1}}{(n-k-1)!} + C_2 \sum_{k=0}^{n}\sum_{r=0}^{k} g_{k-r}p_r \frac{B^{n-k}}{(n-k)!} \\
&+ C_1 \sum_{k=0}^{n-1} g_{n-k-1}p_k + C_0 \sum_{k=0}^{n} g_{n-k}p_k + D_1 \sum_{k=0}^{n} p_k \frac{B^{n-k}}{(n-k)!} + D_0 p_n \\
&+ E_2 \sum_{k=0}^{n-1}\sum_{r=0}^{k}(k-r+1)g_{k-r+1}p_r \frac{B^{n-k-1}}{(n-k-1)!} + E_1 \sum_{k=0}^{n-1}(n-k)g_{n-k}p_k + E_0 \sum_{k=0}^{n}(n-k+1)g_{n-k+1}p_k \\
&- \sum_{k=0}^{n}(n-k+1)(n-k+2)g_{n-k+2}p_k + F_1 \sum_{k=0}^{n} g_k \frac{B^{n-k}}{(n-k)!} + F_0 g_n \qquad (71)
\end{aligned}
$$

3.3 Convergence and error analysis

We begin by writing the second order linear differential equation (68) in the standard form:

$$
\begin{aligned}
p'' &- \left[(A_2 s e^{Bs} + A_1 s + A_0) + B_0 \frac{g'(s)}{g(s)} \right] p' \\
&- \left[(C_3 s e^{Bs} + C_2 e^{Bs} + C_1 s + C_0) + (D_1 e^{Bs} + D_0)\frac{1}{g(s)} \right. \\
&\quad \left. + (E_2 s e^{Bs} + E_1 s + E_0)\frac{g'(s)}{g(s)} - \frac{g''(s)}{g(s)} \right] p = F_1 e^{Bs} + F_0 \qquad (72)
\end{aligned}
$$

Next we find a power series for the functions appearing in the coefficients in the ODE (72)

$$
\frac{1}{g(s)} = \sum_{n=0}^{\infty} c_n^{(0)} s^n, \qquad \frac{g'(s)}{g(s)} = \sum_{n=0}^{\infty} c_n^{(1)} s^n, \qquad \frac{g''(s)}{g(s)} = \sum_{n=0}^{\infty} c_n^{(2)} s^n,
$$

$$
\frac{e^{Bs}}{g(s)} = \sum_{n=0}^{\infty} c_n^{(3)} s^n, \qquad \frac{e^{Bs} g'(s)}{g(s)} = \sum_{n=0}^{\infty} c_n^{(4)} s^n. \qquad (73)
$$

We need to construct estimates for the coefficients in the ODE (72) to find an estimate of the error for the price-dividend function. First, the following estimates for $c_n^{(0)}$, $c_n^{(1)}$, $c_n^{(2)}$, $c_n^{(3)}$, and

$c_n^{(4)}$ are provided in the Appendix

$$|c_n^{(0)}| \le (1/|g_0|)(3M_g)^n, \qquad |c_n^{(1)}| \le (3M_g)(3M_g)^n, \qquad |c_n^{(2)}| \le (9M_g^2)(3M_g)^n,$$
$$|c_n^{(3)}| \le \frac{e^{|B|/(2M_g)}}{|g_0|}(3M_g)^n, \qquad |c_n^{(4)}| \le [3M_g e^{|B|/(3M_g)}](3M_g)^n, \tag{74}$$

where $M_g \ge 1$ is given by Theorem 3.4.

We can now establish bounds for each of the coefficients in the ODE (72). The power series representation for the coefficient of p' is given by

$$-\left(A_0 + B_0 c_0^{(1)}\right) - \left(A_2 + A_1 + B_0 c_1^{(1)}\right) s - \sum_{n=2}^{\infty}\left[A_2 \frac{B^{n-1}}{(n-1)!} + B_0 c_n^{(1)}\right] s^n. \tag{75}$$

The power series representation for the coefficient of p is given by

$$\begin{aligned}
&-\left(C_2 + C_0 + D_1 c_0^{(3)} + D_0 c_0^{(0)} + E_0 c_0^{(1)} - c_0^{(2)}\right)\\
&-\left(C_3 + BC_2 + C_1 + D_1 c_1^{(3)} + D_0 c_1^{(0)} + E_2 c_0^{(4)} + E_1 c_0^{(1)} + E_0 c_1^{(1)} - c_1^{(2)}\right) s\\
&-\sum_{n=2}^{\infty}\left[C_3 \frac{B^{n-1}}{(n-1)!} + C_2 \frac{B^n}{n!} + D_1 c_n^{(3)} + D_0 c_n^{(0)} + E_2 c_{n-1}^{(4)} + E_1 c_{n-1}^{(1)} + E_0 c_n^{(1)} - c_n^{(2)}\right] s^n.
\end{aligned} \tag{76}$$

The values of $c_i^{(j)}$ are given in the following table:

$c_0^{(0)} = 1/g_0$	$c_1^{(0)} = -g_1/g_0^2$
$c_0^{(1)} = g_1/g_0$	$c_1^{(1)} = 2g_2/g_0 - g_1^2/g_0^2$
$c_0^{(2)} = 2g_2/g_0$	$c_1^{(2)} = 6g_3/g_0 - 2g_1g_2/g_0^2$
$c_0^{(3)} = 1/g_0$	$c_1^{(3)} = B/g_0 - g_1/g_0^2$
$c_0^{(4)} = g_1/g_0$	

When $n \ge 2$, we can estimate the coefficient of s^n in (75) as follows:

$$\left|A_2 \frac{B^{n-1}}{(n-1)!} + B_0 c_n^{(1)}\right| \le \left[\frac{|A_2|(e^{|B|}-1)}{9M_g^2} + 3M_g|B_0|\right](3M_g)^n \tag{77}$$

Also, when $n \ge 2$, we can estimate the coefficient of s^n in (76) as follows:

$$\begin{aligned}
&\left|C_3 \frac{B^{n-1}}{(n-1)!} + C_2 \frac{B^n}{n!} + D_1 c_n^{(3)} + D_0 c_n^{(0)} + E_2 c_{n-1}^{(4)} + E_1 c_{n-1}^{(1)} + E_0 c_n^{(1)} - c_n^{(2)}\right|\\
&\le \left[\frac{(|C_3|+|C_2|)(e^{|B|}-1) - |BC_2|}{9M_g^2} + \left|\frac{D_1}{g_0}\right| e^{|B|/(2M_g)} + \left|\frac{D_0}{g_0}\right|\right.\\
&\qquad \left. + |E_2| e^{|B|/(3M_g)} + |E_1| + 3M_g|E_0| + 9M_g^2\right](3M_g)^n.
\end{aligned} \tag{78}$$

Now, defining

$$
\begin{aligned}
\mathcal{C}_1 &= |A_0 + B_0 c_0^{(1)}|, \\
\mathcal{C}_2 &= |A_2 + A_1 + B_0 c_1^{(1)}|/(3M_g), \\
\mathcal{C}_3 &= \frac{|A_2|(e^{|B|}-1)}{9M_g^2} + 3M_g|B_0|, \\
\mathcal{C}_4 &= |C_2 + C_0 + D_1 c_0^{(3)} + D_0 c_0^{(0)} + E_0 c_0^{(1)} - c_0^{(2)}|, \\
\mathcal{C}_5 &= |C_3 + BC_2 + C_1 + D_1 c_1^{(3)} + D_0 c_1^{(0)} + E_2 c_0^{(4)} + E_1 c_0^{(1)} + E_0 c_1^{(1)} - c_1^{(2)}|/(3M_g), \\
\mathcal{C}_6 &= \frac{(|C_2| + |C_3|)(e^{|B|}-1) - |BC_2|}{9M_g^2} + \left|\frac{D_1}{g_0}\right| e^{|B|/(2M_g)} + \left|\frac{D_0}{g_0}\right| \\
&\quad + |E_2| e^{|b|/(3M_g)} + |E_1| + 3M_g|E_0| + 9M_g^2
\end{aligned}
\tag{79}
$$

and proceeding as in the proofs of Theorem 3.1 and Corollary 3.2 in the appendix of CCH (2009) we obtain the following result.

Theorem 3.6. *Let $r_p = \epsilon\sigma_0\varphi_e/(3M_g)$ and $M = \max\{\,\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6\,\}$. Then the power series solution of the initial value problem* (68)

$$
p(x) = \sum_{n=0}^{\infty} \frac{p_n}{(\epsilon\sigma_0\varphi_e)^n} x^n \tag{80}
$$

converges in the open interval $-r_p < x < r_p$, where the p_n's are determined by the recurrence relation given in (71).

Corollary 3.7. *Let $T_{p,n}(x)$ be the n^{th} order Taylor polynomial of the price-dividend function $p(x)$, that is*

$$
T_{p,n}(x) = \sum_{k=0}^{n} \left[\frac{p_k}{(\epsilon\sigma_0\varphi_e)^k}\right] x^k.
$$

Then, for any positive number $\nu < 1$, the Taylor's series remainder $p(x) - T_{p,n}(x)$ satisfies the estimate

$$
\begin{aligned}
&\max_{|x|\le \nu r_p} |p(x) - T_{p,n}(x)| \\
&\le \frac{1}{2}\left\{\tilde{M} + [(1+r_p)|p_1| + |p_0|]M\right\} \sum_{k=n+1}^{\infty} \prod_{l=2}^{k-1} \left[\frac{l-1}{r_p(l+1)} + \frac{l+r_p}{l(l+1)}M\right] |\nu r_p|^k,
\end{aligned}
\tag{81}
$$

where $\tilde{M} = \max\{|F_0 + F_1|, |F_1|(e^{|B|/(3M_g)} - 1)\}$ and $M = \max\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6\}$.

4 Simulation

In this section the analytic method is implemented for the continuous time long run risk model of BY. The parameters are based on the data in Bansal, Kiku, and Yaron (2007), which are chosen to match annual data over the time period 1930 to 2002. As in Bansal and Yaron (2004) they calibrate their model at monthly frequency such that the time-aggregated annual data match the empirical observations. Here, we convert the parameters from the discrete time of BY to the continuous time, considered here, using the method of Bergstrom (1984), Campbell and Kyle (1993) and Campbell, *et al.* (2004). In these papers the discrete data is assumed to be generated by an underlying continuous time model such as (1), (2), and (3). The stochastic processes are then integrated over the discrete time period which is a month for Bansal, Kiku, and Yaron's (2007) data. Finally, we match the estimated monthly parameters with the associated parameters from the continuous time processes. These calculations yield the parameters $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, and $\rho_{cd} = 0.2465$.[17]

The volatility parameters $a = 10.9$, and $b = 175.33$ in (4) are set so that the volatility (standard deviation) of consumption growth matches the empirical evidence from Bansal and Yaron (2004, Table III). The volatility of consumption growth, $\sqrt{\sigma^2(0)}\sigma^2(0)b/(a+1)$, matches the value for its standard deviation in BY $\sigma_w = 0.23 \times 10^{-5}$ at the stationary mean of the long run risk variable. In addition, Bansal and Yaron (2004, Table III) report that a 1% change in the price-dividend function leads to a 0.11 decline in the volatility of consumption growth for a two year horizon. This information is also used to set the parameters a and b for the logistic form of the variance of consumption growth (4). In particular, Figure 1 is a parametric plot of the volatility of consumption growth versus the price-dividend ratio as the long run risk variable varies in the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$. Consequently, the model for the variance of consumption growth (4) in the BY model implies that the volatility of consumption growth decreases as the price-dividend ratio increases. In addition, Figure 2 graphs the derivative of

[17] See Chen, Cosimano, Himonas and Kelly (2009) for details.

the volatility of consumption growth with respect to the logarithm of the price-dividend ratio. The parameters a and b are chosen so that this derivative is near -0.11 on average, which is the value found in the empirical work by Bansal and Yaron (2004). Thus, the variance of consumption growth (4) is chosen so that it matches the empirical properties of consumption volatility, found in Bansal and Yaron (2004).

Given the choice of the parameters above, the rate of discount per month for the representative investor,

$$\begin{aligned}\beta =& R^b(0) + (\rho - 1)\bar{x} + \frac{\sigma^2}{2}(\alpha\rho - 2\alpha + 1) \\ &+ \rho_{xc}\phi_e\sigma_0^2 \frac{\rho - \alpha}{\rho}\frac{g'(0)}{g(0)} + \frac{\phi_e^2\sigma_0^2}{2}\frac{\rho-\alpha}{\rho}\left(\frac{g'(0)}{g(0)}\right)^2 = 0.0006903, \end{aligned} \tag{82}$$

is chosen so that the risk free interest rate, given by equation (31), matches the historic average in Bansal, Kiku and Yaron (2007), see our Table 1, at the stationary mean for the long run risk variable $x = 0$.

4.1 Lifetime Utility

The lifetime utility of the representative investor is approximated using a 100^{th} order Taylor polynomial approximation (25).[18] The initial conditions are $g_0 = 1.4330$ and $g_1 = 0.03527$. The first initial condition is an estimate of $g(0)$ using (20) evaluated at the risk neutral stationary mean of the long run risk variable, $x_g = -0.0005892$ so that $\mu_g(x_g) = 0$. This value of the long run risk variable is over two standard deviations below the actual stationary mean of the long run risk variable $x = 0$, see Figure 3. The discount rate for the lifetime utility is estimated using $r_g(x)$ in (21), see Figure 4, evaluated at the risk neutral stationary mean, $r_g(x_g) = 0.06591$. This risk neutral discount rate places less weight on the future then the actual rate of discount $\beta = 0.0006903$. Consequently, the risk neutral adjustment lowers the expected lifetime utility. The second initial condition $g'(0)$ is chosen as in (49). Thus,

[18] The simulations and graphs in this paper are calculated in 27 seconds using Maple on a PC with an Intel Core2 Duo CPU with speed 2.66 GHz.

the equity premium and its standard deviation from Bansal, Kiku, and Yaron (2007), see our Table 1, are matched at the stationary mean of the long run risk variable.

The accuracy of the Taylor polynomial approximation for the analytic solution to the investor's lifetime utility (65) in Figure 6 is determine by applying Theorem 3.4 and Corollary 3.5. The radius of convergence of the lifetime utility is at least $r_g = 0.001583$ so that the lifetime utility is analytic over the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$. The error of the 100^{th} order Taylor polynomial approximation (3.5) is at most 0.001303 which is 0.09091% of g_0. In addition, Figure 6 graphs the relative error between the 90^{th} and 100^{th} order Taylor polynomial which is always less than one in 25 million. However, the errors are as large as 55% when one uses only a fourth order Taylor polynomial approximation over the same interval. Thus, a high order Taylor polynomial is needed to accurately portray the lifetime utility of the representative investor.

The risk neutral mean of the distribution of the long run risk variable $\mu_g(x)$ for the representative investor's lifetime utility, given in (17), is portrayed in Figure 3. This mean along with the standard deviation $\varphi_e\sigma(x)$ determines the probability distribution in the Feynman-Kac formula of the lifetime utility (20). The existence of a solution is dependent on the present value of a unit payoff (21) over a given time interval. The property of this present value is examined using Figure 3. At the stationary mean of the risk neutral probability distribution for the long run risk variable $x_g = -0.0005892$ the rate of discount is $r_g(x_g) = 0.06591$ on a monthly basis. Thus, the adjustment to the investor's lifetime utility for risk lowers the mean value of the long run risk variable x by 2.6 standard deviations. In addition, the rate of discount is higher than the true rate of discount $\beta - \rho\bar{x} = 0.0004379$ so that the present value of lifetime utility is lower as a consequence of risk adjustment.

For the lifetime utility (20) to exist the rate of discount $r_g(x)$ in (21) must be positive for almost all random draws of the long run risk variable. From Figures 3 and 4 one sees that the long run risk variable such that the rate of discount is positive, is $x \in [-7\varphi_e\sigma, 6.4\varphi_e\sigma]$ which goes from 4.4 standard deviations below the risk neutral stationary mean of the long run risk variable to 9 standard deviations above it. In addition, the mean of the long run risk variable in (17) and the discount rate in (21) satisfy the bound and Lipschitz conditions of Duffie (2001, p. 345), since the function $g(x)$ is extended by (27). Both of these functions are determined by the elasticity of the lifetime utility function $\frac{\alpha}{\rho}\frac{g'(x)}{g(x)}$ which is the solid line in Figure 7. As a result, the distribution of the risk neutral long run risk variable would kill off random shocks over any finite period of time in which the discount rate could possibly go positive. Consequently, the probabilistic solution (20) to the investor's lifetime utility exists and the analytic solution (65) is the unique representation of the investor's lifetime utility for the radius of convergence, $r_g = 7\varphi_e\sigma$.

One can use the lifetime utility to portray the consumption to wealth ratio of the representative investor as in Figure 8. Using the optimal consumption decision from Schroder and Skiadas (1998), Campbell and Viceira (2002, p. 146) and Campbell *et al.* (2004, p. 2212), Fisher and Gilles (1998), and Benzoni, Collin-Dufresne and Goldstein (2005) show how to use the state price process to determine the consumption to wealth ratio in the same context as the current paper. Here, the consumption to wealth ratio is given by

$$\frac{C}{W} = \beta^{\frac{1}{1-\rho}} \left(\alpha v(x)\right)^{\frac{\rho}{\alpha(\rho-1)}} = \beta^{\frac{1}{1-\rho}} \left(g(x)\right)^{-\frac{1}{(1-\rho)}} . \tag{83}$$

As a result, the lifetime utility function of the representative investor can be converted into units of consumption per wealth. Figure 8 graphs the consumption to wealth ratio on an annual basis which is 0.01267% and is determined mainly by $12\beta^{\frac{1}{1-\rho}} = 0.02176\%$, which is significantly smaller than the value used in Campbell *et al.* and Bansal and Yaron (2004).[19] In addition, this consumption to wealth ratio is significantly different from the rule of thumb used by investment advisors.[20] Thus, there is a conflict in the model between the $\beta = 0.06903\%$ per month implied by the historic risk free interest rate and the $\beta = 0.33\%$ or 0.5% per month consistent with recommendations by financial advisors or the value used in research, respectively.

[19] Most researchers take $12\beta = 0.06$ which corresponds to a 6% rate of discount so that $12 * \beta^{\frac{1}{1-\rho}} = 0.4243\%$.

[20] For example Vanguard's web site uses a 4% consumption to wealth ratio for calculating expected income for retired investors.

4.2 Financial Market Implications

Having identified an analytic solution to the lifetime utility in the Duffie and Epstein (1992a, 1992b) model within the interval of convergence $[-r_g, r_g]$, the price-dividend ratio is also an analytic solution (80) with radius of convergence $r_p = \frac{r_p}{3}$ thanks to Theorem 3.6. Using Corollary 3.7 the Taylor's series remainder for a 100^{th} order Taylor polynomial is at most 1.71745×10^{-08} for $\nu = 0.05$, so that the long run risk variable is in the interval $[-0.35\varphi_e\sigma, 0.35\varphi_e\sigma]$. As a result, a significant increase in the Taylor polynomial order is necessary to increase this interval towards its maximum. This smaller interval can be attributed to the imprecise bounds (79) on the coefficients of the ODE for the price-dividend function (72), which in turn are based on the estimates (74). Numerical evidence on the accuracy of the price-dividend solution is provided in Figure 10 which compares the 90^{th} and 100^{th} order Taylor polynomial. In this case the relative error between these polynomials is one in 3 million over the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$. Consequently, the numerical evidence suggests that the price-dividend ratio is accurate for a larger interval relative to the theorectical estimate of this interval.

The initial conditions are $p_0 = 617.23$ and $p_1 = 228.24$. The first initial condition $p(0)$ is chosen by estimating the Feynman-Kac risk neutral value of the price-dividend function (40). As a result, the price-dividend ratio $p(0) = 51.44$ on an annual basis is significantly above its historic value 27.94 in Table 1. The reason is that the rate of discount implied by the risk free interest rate is too low. For the price-dividend ratio the rate of discount, $r_p(x_p) = 0.01944$, following (41), is calculated at the risk neutral mean of the stationary distribution for the long run risk variable. Here the long run risk variable is implied by the price-dividend ratio, i.e. $x_p = -0.0008685$ such that $\mu_p(x_p) = 0$ using (37). In this case the risk neutral mean for the long run risk variable is below the value implied by the lifetime utility function, which can be seen by comparing Figure 3 with Figure 11. However, this is necessary for the existence of the Feynman-Kac form of the price-dividend ratio (40), since the rate of discount implied by the price-dividend function, see Figure 12, is lower than that implied by the lifetime utility function, see Figure 4. Thus, the risk neutral distirbution of the long run risk variable has to on average be lower so that the integral in the Feynman-Kac formula (40) exists. The second initial condition $p'(0)$ is chosen so that the elasticity of the price-dividend function (48) is consistent with the standard deviation of the equity premium. Thus, the analytic solution to the lifetime utility function and price-dividend function can be found such that the mean and standard deviation of the equity premium matches the historic values at the stationary mean of the long run risk variable as well as the risk free interest rate.

The main problem with the analytic solution to the BY model is the low rate of discount for the price-dividend ratio. This problem can be removed if the rate of discount β is increased so that the representative individual places less weight on the future.[21] For example in Table 1 column 3 the risk free rate is increased by two standard deviation to 2.8%, so that the price-dividend ratio is now 28.68 per year without affecting most of the other properties of the moments for the financial market data.[22] A possible explanation for this result is that the short term Treasury Bill rate is lower because of liquidity effects which are not incorporated in the long run risk model.[23]

Matching the first two moments of the equity premium is no longer sufficient to distinguish among asset pricing models. For example Table 2 reproduces the results from CCH for the analytic solution of the Campebell and Cochrane model, which does as well as the BY model in matching the mean and standard deviation of the equity premium. As a result, Beeler and Campbell (2009) examine alternative properties such as higher order autocorrelation, as well as the ability of the price-dividend ratio to forecast expected future returns.

Here, the properties of stock returns are examined with regards to the long run risk variable

[21]This problem is also present in other asset pricing models. Weil (1989) pointed out that the risk free interest rate would be too high when the equity premium is matched in the Mehra-Prescott model. CCH (2008) also find that the low real risk free rate limits the range in which the solution to the CC model exists.

[22]It is also possible to lower the price-dividend ratio by raising ρ to 0.9. However, this would increase the intertemporal rate of substitution to $\psi = 10$. There is already substantial questions about this elasticity of substitution, see Beeler and Campbell (2009), so that raising this parameter does not appear to be a viable option.

[23]Wachter (2006) finds real interest rates increase by 1.05%, when going from 1 to 12 month yield to maturity for data from 1952 to 2004.

in the BY model. These properties are compared with the analytic solution of the CC (1999) model, found in CCH (2009), with regards to their surplus consumption variable S which measures the deviation of consumption from its habitual level.[24] The key difference is that the BY model yields a convex price-dividend function with respect to its state variable, see Figure 9, while in Figure 13 the CC price-dividend function is concave in its state variable. As a result, the mean and standard deviation of the equity premium behave differently over time between the two models. During expansions in the economy both the long run risk variable of BY and the surplus consumption ratio of CC increase so that the price-dividend ratio increases. However, the mean and standard deviation of the equity premium increases in the BY model, see Figure 16, while they decrease in the CC model, see Figure 14. Consequently, a high price-dividend ratio predicts higher returns in the BY model and lower returns in the CC model following (45). These results suggest that a detailed statistical analysis of these two analytic solutions should be undertaken now that a quick and accurate procedure has been established to solve these models.

4.3 Long Term Risk

Hansen, Heaton and Li (2008) emphasize the long term implications of asset pricing models by identifying and approximating an asymptotic rate of return using techniques developed by Hansen and Scheinkman (2009). This asymptotic rate of return is equal to the sum of the mean of the stochastic discount factor, the mean of the growth rate for the cash flows, and an adjustment for long term risk. Here, we argue that this asymptotic rate of return is conceptually similar to risk neutral discount factor $r_p(x)$ evaluated at the risk neutral stationary mean for the long run risk variable x_p implied by the financial asset.

Hansen and Scheinkman (2009) develop a method to decompose the present value of cash flows such as (40) into three parts 1.) a deterministic trend, 2.) a martingale, and 3.) the transitory effects of changes in the long run risk variable. They call the deterministic trend

[24] One may argue that the differences arise because of the use of a more recent time period for the financial market data. However, the convexity of the price-dividend function does not change when the financial market data of CC is used to calibrate the BY model.

the asymptotic rate of return for the asset. Applying their method to the stock price problem involves the solution to the eigenvalue problem

$$\nu e(x) = r_p(x)e(x) + \mathcal{D}e(x), \tag{84}$$

where the differential operator is given by (36).[25] They want to find the smallest eigenvalue ν for problem (84) which is consistent with the stationary distribution implied by (38). Rather than solving the eigenvalue problem (84) the asymptotic rate of return can be found using the Feyman-Kac formula (40) and the stochastic process (38).

Hansen and Scheinkman want to know a deterministic rate of return which can be factored out of the probabilistic solution for the price-dividend ratio. Given the solution to the price-dividend function the Feynman-Kac formula (40) has a deterministic trend which is given by the risk neutral discount factor evaluated at x_p:

$$r_p(x_p) = \phi x_p + \bar{x} + \frac{\varphi_d^2 \sigma^2(x_p)}{2} - R^b(x_p) + \sigma^2(x_p)\varphi_d \rho_{cd}\left(\alpha - 1 + \frac{\varphi_e \rho_{xc}(\alpha - \rho)}{\rho}\frac{g'(x_p)}{g(x_p)}\right). \tag{85}$$

Given a shock to the long run risk value the discount factor eventually returns to this value, so that this term can be factored out of (40) and only transitory deviations of the discount factor from this stationary value is left within the expectations, i.e.,

$$p(x) = E_{x,p}\left[\int_t^\infty e^{-r_p(x_p)(s-t)} \exp\left[-\int_t^s \left\{r_p\left(x_\tau\right) - r_p(x_p)\right\} d\tau\right] ds\right]. \tag{86}$$

The stationary point in Figure 11 is $x_p = -0.0008685$ and the long term discount factor is $r_p(x_p) = 1.944\%$ on an annual basis. Under the higher interest rate of 2.8% the long term discount factor is $r_p(x_p) = 3.487\%$ on an annual basis, which is much closer to the equity premium.[26] This discount factor has three components as in Hansen, Heaton, and Li (2008): 1.) The mean of the stochastic discount factor $R^b(x_p)$, 2.) the mean of the growth rate for the cash flows (dividends) $\phi x_p + \bar{x}$, and 3.) an adjustment for long term risk $\frac{\varphi_d^2\sigma^2(x_p)}{2} + \frac{\varphi_e\rho_{xc}(\alpha-\rho)}{\rho}\frac{g'(x_p)}{g(x_p)}$. In addition, this risk neutral discount factor is consistent with the

[25] The operator $\mathcal{D}$ is the extended generator of Hansen and Scheinkman. Intuitively, this generator is the expected derivative of the price-dividend ratio.

[26] Note that $p(0) = \frac{1}{r_p(x_p)}$ is the long term price-dividend ratio.

risk neutral stochastic process for the long run risk variable (38). However, this discount factor was not arrived at using the operator method of Hansen and Scheinkman (2009). Hansen and Scheinkman set up the eigenvalue problem (84) whose solution directly leads to their asymptotic rate of return. The asymptotic rate of return arrived at here was found by solving the ODE for the lifetime utility and using this solution in standard asset pricing formulas. This procedure also provides the discount factor for all possible values of the long run risk variable, see Figure 12, as well as the stochastic process for the long run risk variable, see Figure 11 for the mean of this process, which characterizes the behavior of this discount factor over time.

The solution of the lifetime utility function can also be used to find the long term mean and variance of other financial asset such as Cochrane (2008)'s long run risk free asset. In continuous time he defines the long run mean of any payoff $\{x_t\}_{t=0}^{\infty}$ as

$$\mathcal{E}(x) = \beta E\left[\int_0^{\infty} e^{-\beta t} x_t dt\right] . \tag{87}$$

Consequently, we can price the long term risk free asset which promises the same real payout each period $x_t = 1$ for all t. Using the state price process (30) and the consumption growth process (3), the price of the long term risk free asset solves the ODE

$$1 - R^b(x)k(x) + \left(-\kappa x + \frac{\varphi_e^2\sigma^2(x)(\alpha-\rho)}{\rho}\frac{g'(x)}{g(x)}\right)k'(x) + \frac{\varphi_e^2\sigma^2(x)}{2}k''(x) = 0, \tag{88}$$

where

$$k(x) = \frac{1}{\beta}\mathcal{E}\left(\frac{\Lambda_s}{\Lambda_t}\right) = E\left[\int_t^{\infty}\frac{\Lambda_s}{\Lambda_t}ds\right].$$

The Feynman-Kac solution to (88) is given by

$$k(x) = \lim_{T\to\infty} E_{x,k}\left[\int_t^T \beta_{t,s}^k ds + \beta_{t,T}^k k_T\right] = E_{x,p}\left[\int_t^{\infty}\beta_{t,s}^k ds\right], \tag{89}$$

where

$$\beta_{t,s}^k = \exp\left[-\int_t^s R^b(x_\tau)d\tau\right]. \tag{90}$$

Following Cochrane (2008) the risk free yield is given by $y^f(x) = \frac{1}{k(x)}$.

To define the risk neutral conditional expectations $E_{x,k}[\cdot]$, introduce the differential operator for the long run risk free asset

$$\mathcal{D}k(x) = \mu_k(x)k'(x) + \frac{\varphi_e^2\sigma^2(x)}{2}k''(x), \tag{91}$$

where

$$\begin{aligned}\mu_k(x) &= \left[-\kappa x + \frac{\varphi_e^2\sigma^2(x)(\alpha-\rho)}{\rho}\frac{g'(x)}{g(x)}\right] \\ &= \mu_g(x) + \sigma^2(x)\varphi_e\rho_{xc}\alpha - \left(\frac{\alpha}{\rho}-1\right)\frac{\varphi_e^2\sigma^2(x)}{2}\frac{g'(x)}{g(x)}.\end{aligned} \tag{92}$$

The long run risk variable now follows the stochastic differential equation (SDE)

$$dx = \mu_k(x)dt + \varphi_e\sigma(x)d\hat{\omega}_3, \tag{93}$$

where the twisted Brownian motion is

$$d\hat{\omega}_3 = d\tilde{\omega}_3 - \frac{\mu_k(x)+\kappa x}{\varphi_e\sigma(x)}dt. \tag{94}$$

Following the logic of Hansen and Scheinkman (2009) and Hansen, Heaton, and Li (2008) one can define the stationary point of the risk neutral distribution for the long term risk free asset as x_k such that $\mu_k(x_k) = 0$. Figure 17 plots the instantaneous mean of the risk neutral stochastic process for x under $k(x)$ so that $x_k = -0.0007814 = -3.5\varphi_e\sigma$. Thus, the long term rate of return at this stationary point $R^b(x_k) = 0.3497\%$ on an annual basis which again reflects the low value of the rate of discount β (82). In particular, for the case in which $R^b(0) = 2.8\%$ the long term risk free rate of return at the stationary point is $R^b(x_k) = 2.390\%$ on an annual basis, while the risk neutral discount factor for equity is $r_p(x_p) = 3.487\%$.

5 Conclusion

This paper finds the analytic solution to the continuous time version of Bansal and Yaron's long run risk model. As a result the lifetime utility, price-dividend ratio, the consumption to wealth ratio, the expected equity premium, its standard deviation and the risk free interest rate are power series in the long run risk variable near its stationary mean. The radius of convergence for the lifetime utility is at least seven standard deviations of the long run risk variable, while the radius of convergence for the price-dividend function is one third of this value. This allows these functions to be accurately represented by Taylor polynomials within the radius of convergence. In addition, a comparison between the 90^{th} and 100^{th} order Taylor polynomial approximation yields a relative error smaller than one in 25 million for the lifetime utility and one in 3 million for the price-dividend ratio, as long as the long run risk variable stays within the radius of convergence for the lifetime utility. On the other hand these errors can be as high as 55% when only a fourth order approximation is used. Thus, a higher order polynomial approximation needs to be used to accurately represent the solution to the long run risk model.

The Bansal and Yaron model assumes recursive preferences introduced by Epstein and Zin (1989, 1990, 1991). In continuous time Duffie and Epstein (1992a, 1992b), and Duffie and Lions (1992) show that the lifetime utility under recursive preferences is the solution of a backward stochastic differential equation (6), which involves solving a second order nonlinear differential equation (15). In order to solve the long run risk model the analytic methods in CCH (2009) had to be extended to handle a second order nonlinear differential equation. While the Cauchy-Kovalevsky Theorem 3.1, which was used by CCH (2009) to show the price-dividend function is analytic near the mean of the state variable, holds for nonlinear and higher dimensional differential equations, it does not provide much information about the radius of convergence for solutions to nonlinear differential equations. To find a good estimate for the radius of convergence (see Theorem 3.4) one must choose a dominant convergent power series carefully. Subsequently, the estimate for the radius of convergence is at least the radius of convergence of this dominant power series. For the Bansal and Yaron model this radius of convergence is at least seven standard deviations of the long run risk variable around the stationary mean of the long run risk variable. The procedure of CCH (2009) is then used in Theorem 3.6 to show that the price-dividend function is an analytic function near this stationary mean with radius of convergence at least one third the radius of convergence for the lifetime utility.

For the Cauchy-Kovalevsky Theorem to yield the unique analytic solution to a second order initial value problem one must provide two initial conditions for the ODE of the lifetime utility (15) and the price-dividend ratio (35), i.e. the value of the function and its derivative at a particular point. Duffie (2001) determines conditions under which the solution to these ODEs have solutions given by the Feynman-Kac formulas (20) and (40), respectively. As a result, the first initial conditions for the lifetime utility function $g(0)$ and price-dividend function $p(0)$ are chosen to be consistent with the Feynman-Kac formulas (20) and (40) at the stationary mean of the long run risk variable $x = 0$. Next it was shown that the elasticity of the lifetime utility (49) and the price-dividend function (48) are uniquely determine by the expected equity premium and its standard deviation. In addition, the discount factor β can be chosen using the risk free interest rate (82). Consequently, this information is used to set the second initial conditions for the lifetime utility function $g'(0)$ and the price-dividend function $p'(0)$. Thus, the initial conditions can be chosen so that the representative investor has behavior consistent with financial market data.

We found that the long run risk model is subject to the risk free interest rate problem identified by Weil (1989). In particular, the value of β influences the discount factor for the price-dividend function (41), as well as the price-dividend solution using the Feynman-Kac formula (40). For the Bansal and Yaron parameters the risk free rate has to be about 2% higher than its historic average for the stationary value of the price-dividend ratio to equal its historic average. Thus, the risk free interest rate is too low relative to its theorectical value consistent with the price-dividend ratio.

The paper concludes with an examination of the short term and long term properties of the expected returns on stocks in the long run risk model. After matching the average values for key financial variables within the data, we focused on the time series properties of the expected equity premium and its standard deviation. Over time, these moments depend on the convexity of the price-dividend function. An increase in the long run risk variable, which is expected to persist for a half life of 38 months, means that the price-dividend function would increase and then slowly revert toward its stationary mean. Consequently, we find predictable moments in stock prices. The convexity of the price-dividend function, when we use the parameters of Bansal, Kiku and Yaron (2007) means that both the expected equity premium and its standard deviation increase during expansions. This is contrary to the result found in CCH (2009) for the Campbell and Cochrane model using their parameters. Finally, the asymptotic rate of return on stocks (82) for the Bansal and Yaron model, as in Hansen, Heaton and Li (2008), is found without using the procedure of Hansen and Scheinkman (2009). This asymptotic rate of return includes the mean of the stochastic discount factor, the mean of the growth rate of dividends and an adjustment for long term risk. Finally, this procedure is used to price the long run risk free asset of Cochrane (2008), as well as its risk free rate of return.

These results for the high order polynomial approximations for the solutions of the long run risk model and the external habit model suggest that the debate over the properties of these models will be altered when these approximations are used to estimate the underlying parameters. Consequently, future research should implement the simulated method of moment estimation for these models, following Bansal, Gallant and Tauchen (2007), in which the high order polynomial approximations replace the quadratic approximations to these models. The Nonparametric simulation estimation of Bansal, Gallant, Hussey, and Tauchen (1993, 1995) can be undertaken since the simulated data from the high order polynomial approximations can be obtained in less than a minute.[27] Thus, one can merge the quick and accurate polynomial approximations of the long run risk model and external habit model, developed here and in CCH (2009), with an estimation method which can discriminate effectively between these models.

[27]This speed compares favorably with the speed reported in Bansal, Gallant, Hussey, and Tauchen (1993). See Gallant and Tauchen (2009) for a survey of these methods.

Table 1. Comparison of BY Model Relative to Data

Statistic	BY low $E_t(R^b)$	BY high $E_t(R^b)$	Bansal, Kiku Yaron Data
$E_t(R^e)$	0.083	0.083	0.083
$\sigma(R^e)$	0.20	0.20	0.20
$E_t(R^b)$	0.008	0.028	0.008
$E_t(R^e - R^b)$	0.075	0.055	0.075
$Sharpe$	0.375	0.275	0.0375
p	51.44	28.68	27.94

$Notes$: R^e is the real return on stocks and R^b is the real return on bonds, and p is the price-dividend ratio. E_t is the conditional expectation operator and σ is the standard deviation. The statistics for the theoretical solutions are evaluated at the stationary mean of the long run risk variable. The parameters for BY model are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The preference parameters are $\alpha = -9$, $\rho = \frac{1}{3}$ and $\beta = 0.0006902$. The initial conditions for the low interest rate case are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The initial conditions for the higher interest rate case are $g_0 = 1.1034$, $g_1 = 0.01864$, $p_0 = 344.16$, and $p_1 = 127.26$. The data for BY model is taken from Bansal, Kiku, and Yaron (2007) their Table 1 and 5.

Table 2. Comparison of Campbell and Cochrane Model Relative to Data

Statistic	Campbell Cochrane	Campbell Cochrane Data
$E_t(R^e)$	0.075	0.076
$\sigma(R)$	0.133	0.157
$E_t(R^b)$	0.009	0.009
$E_t(R^e - R^b)$	0.066	0.067
$Sharpe$	0.56	0.34
P	18.3	24.7

$Notes$: R^e is the real return on stocks and R^b is the real return on bonds, and P is the price-dividend ratio. E_t is the conditional expectation operator and σ is the standard deviation. The statistics for the theoretical solutions are evaluated at the historic average for the state variable. The parameters for Campbell and Cochrane's model from CCH are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9896$, $\gamma = 2$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 111.76$, $\bar{S} = 0.0448$ and $\mu r = 0.32$. The data for Campbell and Cochrane is taken from their Table 4. We use the Postwar Sample from 1947 to 1995 for the U.S.

Figure 1 displays a parametric plot of the standard deviation of consumption growth relative to the price-dividend ratio for the BY model. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The preference parameters are $\alpha = -9$, $\rho = \frac{1}{3}$ and $\beta = 0.0006902$. The initial conditions for the low interest rate case are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the price-dividend ratio and the y-axis records the standard deviation of consumption growth. The long run risk variable varies over the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$.

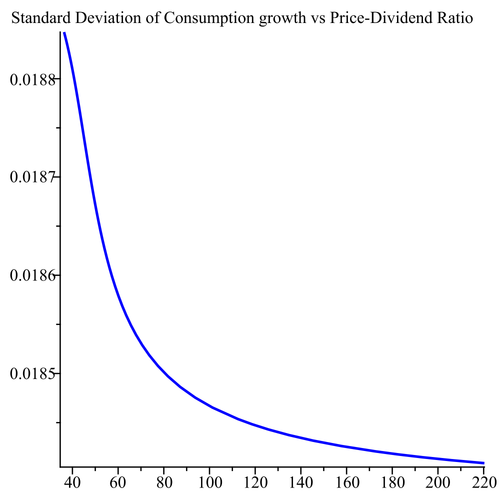

Figure 1

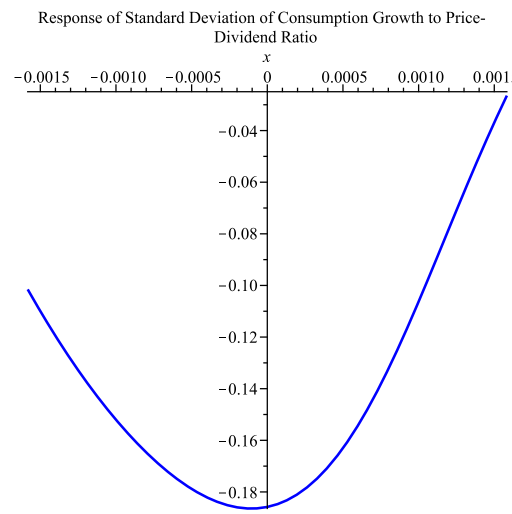

Figure 2

Figure 2 plots the derivative of the standard deviation of consumption growth with respect to the logarithm of the price-dividend ratio as the long run risk variable changes. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The initial conditions are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the long run risk variable in the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$. The y-axis records the derivative of the consumption growth with respect to the logarithm of the price-dividend ratio.

Figure 3 displays the risk neutral mean of the long run risk variable $\mu_g(x)$ for the investor's lifetime utility. of the representative investor for the BY model. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The preference parameters are $\alpha = -9$, $\rho = \frac{1}{3}$ and $\beta = 0.0006902$. The initial conditions for the low interest rate case are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the long run risk variable on the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$. The y-axis records the risk neutral mean of the long run risk variable.

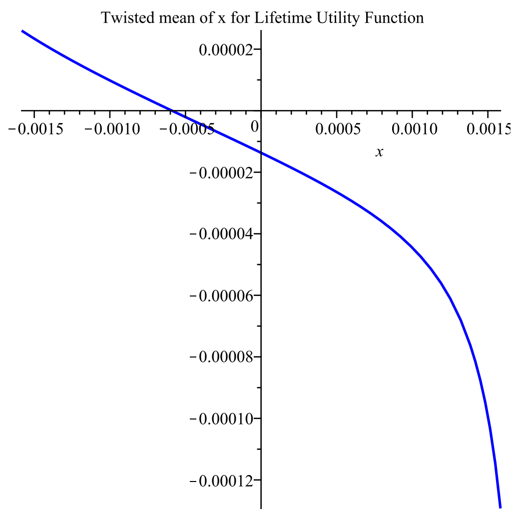

Figure 3

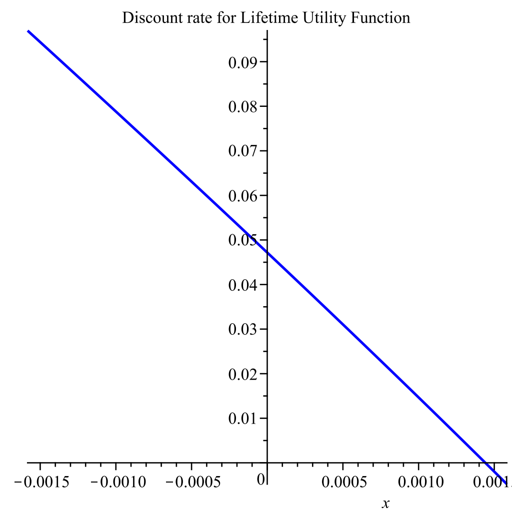

Figure 4

Figure 4 displays the rate of discount for the lifetime utility of the representative investor. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The initial conditions are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the long run risk variable in the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$. The y-axis records discount rate for the lifetime utility for the representative investor.

Figure 5 displays the lifetime utility of the representative investor for the BY model. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The preference parameters are $\alpha = -9$, $\rho = \frac{1}{3}$ and $\beta = 0.0006902$. The initial conditions are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the long run risk variable on the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$. The y-axis records the lifetime utility.

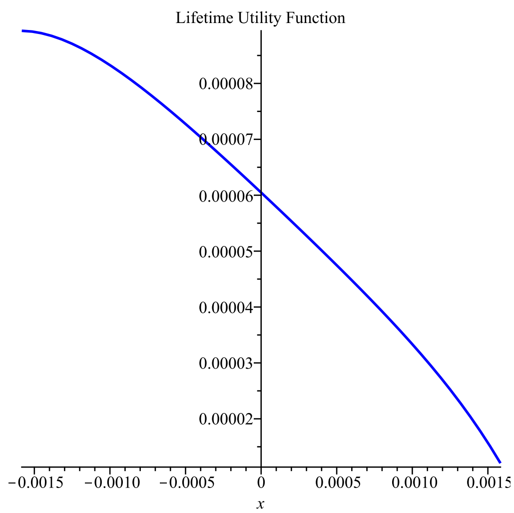

Figure 5

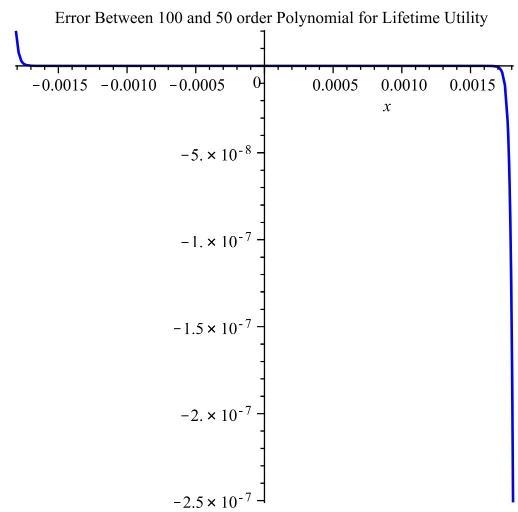

Figure 6

Figure 6 displays the relative error between a 90^{th} order and 100^{th} order Taylor polynomial for the representative investor's lifetime utility in the BY model. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The preference parameters are $\alpha = -9$, $\rho = \frac{1}{3}$ and $\beta = 0.0006902$. The initial conditions are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the long run risk variable in the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$. The y-axis records the lifetime utility for the representative investor.

Figure 7 displays the elasticity of the investor's lifetime utility for the BY model. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The preference parameters are $\alpha = -9$, $\rho = \frac{1}{3}$ and $\beta = 0.0006902$. The initial conditions are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the long run risk variable on the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$. The y-axis records the elasticity of the lifetime utility.

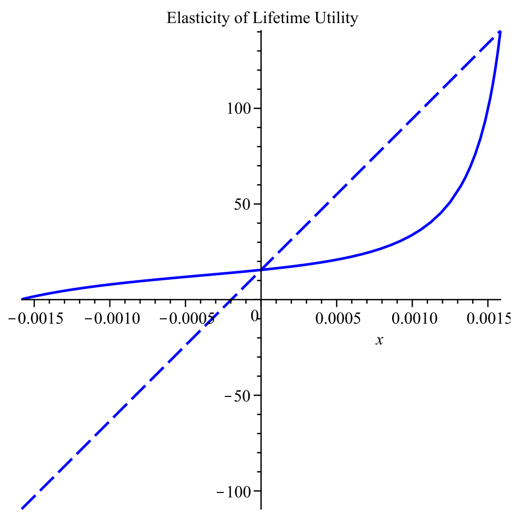

Figure 7

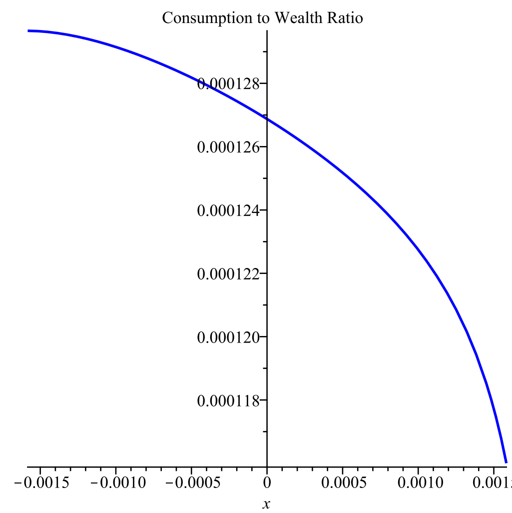

Figure 8

Figure 8 displays the consumption-wealth ratio for the representative investor in the BY model. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The preference parameters are $\alpha = -9$, $\rho = \frac{1}{3}$ and $\beta = 0.0006902$. The initial conditions are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the long run risk variable in the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$. The y-axis records the consumption to wealth ratio for the representative investor.

Figure 9 displays the price-dividend ratio for the BY model. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The preference parameters are $\alpha = -9$, $\rho = \frac{1}{3}$ and $\beta = 0.0006902$. The initial conditions are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the long run risk variable on the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$. The y-axis records the price-dividend ratio

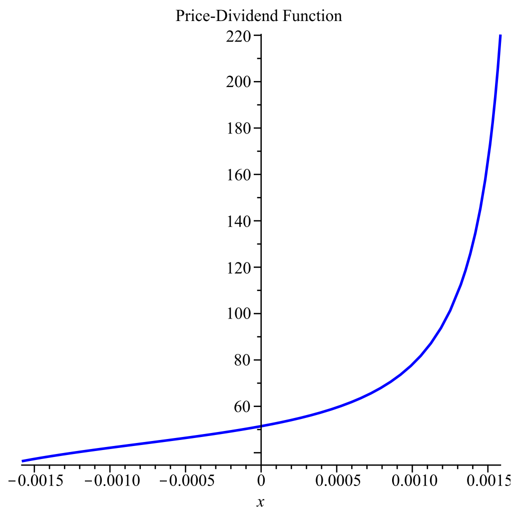

Figure 9

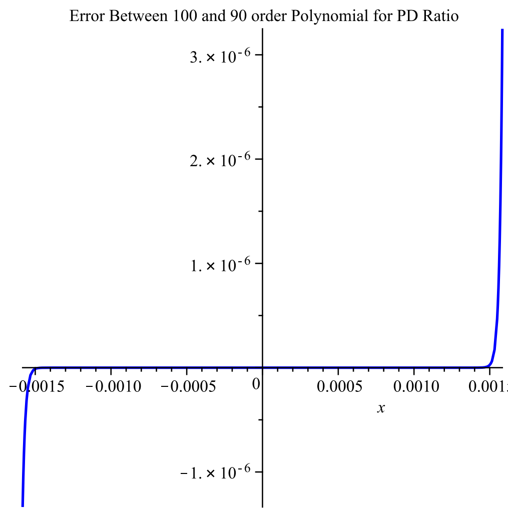

Figure 10

Figure 10 displays the relative error between a 90^{th} order and 100^{th} order Taylor polynomial for the price-dividend ratio in the BY model. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The preference parameters are $\alpha = -9$, $\rho = \frac{1}{3}$ and $\beta = 0.0006902$. The initial conditions are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the long run risk variable in the interval $[-7\varphi_e\sigma, -7\varphi_e\sigma]$. The y-axis records the price-dividend ratio.

Figure 11 displays the risk neutral mean of the long run risk variable $\mu_p(x)$ for the price-dividend ratio for the BY model. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The preference parameters are $\alpha = -9$, $\rho = \frac{1}{3}$ and $\beta = 0.0006902$. The initial conditions for the low interest rate case are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the long run risk variable on the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$. The y-axis records the risk neutral mean of the long run risk variable for the price-dividend ratio.

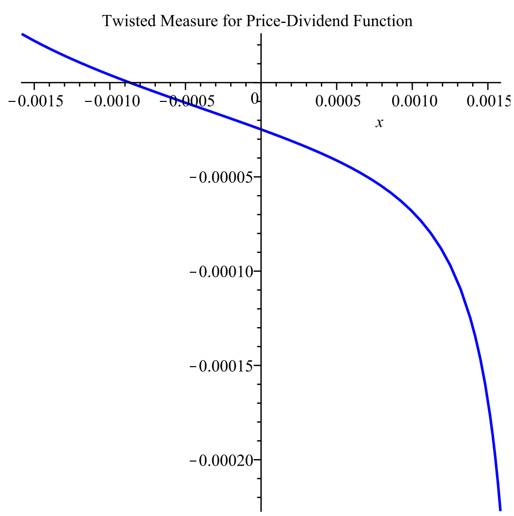

Figure 11

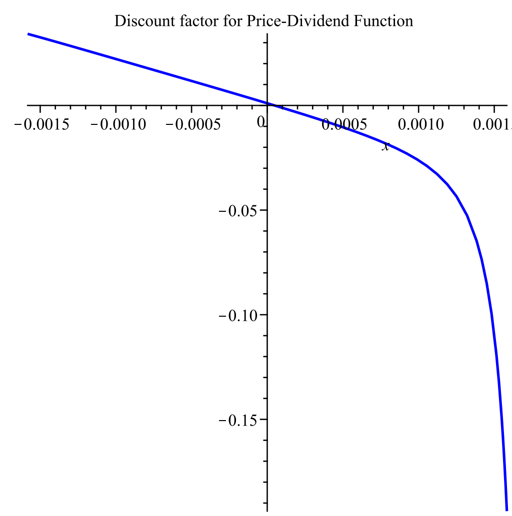

Figure 12

Figure 12 displays the rate of discount for the price-dividend ratio for the BY model. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The initial conditions are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the long run risk variable in the interval $[-7\varphi_e\sigma, 7\varphi_e\sigma]$. The y-axis records discount rate for the price-dividend ratio.

Figure 13 displays the price-dividend function in the Campbell and Cochrane model. The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9896$, $\gamma = 2$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 111.76$, $\bar{S} = 0.0448$ and $\mu r = 0.32$ come from CCH. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}] = [0.032, 0.061]$. The y-axis records the price-dividend ratio.

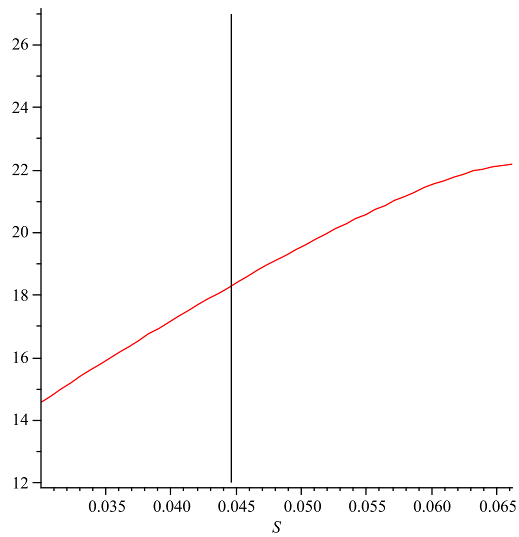

Figure 13

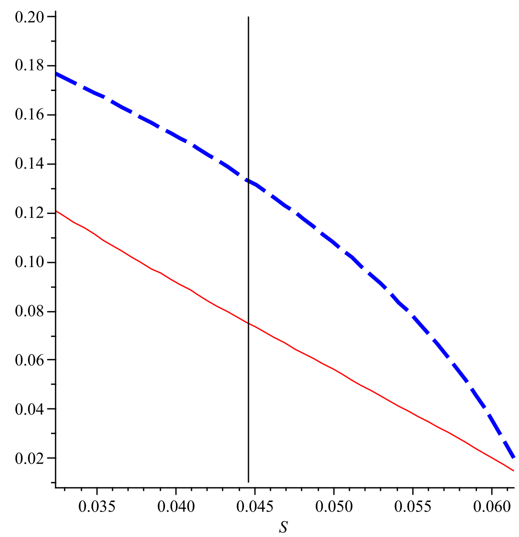

Figure 14

Figure 14 portrays the equity premium and standard deviation of equity in the continuous time model of Campbell and Cochrane. The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9896$, $\gamma = 2$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 111.76$, $\bar{S} = 0.0448$ and $\mu r = 0.32$ comes from CCH. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}] = [0.032, 0.061]$. The y-axis records the equity premium and standard deviation. The equity premium line is the solid line, while the dotted line represents the standard deviation.

Figure 15 displays the risk interest rate for the BY model. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The preference parameters are $\alpha = -9$, $\rho = \frac{1}{3}$ and $\beta = 0.0006902$. The initial conditions are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the long run risk variable on the interval $[-4\varphi_e\sigma, 4\varphi_e\sigma]$. The y-axis records the risk free interest rate.

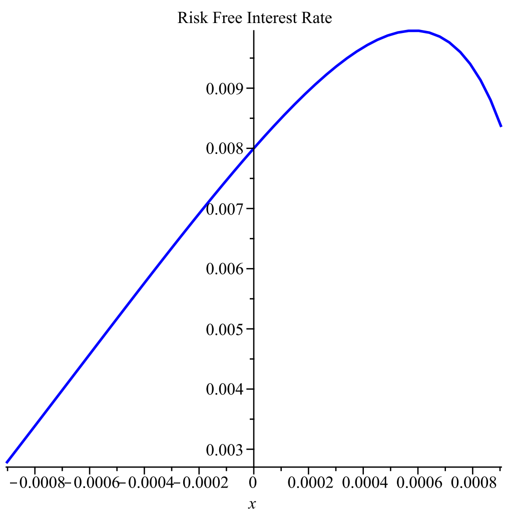

Figure 15

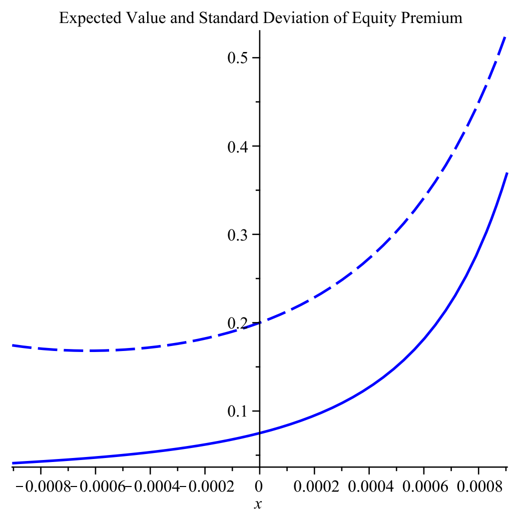

Figure 16

Figure 16 portrays the equity premium and standard deviation of equity premium for the BY model. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The preference parameters are $\alpha = -9$, $\rho = \frac{1}{3}$ and $\beta = 0.0006902$. The initial conditions are $g_0 = 1.4340$, $g_1 = 0.03527$, $p_0 = 617.23$, and $p_1 = 228.24$. The x-axis gives the long run risk variable on the interval $[-4\varphi_e\sigma, 4\varphi_e\sigma]$. The y-axis records the equity premium (solid line) and the standard deviation (dash line) for the equity premium.

Figure 17 displays risk neutral mean of x for the long run risk free asset for the BY model. The parameters are $\bar{x} = 0.0007573$, $\sigma_0 = 0.005385$, $\phi = 2.3$, $\varphi_d = 3.8$, $\kappa = 0.01816$, $\varphi_e = 0.042$, $\rho_{cx} = 0.2328$, $\rho_{cd} = 0.2465$, $a = 10.9$, and $b = 175.33$. The preference parameters are $\alpha = -9$, $\rho = \frac{1}{3}$ and $\beta = 0.0006902$. The initial conditions are $g_0 = 1.4340$, and $g_1 = 0.03527$. The x-axis gives the long run risk variable on the interval $[-6\varphi_e\sigma, 6\varphi_e\sigma]$. The y-axis records the risk free interest rate.

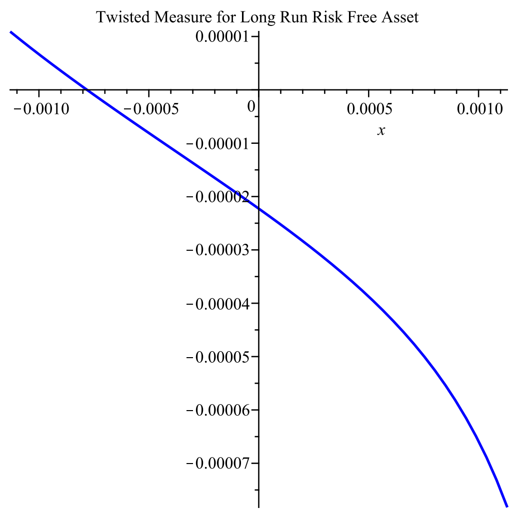

Figure 17

6 Appendix

Derivation of lifetime utility ODE (15). By the stochastic process (1) and Ito's lemma, we have

$$dC = \left[x + \bar{x} + \frac{\sigma^2(x)}{2}\right] C\,dt + \sigma(x)C\,d\tilde{\omega}_1, \tag{95}$$

where $\bar{C} = e^{\bar{x}}$ is the steady state consumption. The long run risk variable x follows the stochastic process (3). Base on (3), (95), and the change of variable which leads to the equivalent aggregator (10), the lifetime utility follows the backward stochastic differential equation (12).

By (12), the lifetime utility satisfies

$$E[dV] = -f(C,V)\,dt. \tag{96}$$

Duffie and Lions (1992) assume that the lifetime utility is a function of consumption. We will make the same assumption that $V = V(C,x)$, where x is included in the lifetime utility function because it is a driving force of consumption. Use Ito's lemma to compute $E[dV(C,x)]$ and equate this with $-f(C,V)$ to get the differential equation.

$$dV(C,x) = \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial C}\,dC + \frac{\partial V}{\partial x}\,dx + \frac{1}{2}\frac{\partial^2 V}{\partial C^2}(dC)^2 + \frac{\partial^2 V}{\partial C \partial x}(dC)(dx) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 \tag{97}$$

After plugging in (3) and (95), setting the expression equal to $-f(C,V)$, and dividing through by dt, one arrives at the line below.

$$\begin{aligned} 0 = f(C,V) &+ \left[x + \bar{x} + \frac{\sigma^2(x)}{2}\right] CV_C - \kappa x V_x \\ &+ \frac{\sigma^2(x)}{2} C^2 V_{CC} + \sigma^2(x)\varphi_e \rho_{xc} C V_{Cx} + \frac{\sigma^2(x)}{2}\varphi_e^2 V_{xx}. \end{aligned} \tag{98}$$

Recall (13) $V(C,x) = C^\alpha v(x)$ so that

$$f(C, V(C,x)) = C^\alpha f(1, v(x)). \tag{99}$$

Compute the second order partial derivatives of V.

$$V_C = \alpha C^{\alpha-1} v, \qquad V_x = C^\alpha v', \tag{100}$$

$$V_{CC} = \alpha(\alpha-1)C^{\alpha-2}v, \quad V_{Cx} = \alpha C^{\alpha-1}v', \quad V_{xx} = C^{\alpha}v''. \tag{101}$$

The equation (98) is equivalent to

$$0 = \frac{\beta}{\rho}(\alpha v)^{1-\rho/\alpha} + \left[\alpha(x+\bar{x}) + \frac{\sigma^2(x)}{2}\alpha^2 - \frac{\alpha\beta}{\rho}\right] v + [\sigma^2(x)\varphi_e\rho_{xc}\alpha - \kappa x]v' + \frac{\sigma^2(x)}{2}\varphi_e^2 v''. \tag{102}$$

Next use the change of variable (14) so that

$$v'(x) = \frac{1}{\rho}g(x)^{\alpha/\rho-1}g'(x) \tag{103}$$

and

$$v''(x) = \frac{1}{\rho}\left(\frac{\alpha}{\rho}-1\right)g(x)^{\alpha/\rho-2}g'(x)^2 + \frac{1}{\rho}g(x)^{\alpha/\rho-1}g''(x). \tag{104}$$

Substitute these results into (102) to get the ODE (15)

$$\begin{aligned} 0 = \frac{\beta}{g} + \left[\rho x + \rho\bar{x} + \frac{\sigma^2(x)}{2}\alpha\rho - \beta\right] + \left[\sigma^2(x)\varphi_e\rho_{xc}\alpha - \kappa x\right]\frac{g'}{g} \\ + \frac{\sigma^2(x)}{2}\varphi_e^2\left(\frac{\alpha}{\rho}-1\right)\left(\frac{g'}{g}\right)^2 + \frac{\sigma^2(x)}{2}\varphi_e^2\frac{g''}{g}. \end{aligned} \tag{105}$$

Derivation of SDE for state prices (30). The SDE (30) follows from (28). To find the differential in (28) first recall (97). Plugging in the partial derivatives of V (100) and (101), and dividing the result by V, one arrives at

$$\begin{aligned} \frac{dV}{V} &= \alpha\frac{dC}{C} + \frac{v'}{v}\,dx + \frac{1}{2}\alpha(\alpha-1)\left(\frac{dC}{C}\right)^2 + \alpha\frac{v'}{v}\left(\frac{dC}{C}\right)(dx) + \frac{1}{2}\frac{v''}{v}(dx)^2 \\ &= \left\{\alpha x + \alpha\bar{x} + \frac{\sigma^2(x)}{2}\alpha^2 + [\sigma^2(x)\varphi_e\rho_{xc}\alpha - \kappa x]\frac{v'}{v} + \frac{\sigma^2(x)}{2}\varphi_e^2\frac{v''}{v}\right\}dt \\ &\quad + \sigma(x)\alpha\,d\tilde{\omega}_1 + \sigma(x)\varphi_e\frac{v'}{v}\,d\tilde{\omega}_3. \end{aligned} \tag{106}$$

Remember that

$$f(C,V) = \frac{\beta}{\rho}\frac{C^{\rho} - (\alpha V)^{\rho/\alpha}}{(\alpha V)^{\rho/\alpha-1}}. \tag{107}$$

This implies that

$$f_V(C,V) = -\frac{\beta}{\rho}\left[\frac{(\rho-\alpha)C^{\rho}}{(\alpha V)^{\rho/\alpha}} + \alpha\right] = -\frac{\beta}{\rho}\left[\frac{\rho-\alpha}{(\alpha v)^{\rho/\alpha}} + \alpha\right] \tag{108}$$

and

$$f_C(C,V) = \beta \frac{C^{\rho-1}}{(\alpha V)^{\rho/\alpha - 1}} = \beta \frac{C^{\alpha-1}}{(\alpha v)^{\rho/\alpha-1}}. \tag{109}$$

Now take the partial derivatives of $f_C(C,V)$ which will be used in the computation of $d\Lambda$, and then evaluate using the functional form $V(C,x) = C^\alpha v(x)$.

The first order partial derivatives of $f_C(C,V)$ are

$$\begin{aligned}
\frac{\partial f_C}{\partial C}(C,V) &= \beta(\rho-1)\frac{C^{\rho-2}}{(\alpha V)^{\rho/\alpha-1}} = \frac{\rho-1}{C} f_C(C,V), \\
\frac{\partial f_C}{\partial V}(C,V) &= \beta(\alpha-\rho)\frac{C^{\rho-1}}{(\alpha V)^{\rho/\alpha}} = \frac{\alpha-\rho}{\alpha V} f_C(C,V).
\end{aligned} \tag{110}$$

The second order partial derivatives of $f_C(C,V)$ are

$$\begin{aligned}
\frac{\partial^2 f_C}{\partial C^2}(C,V) &= \beta(\rho-1)(\rho-2)\frac{C^{\rho-3}}{(\alpha V)^{\rho/\alpha-1}} = \frac{(\rho-1)(\rho-2)}{C^2} f_C(C,V), \\
\frac{\partial^2 f_C}{\partial C \partial V}(C,V) &= \beta(\alpha-\rho)(\rho-1)\frac{C^{\rho-2}}{(\alpha V)^{\rho/\alpha}} = \frac{(\alpha-\rho)(\rho-1)}{\alpha C V} f_C(C,V), \\
\frac{\partial^2 f_C}{\partial V^2}(C,V) &= \beta\rho(\rho-\alpha)\frac{C^{\rho-1}}{(\alpha V)^{\rho/\alpha+1}} = \frac{\rho(\rho-\alpha)}{\alpha^2 V^2} f_C(C,V).
\end{aligned} \tag{111}$$

By Ito's lemma, the pricing kernel Λ follows the stochastic process

$$\begin{aligned}
d\Lambda = \exp\left[\int_0^t f_V(C_s,V_s)\,ds\right] &\left[f_V f_C\,dt + \frac{\partial f_C}{\partial C} dC + \frac{\partial f_C}{\partial V} dV \right. \\
&\left. + \frac{1}{2}\frac{\partial^2 f_C}{\partial C^2}(dC)^2 + \frac{\partial^2 f_C}{\partial C \partial V}(dC)(dV) + \frac{1}{2}\frac{\partial^2 f_C}{\partial V^2}(dV)^2 \right] \\
= \Lambda &\left[f_V\,dt + (\rho-1)\frac{dC}{C} + \frac{\alpha-\rho}{\alpha}\frac{dV}{V} + \frac{(\rho-1)(\rho-2)}{2}\left(\frac{dC}{C}\right)^2 \right. \\
&\left. + \frac{(\alpha-\rho)(\rho-1)}{\alpha}\left(\frac{dC}{C}\right)\left(\frac{dV}{V}\right) + \frac{\rho(\rho-\alpha)}{2\alpha^2}\left(\frac{dV}{V}\right)^2 \right].
\end{aligned} \tag{112}$$

Divide the equation (112) by Λ to get

$$\begin{aligned}
\frac{d\Lambda}{\Lambda} = &\left\{ (\alpha-1)x + (\alpha-1)\bar{x} - \frac{\alpha\beta}{\rho} + \frac{\sigma^2(x)}{2}(\alpha-1)^2 - \frac{\beta(\rho-\alpha)}{\rho(\alpha v)^{\rho/\alpha}} \right. \\
&+ \left[\frac{\kappa(\rho-\alpha)}{\alpha}x - \sigma^2(x)\varphi_e\rho_{xc}\frac{(\rho-\alpha)(\alpha-1)}{\alpha}\right]\frac{v'}{v} \\
&\left. + \frac{\sigma^2(x)}{2}\frac{\varphi_e^2\rho(\rho-\alpha)}{\alpha^2}\left(\frac{v'}{v}\right)^2 + \frac{\sigma^2(x)}{2}\frac{\varphi_e^2(\alpha-\rho)}{\alpha}\frac{v''}{v} \right\} dt \\
&+ \sigma(x)(\alpha-1)\,d\tilde{\omega}_1 + \frac{\sigma(x)\varphi_e(\alpha-\rho)}{\alpha}\frac{v'}{v}\,d\tilde{\omega}_3.
\end{aligned} \tag{113}$$

Apply the substitution described in (14), (103), and (104).

$$\begin{aligned}\frac{d\Lambda}{\Lambda} &= \Bigg\{(\alpha-1)x + (\alpha-1)\bar{x} - \frac{\alpha\beta}{\rho} + \frac{\sigma^2(x)}{2}(\alpha-1)^2 - \frac{\beta(\rho-\alpha)}{\rho g} \\ &\quad + \left[\frac{\kappa(\rho-\alpha)}{\rho}x - \sigma^2(x)\varphi_e\rho_{xc}\frac{(\rho-\alpha)(\alpha-1)}{\rho}\right]\frac{g'}{g} \\ &\quad + \frac{\sigma^2(x)}{2}\frac{\varphi_e^2(\rho-\alpha)(2\rho-\alpha)}{\rho^2}\left(\frac{g'}{g}\right)^2 + \frac{\sigma^2(x)}{2}\frac{\varphi_e^2(\alpha-\rho)}{\rho}\frac{g''}{g}\Bigg\}\,dt \\ &\quad + \sigma(x)(\alpha-1)\,d\tilde{\omega}_1 + \frac{\sigma(x)\varphi_e(\alpha-\rho)}{\rho}\frac{g'}{g}\,d\tilde{\omega}_3 \\ &= \mu_\Lambda(x)\,dt + \sigma(x)(\alpha-1)\,d\tilde{\omega}_1 + \frac{\sigma(x)\varphi_e(\alpha-\rho)}{\rho}\frac{g'}{g}\,d\tilde{\omega}_3. \end{aligned} \tag{114}$$

Rewrite the differential equation (15) for the lifetime utility function as

$$\begin{aligned}\frac{\sigma^2(x)}{2}\frac{\varphi_e^2(\rho-\alpha)}{\rho}\left(\frac{g'}{g}\right)^2 &= \frac{\beta}{g} + \left[\rho x + \rho\bar{x} + \frac{\sigma^2(x)}{2}\alpha\rho - \beta\right] - \left[\kappa x - \sigma^2(x)\varphi_e\rho_{xc}\alpha\right]\frac{g'}{g} \\ &\quad + \frac{\sigma^2(x)}{2}\varphi_e^2\frac{g''}{g}. \end{aligned} \tag{115}$$

Substitute this result into (114) to get a new expression for the instantaneous mean of Λ.

$$\begin{aligned}\mu_\Lambda(x) &= (2\rho-1)x + (2\rho-1)\bar{x} - 2\beta + \frac{\sigma^2(x)}{2}(2\alpha\rho - 2\alpha + 1) + \frac{\beta}{g(x)} \\ &\quad + \left[\sigma^2(x)\varphi_e\rho_{xc}\frac{\alpha\rho+\rho-\alpha}{\rho} - \kappa x\right]\frac{g'(x)}{g(x)} + \frac{\sigma^2(x)}{2}\varphi_e^2\frac{g''(x)}{g(x)} \end{aligned} \tag{116}$$

The rate of return for the risk-free bonds is determined by

$$R^b(C_t)\,dt = -E_t\left[d\Lambda_t/\Lambda_t\right] = -\mu_\Lambda(x_t)\,dt. \tag{117}$$

This completes the derivation of the SDE for the state prices (30) and its mean (31).

Derivation of price-dividend ratio ODE (35). The differential equation for the stock price (34) is equivalent to

$$\Lambda(t)D(t)\,dt + E_t\left[d(\Lambda(t)p(t)D(t))\right] = 0. \tag{118}$$

By Ito's lemma

$$\frac{d(\Lambda pD)}{\Lambda pD} = \frac{d\Lambda}{\Lambda} + \frac{dp}{p} + \frac{dD}{D} + \frac{d\Lambda dp}{\Lambda p} + \frac{dDdp}{Dp} + \frac{d\Lambda dD}{\Lambda D}. \tag{119}$$

Consequently, (118) is equivalent further to

$$\frac{dt}{p} + E_t\left[\frac{d\Lambda}{\Lambda} + \frac{dp}{p} + \frac{dD}{D} + \frac{d\Lambda dp}{\Lambda p} + \frac{dDdp}{Dp} + \frac{d\Lambda dD}{\Lambda D}\right] = 0. \tag{120}$$

The dividend growth is found by applying Ito's lemma to (2) so that

$$\frac{dD}{D} = \left[\phi x + \bar{x} + \frac{\sigma^2(x)}{2}\varphi_d^2\right] dt + \sigma(x)\varphi_d\, d\tilde{\omega}_2. \tag{121}$$

The price-dividend ratio function $p = p(x)$ is assumed to be dependent on the long run risk variable, since it is the only variable impacting dividend growth (33) or the state price process (30). By Ito's Lemma

$$dp = p'(x)dx + \frac{1}{2}p''(x)(dx)^2. \tag{122}$$

The stochastic process (3) for the long run risk implies $(dx)^2 = \sigma^2(x)\varphi_e^2\, dt$.

$$dp = \left[-\kappa x p'(x) + \frac{\sigma^2(x)}{2}\varphi_e^2 p''(x)\right] dt + \sigma(x)\varphi_e p'(x)\, d\tilde{\omega}_3 \tag{123}$$

Calculate $E_t[d\Lambda/\Lambda]$, $E_t[dp/p]$, $E_t[dD/D]$, $E_t[d\Lambda dp/\Lambda p]$, $E_t[dDdp/Dp]$, and $E_t[d\Lambda dD/\Lambda D]$.

$$\begin{aligned}
E_t\left[\frac{d\Lambda}{\Lambda}\right] &= F(x)dt, \\
E_t\left[\frac{dp}{p}\right] &= \left[-\kappa x\frac{p'(x)}{p(x)} + \frac{\sigma^2(x)}{2}\varphi_e^2\frac{p''(x)}{p(x)}\right] dt, \\
E_t\left[\frac{dD}{D}\right] &= \left[\phi x + \bar{x} + \frac{\sigma^2(x)}{2}\varphi_d^2\right] dt, \\
E_t\left[\frac{d\Lambda dp}{\Lambda p}\right] &= \sigma^2(x)\left[\varphi_e\rho_{xc}(\alpha-1) + \frac{\varphi_e^2(\alpha-\rho)}{\rho}\frac{g'(x)}{g(x)}\right]\frac{p'(x)}{p(x)}\, dt, \\
E_t\left[\frac{dDdp}{Dp}\right] &= \sigma^2(x)\varphi_e\varphi_d\rho_{xc}\rho_{cd}\frac{p'(x)}{p(x)}\, dt, \\
E_t\left[\frac{d\Lambda dD}{\Lambda D}\right] &= \sigma^2(x)\varphi_d\rho_{cd}\left[\alpha - 1 + \frac{\varphi_e\rho_{xc}(\alpha-\rho)}{\rho}\frac{g'(x)}{g(x)}\right] dt.
\end{aligned} \tag{124}$$

By (120), one obtains the ordinary differential equation

$$\begin{aligned}
1 &+ \left\{\mu_\Lambda(x) + \phi x + \bar{x} + \frac{\sigma^2(x)}{2}\varphi_d^2 + \sigma^2(x)\varphi_d\rho_{cd}\left[\alpha - 1 + \frac{\varphi_e\rho_{xc}(\alpha-\rho)}{\rho}\frac{g'(x)}{g(x)}\right]\right\} p \\
&+ \left\{\sigma^2(x)\varphi_e\rho_{xc}(\alpha - 1 + \varphi_d\rho_{cd}) - \kappa x + \sigma^2(x)\frac{\varphi_e^2(\alpha-\rho)}{\rho}\frac{g'(x)}{g(x)}\right\} p' \\
&+ \frac{\sigma^2(x)}{2}\varphi_e^2 p'' = 0.
\end{aligned} \tag{125}$$

Finally, use the expression for $\mu_\Lambda(x)$ in (31) to find the final form of the differential equation for the price-dividend ratio function.

$$
\begin{aligned}
1 &+ \left\{ \rho x + 2\rho\bar{x} - 2\beta + \frac{\sigma^2(x)}{2}[\varphi_d^2 + 2\varphi_d\rho_{cd}(\alpha-1) + 2\alpha\rho - 2\alpha + 1] + \frac{\beta}{g(x)} \right. \\
&\left. + \left[\frac{\sigma^2(x)\varphi_e\rho_{xc}}{\rho}\left(\rho\alpha + \rho - \alpha + \varphi_d\rho_{cd}(\alpha-\rho)\right) - \kappa x \right] \frac{g'(x)}{g(x)} + \frac{\sigma^2(x)}{2}\varphi_e^2 \frac{g''(x)}{g(x)} \right\} p \\
&+ \left\{ \sigma^2(x)\varphi_e\rho_{xc}(\alpha - 1 + \varphi_d\rho_{cd}) - \kappa x + \sigma^2(x)\frac{\varphi_e^2(\alpha-\rho)}{\rho}\frac{g'(x)}{g(x)} \right\} p' \\
&+ \frac{\sigma^2(x)}{2}\varphi_e^2 p'' = 0.
\end{aligned} \tag{126}
$$

Derivation of long term risk free asset ODE (88). The differential equation for the long term risk free asset (34) is equivalent to

$$\left(\frac{dt}{k} - R^b\right) dt + E_t\left[\frac{dk}{k} + \frac{d\Lambda dk}{\Lambda k}\right] = 0, \tag{127}$$

since dividends is always one unit of consumption $D(t) = 1$.

The price of the long term risk free asset $k = k(x)$ is assumed to be dependent on the long run risk variable, since it is the only variable impacting the state price process (30). By Ito's Lemma

$$dk = k'(x)dx + \frac{1}{2}k''(x)(dx)^2. \tag{128}$$

The stochastic process (3) for the long run risk implies $(dx)^2 = \varphi_e^2\sigma^2(x)\,dt$ so that

$$\frac{dk}{k} = \left[-\kappa x\frac{k'(x)}{k(x)} + \frac{\varphi_e^2\sigma^2(x)}{2}\frac{k''(x)}{k(x)}\right] dt + \sigma(x)\varphi_e\frac{k'(x)}{k(x)}\,d\tilde{\omega}_3 \tag{129}$$

As a result,

$$E_t\left[\frac{d\Lambda dk}{\Lambda k}\right] = \frac{\varphi_e^2\sigma^2(x)(\alpha-\rho)}{\rho}\frac{g'(x)}{g(x)}\frac{k'(x)}{k(x)}\,dt. \tag{130}$$

Substituting (129) and (130) into (127) yields (88).

Proof of The Cauchy-Kovalevsky Theorem in $\mathbb{R}^2$. Consider the initial value problem (IVP) for second-order linear partial differential equation of the form

$$\frac{\partial^2 u}{\partial t^2} = A(x,t)\frac{\partial^2 u}{\partial x^2} + B(x,t)\frac{\partial^2 u}{\partial x\partial t} + C(x,t)\frac{\partial u}{\partial x} + D(x,t)\frac{\partial u}{\partial t} + E(x,t)u + g(x,t) \tag{131}$$

$$u(x,0) = u_0(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \tag{132}$$

where the coefficients are analytic functions about $(x,t) = (0,0)$.

Theorem 6.1. *There is a unique analytic solution $u(x,t)$ to the initial value problem (131)–(132) near $(0,0)$. If the coefficients A, B, C, D, E and the force term g are analytic in the square $\{(x,t) \in \mathbf{R}^2 : |x| < r,\, |t| < r\}$, and furthermore the coefficients are bounded in absolute value by M and the force term g is bounded in absolute value by L, then the region of analyticity of the solution contains the set*

$$\left\{(x,t) \in \mathbf{R}^2 : |x + \rho t| < r\left(1 - \frac{M(\rho+1)}{\rho^2}\right)\right\}, \tag{133}$$

where $\rho > 1$ and large enough so that $M(\rho+1)/\rho^2 < 1$.

Proof: This proof is written along the lines of the proof in the book of Mizohata (1973), in which the Cauchy-Kovalevsky Theorem is proved for general linear PDE. Letting

$$a_1(x,t,\frac{\partial}{\partial x}) = B(x,t)\frac{\partial}{\partial x} + D(x,t) \tag{134}$$

and

$$a_0(x,t,\frac{\partial}{\partial x}) = A(x,t)\frac{\partial^2}{\partial^2 x} + C(x,t)\frac{\partial}{\partial x} + E(x,t) \tag{135}$$

equation (131) can be written as

$$\frac{\partial^2 u}{\partial t^2} = a_1(x,t,\frac{\partial}{\partial x})\frac{\partial u}{\partial t} + a_0(x,t,\frac{\partial}{\partial x})u + g(x,t) \tag{136}$$

Now, making the following change of the dependent variable

$$\tilde{u}(x,t) = u(x,t) - u_1(x)t - u_0(x), \text{ or } u(x,t) = \tilde{u}(x,t) + u_1(x)t + u_0(x) \tag{137}$$

equation (136) becomes

$$\frac{\partial^2 \tilde{u}}{\partial t^2} = a_1(x,t,\frac{\partial}{\partial x})\frac{\partial \tilde{u}}{\partial t} + a_0(x,t,\frac{\partial}{\partial x})\tilde{u} + f(x,t)$$

where

$$f(x,t) = g(x,t) + a_1(x,t,\frac{\partial}{\partial x})u_1(x) + a_0(x,t,\frac{\partial}{\partial x})[u_1(x)t + u_0(x)]. \tag{138}$$

Also, we have

$$\tilde{u}(x,0) = 0 \quad and \quad \frac{\partial \tilde{u}}{\partial t}(x,0) = 0.$$

Thus, after dropping the the "tildes" the initial value problem (131)–(132) takes the form

$$\frac{\partial^2 u}{\partial t^2} = a_1(x,t,\frac{\partial}{\partial x})\frac{\partial u}{\partial t} + a_0(x,t,\frac{\partial}{\partial x})u + f(x,t) \tag{139}$$

$$u(x,0) = 0, \text{ and } \frac{\partial u}{\partial t}(x,0) = 0, \tag{140}$$

We are looking for a solution which is analytic near the origin. Therefore it should have a series expansion of the form:

$$u(x,t) = \sum_{k,j=0}^{\infty} c_{k,j} x^k t^j. \tag{141}$$

Since the initial data are zero, we must have

$$u(x,0) = \sum_{k,j=0}^{\infty} c_{k,0} x^k = 0, \text{ and } \frac{\partial u}{\partial t}(x,0) = \sum_{k=0}^{\infty} c_{k,1} x^k = 0,$$

which gives

$$c_{k,0} = 0 \text{ and } c_{k,1} = 0, \text{ for all } k = 0,1,2,\cdots.$$

Therefore the solution is of the form:

$$u(x,t) = \sum_{k=0,\, j=2}^{\infty} c_{k,j} x^k t^j. \tag{142}$$

Next, letting $t = 0$ in PDE (139) and using the data $u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = 0$, gives

$$\frac{\partial^2 u}{\partial t^2}(x,0) = \Big(a_1(x,t,\frac{\partial}{\partial x})\frac{\partial u}{\partial t}(x,t) + a_0(x,t,\frac{\partial}{\partial x})u(x,t) + f(x,t)\Big)\Big|_{t=0} = f(x,0). \tag{143}$$

Since

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \sum_{k=0,\, j=2}^{\infty} j(j-1) c_{k,j} x^k t^{j-2},$$

in terms of power series the last relation reads as

$$\sum_{k=0}^{\infty} 2! c_{k,2} x^k = \sum_{k=0}^{\infty} c_{k,0}^f x^k,$$

or

$$c_{k,2} = \frac{1}{2!}\, c^f_{k,0},\ k = 0, 1, 2, \cdots . \tag{144}$$

Next, differentiating PDE (139) with respect to t gives

$$\begin{aligned}\frac{\partial^3 u}{\partial t^3}(x,t) = a_1(x,t,\frac{\partial}{\partial x})\frac{\partial^2 u}{\partial t^2}(x,t) + a_0(x,t,\frac{\partial}{\partial x})\frac{\partial u}{\partial t}(x,t) + \frac{\partial f}{\partial t}(x,t)\\ +\frac{\partial a_1}{\partial t}(x,t,\frac{\partial}{\partial x})\frac{\partial u}{\partial t} + \frac{\partial a_0}{\partial t}(x,t,\frac{\partial}{\partial x})u\end{aligned}$$

Evaluating both sides of this equation at $t = 0$ and using the zero-initial data and relation (143) gives

$$\frac{\partial^3 u}{\partial t^3}(x,0) = a_1(x,t,\frac{\partial}{\partial x})f(x,t)\Big|_{t=0} + \frac{\partial f}{\partial t}(x,t)\Big|_{t=0},$$

or

$$\frac{\partial^3 u}{\partial t^3}(x,0) = B(x,t)\frac{\partial f}{\partial x}(x,t)\Big|_{t=0} + D(x,t)f(x,t)\Big|_{t=0} + \frac{\partial f}{\partial t}(x,t)\Big|_{t=0}, \tag{145}$$

which in terms of power series reads as follows:

$$\begin{aligned}\sum_{k=0}^{\infty} 3!c_{k,3}x^k =& \Big(\sum_{k=0}^{\infty} c^B_{k,0}x^k\Big)\Big(\sum_{k=1}^{\infty} k\, c^f_{k,0}x^{k-1}\Big)\\ &+\Big(\sum_{k=0}^{\infty} c^D_{k,0}x^k\Big)\Big(\sum_{k=0}^{\infty} c^f_{k,0}x^k\Big) + \sum_{k=0}^{\infty} c^f_{k,1}x^k\end{aligned} \tag{146}$$

After, multiplying and equating the coefficients of same powers we obtain the relation

$$c_{k,3} = \frac{1}{3!}\Big\{\sum_{\ell=0}^{k}(c^B_{k-\ell,0}\cdot(\ell+1)\, c^f_{\ell+1,0} + c^D_{k-\ell,0}c^f_{\ell,0}) + c^f_{k,1}\Big\},\ k = 0, 1, 2, \cdots . \tag{147}$$

Continuing this way its easy to see that for any fixed $j \geq 2$ we have that the coefficients of the solution $c_{k,j}$ are determined uniquely. Also, each $c_{k,j}$ is expressed as a polynomial $Q_{k,j}$ with positive coefficients and with variables the coefficients of $c^A_{m,\ell}$, $c^B_{m,\ell}$, $c^C_{m,\ell}$, $c^D_{m,\ell}$, $c^E_{m,\ell}$ and $c^f_{m,\ell}$ with $m \leq k+2$ and $\ell < j$. That is,

$$c_{k,j} = Q_{k,j}(c^A_{m,\ell}, c^B_{m,\ell}, c^C_{m,\ell}, c^D_{m,\ell}, c^E_{m,\ell}, c^f_{m,\ell})_{m\leq k+2,\,\ell<j}. \tag{148}$$

Next, let us assume that all coefficients and the force function are analytic in the square defined by $|x| \leq r$ and $|t| \leq r$. Also, assume that the coefficients are bounded in abulute value by M and the function f by L. Then, using the Cauchy's integral formula in $\mathbb{C}^2$

$$h(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{|\zeta_1|=r} \int_{|\zeta_2|=r} \frac{h(\zeta_1, \zeta_2)}{(z-\zeta_1)(z-\zeta_2)} d\zeta_1 d\zeta_2$$

for a holomorphic function $h(z_1, z_2)$ on the polydisc $D_2 = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \leq r, |z_2| \leq r\}$ it follows that

$$\frac{\partial^{k+j} h}{\partial z_1^k \partial z_2^j}(z_1, z_2) = \frac{k!j!}{(2\pi i)^2} \int_{|\zeta_1|=r} \int_{|\zeta_2|=r} \frac{h(\zeta_1, \zeta_2)}{(z-\zeta_1)^{k+1}(z-\zeta_2)^{j+1}} d\zeta_1 d\zeta_2.$$

If $|h(z_1, z_2)| \leq M$ on the polydisc then at the origin this relation gives

$$|\frac{\partial^{k+j} h}{\partial z_1^k \partial z_2^j}(0,0)| \leq M \frac{k!j!}{r^{k+j}}. \tag{149}$$

Using inequality (149) we see that the the coefficients of the power series of the function

$$H(z_1, z_2) = \frac{M}{(1 - z_1/r)(1 - z_2/r)} \tag{150}$$

dominate the coefficients of the power series of the function $h(z_1, z_2)$ in the polydiss D_2.

Therefore, the power series of the coefficients of our PDE are dominated by the power series of the function $M/[(1 - z_1/r)(1 - z_2/r)]$ and the power series of the force term $f(x,t)$ is dominated by the power series of the function $L/[(1 - z_1/r)(1 - z_2/r)]$. Furthermore, observe that the power series of $1/[(1 - z_1/r)(1 - z_2/r)]$ is dominated by the power series of $1/[1 - (z_1 + z_2)/r)]$. Also, for any $\rho > 1$ the power series of $1/[1 - (z_1 + z_2)/r)]$ is dominated by the power series of $1/[1 - (z_1 + \rho z_2)/r)]$. Therefore, the following holds:

(I) *The power series of the coefficients $A(x,t), B(x,t), C(x,t), D(x,t)$ and $E(x,t)$ of our PDE (139) are dominated by the power series of:*

$$\frac{M}{1 - (x + \rho t)/r}, \quad \text{for } |\frac{x + \rho t}{r}| < 1, \tag{151}$$

and

(II) *The power series of the force term $f(x,t)$ is dominated by the power series of:*

$$\frac{L}{1-(x+\rho t)/r}, \quad \text{for } \left|\frac{x+\rho t}{r}\right| < 1. \tag{152}$$

Replacing all the coefficients of the PDE (139) with $M/[1-(x+\rho t)/r]$ and the force term $f(x,t)$ with $L/[1-(x+\rho t)/r]$ we obtain the following "dominant" PDE

$$\frac{\partial^2 w}{\partial t^2} = \tilde{a}_1(x,t,\frac{\partial}{\partial x})\frac{\partial w}{\partial t} + \tilde{a}_0(x,t,\frac{\partial}{\partial x})w + \tilde{f}(x,t) \tag{153}$$

$$\tilde{a}_1(x,t,\frac{\partial}{\partial x}) = \frac{M}{1-(x+\rho t)/r}\Big[\frac{\partial}{\partial x}+1\Big], \tag{154}$$

$$\tilde{a}_0(x,t,\frac{\partial}{\partial x}) = \frac{M}{1-(x+\rho t)/r}\Big[\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial x}+1\Big], \tag{155}$$

and

$$\tilde{f}(x,t) = \frac{L}{1-(x+\rho t)/r}. \tag{156}$$

Look for a solution $w(x,t)$ to (153) of the form

$$w(x,t) = \varphi(\frac{x+\rho t}{r}), \tag{157}$$

where $\varphi(s)$ is a function of a single variable, then PDE (153) reduces to the following ODE:

$$\left(\frac{\rho}{r}\right)^2\varphi'' = \frac{1}{1-s}\left(\frac{M\rho}{r^2}\varphi'' + \frac{M\rho}{r}\varphi' + \frac{M}{r^2}\varphi'' + \frac{M}{r}\varphi' + M\varphi + L\right). \tag{158}$$

Solving for φ'' the last equation gives

$$[(1-s)\rho^2 - M(\rho+1)]\varphi'' = Mr(\rho+1)\varphi' + Mr^2\varphi + Lr^2,$$

or

$$\varphi'' = \frac{1/\rho^2}{1-M(\rho+1)/\rho^2-s}[Mr(\rho+1)\varphi' + Mr^2\varphi + Lr^2]. \tag{159}$$

Using the analyticity theorem for ODE (presented in CCH) one finds that no matter what are the initial conditions at zero differental equation (159) has an analytic solution with radius of convergence given by:

$$|s| < 1 - \frac{M(\rho+1)}{\rho^2}$$

which gives the following (x, t)-region of convergence for the solution w of the partial differential equation (153)

$$|x + \rho t| < r\left(1 - \frac{M(\rho+1)}{\rho^2}\right).$$

The constant $\rho > 1$ should be chosen so that

$$\frac{M(\rho+1)}{\rho^2} < 1.$$

In fact, it should be chosen in a way that makes the region of convergence:

$$\left\{(x,t) \in \mathbf{R}^2 : |x + \rho t| < r\left(1 - \frac{M(\rho+1)}{\rho^2}\right)\right\},$$

largest.

Finally, choosing initial data $\varphi(0) \geq 0$ and $\varphi'(0) \geq 0$ and using equation (159) one finds that

$$\varphi''(0) = \frac{1/\rho^2}{1 - M(\rho+1)/\rho^2}[Mr(\rho+1)\varphi'(0) + Mr^2\varphi(0) + Lr^2] \geq 0. \tag{160}$$

Differentiating equation (159) repeatedly and using previous information it follows that

$$\frac{d^k\varphi}{ds^k}(0) \geq 0, \quad \text{for all} \quad k = 0, 1, 2, \cdots.$$

Thus the coefficients of the power series of $\varphi(s)$ are non-negative. Therefore, the coefficients of the power series of the initial data

$$w(x,0) = \frac{1}{r}\varphi(x/r), \quad \text{and} \quad \frac{\partial w}{\partial t}(x,0) = \frac{\rho}{r}\varphi'(x/r), \tag{161}$$

are nonegative. Therefore, the coefficients of the solution to the initial value problem (153)-(161) dominate those of the solution to the initial value problem (139)–(140), since they are both expressed by the same universal polynomial $Q_{j,k}$, described in (148). Thus, the PDE (131) has a unique analytic solution near zero with region of convergence containing the set described in (133).

□

Remark. To obtain the largest region of convergence for the solution one should begin with the largest region of congergence for the coeefficients the force function and the data. That could be more complicated than a square, at least it could be a rectangle. Also, one should use the least upper bound for the coeeficients M. Finally, one should use the "best" dominant PDE and should try to solve it explicitly or find the best estimate for the region of analyticity of its solution. The PDE's encountered in financial and economics models are special and one could do better analysis for them as in illustrated by the analysis in section 4 of (15).

Proof of Lemmas 3.2 and 3.3. The proof of Lemma 3.2 and Lemma 3.3 follows:

Proof: The sequence $\{\sum_{k=1}^{n}(1/k) - \ln(n+1)\}_{n=1}^{\infty}$ is an increasing sequence and converges to the *Euler-Mascheroni constant* $\gamma \approx 0.5772$. So $\sum_{k=1}^{n} \frac{1}{k} \leq \gamma + \ln(n+1)$ for $n = 1, 2, 3, \ldots$.

$$\sum_{k=a}^{n-b} \frac{1}{(k+c)(n-k+d)} = \frac{1}{n+c+d} \sum_{k=a}^{n-b} \left(\frac{1}{k+c} + \frac{1}{n-k+d} \right) \leq \frac{2}{n+c+d} \sum_{k=1}^{n+c+d} \frac{1}{k}$$
$$\leq \frac{2[1+\ln(n+c+d+1)]}{n+c+d} = U_{n+c+d}$$

□

Proof: Recall that $U_k = 2[1+\ln(k+1)]/k$ for $k = 1, 2, 3, \ldots$.

$$\sum_{k=b}^{n} \frac{1}{k+d} \cdot \frac{B^{n-k}}{(n-k)!} = \frac{1}{n+d} + \sum_{k=b}^{n-1} \frac{1}{k+d} \cdot \frac{B^{n-k}}{(n-k)!}$$
$$= \frac{1}{n+d} + B \sum_{k=b}^{n-1} \frac{1}{(k+d)(n-k)} \cdot \frac{B^{n-k-1}}{(n-k-1)!}$$
$$\leq \frac{1}{n+d} + Be^{B} U_{n+d} = \frac{1}{n+d} + \frac{2Be^{B}[1+\ln(n+d+1)]}{n+d}$$
$$= \frac{(1+2Be^{B}) + 2Be^{B}\ln(n+d+1)}{n+d} = U_{n+d}^{B}$$

□

Power Series Representations and Their Coefficients Suppose that the coefficients of an invertible power series $f(x) = \sum_{n=0}^{\infty} A_n x^n$ satisfy the inequalities:

$$|A_n/A_0| \leq M^n \quad \text{for } n = 0, 1, 2, \ldots \tag{162}$$

where M is a fixed positive number. Find the power series representations around the origin for the functions $1/f(x)$, $f'(x)/f(x)$, $f''(x)/f(x)$, $e^{Bx}/f(x)$, and $e^{Bx}f'(x)/f(x)$, where B is a constant, and then estimate the coefficients of these power series.

$$f'(x) = \sum_{n=0}^{\infty}(n+1)A_{n+1}x^n\,, \quad f''(x) = \sum_{n=0}^{\infty}(n+1)(n+2)A_{n+2}x^n\,,$$

(1) Let $g(x) = f(x)/A_0 = \sum_{n=0}^{\infty}(A_n/A_0)x^n$. Find the power series representation around the origin for the function $1/g(x)$.

Write $1/g(x) = \sum_{n=0}^{\infty} B_n x^n$.

$$1 = \left(\sum_{n=0}^{\infty}\frac{A_n}{A_0}x^n\right)\left(\sum_{n=0}^{\infty}B_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{A_{n-k}}{A_0}B_k\right)x^n$$

$$\sum_{k=0}^{n}\frac{A_{n-k}}{A_0}B_k = \delta_{n,0}; \qquad B_0 = 1, \quad B_n = -\sum_{k=0}^{n-1}\frac{A_{n-k}}{A_0}B_k \quad \text{for } n = 1, 2, \ldots$$

Verify that $|B_n| \leq (2M)^n$ for $n = 0, 1, 2, 3, \ldots$.

Proof. The inequality holds when $n = 0$. Suppose that $|B_k| \leq (2M)^k$ for $0 \leq k \leq n-1$.

$$|B_n| \leq \sum_{k=0}^{n-1}\left|\frac{A_{n-k}}{A_0}\right||B_k| \leq \sum_{k=0}^{n-1}M^{n-k}(2M)^k = M^n\sum_{k=0}^{n-1}2^k = (2^n-1)M^n \leq (2M)^n$$

By mathematical induction, we have $|B_n| \leq (2M)^n$ for $n = 0, 1, 2, \ldots$. ∎

$$\frac{1}{f(x)} = \frac{1}{A_0 g(x)} = \sum_{n=0}^{\infty}\frac{B_n}{A_0}x^n$$

(2) Write the power series representation around the origin for $f'(x)/f(x)$ as $\sum_{n=0}^{\infty}C_n x^n$.

$$\frac{f'(x)}{f(x)} = \left(\sum_{n=0}^{\infty}(n+1)A_{n+1}x^n\right)\left(\sum_{n=0}^{\infty}\frac{B_n}{A_0}x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(k+1)\frac{A_{k+1}}{A_0}B_{n-k}\right)x^n$$

$$C_n = \sum_{k=0}^{n}(k+1)\frac{A_{k+1}}{A_0}B_{n-k} \quad \text{for } n = 0, 1, 2, \ldots$$

Verify that $|C_n| \leq (3M)^{n+1}$ for $n = 0, 1, 2, \ldots$.

Proof. The inequality holds when $n = 0$. Suppose that $|C_k| \le (3M)^{k+1}$ for $0 \le k \le n-1$. Note that $\sum_{k=0}^{n}(k+1)2^{n-k} \le 3^{n+1}$ for $n = 0, 1, 2, \dots$.

$$\begin{aligned}
|C_n| &\le \sum_{k=0}^{n}(k+1)\left|\frac{A_{k+1}}{A_0}\right||B_{n-k}| \\
&\le \sum_{k=0}^{n}(k+1)M^{k+1}(2M)^{n-k} \\
&= M^{n+1}\sum_{k=0}^{n}(k+1)2^{n-k} \\
&\le (3M)^{n+1}
\end{aligned}$$

By mathematical induction, we have $|C_n| \le (3M)^{n+1}$ for $n = 0, 1, 2, \dots$. ■

(3) Write the power series representation around the origin for $f''(x)/f(x)$ as $\sum_{n=0}^{\infty} D_n x^n$.

$$\frac{f''(x)}{f(x)} = \left(\sum_{n=0}^{\infty}(n+1)(n+2)A_{n+2}x^n\right)\left(\sum_{n=0}^{\infty}\frac{B_n}{A_0}x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(k+1)(k+2)\frac{A_{k+2}}{A_0}B_{n-k}\right)x^n$$

$$D_n = \sum_{k=0}^{n}(k+1)(k+2)\frac{A_{k+2}}{A_0}B_{n-k} \quad \text{for } n = 0, 1, 2, \dots$$

Verify that $|D_n| \le (3M)^{n+2}$ for $n = 0, 1, 2, \dots$.

Proof. The inequality holds when $n = 0$. Suppose that $|D_k| \le (3M)^{k+2}$ for $0 \le k \le n-1$. Note that $\sum_{k=0}^{n}(k+1)(k+2)2^{n-k} \le 3^{n+2}$ for $n = 0, 1, 2, \dots$.

$$\begin{aligned}
|D_n| &\le \sum_{k=0}^{n}(k+1)(k+2)\left|\frac{A_{k+2}}{A_0}\right||B_{n-k}| \\
&\le \sum_{k=0}^{n}(k+1)(k+2)M^{k+2}(2M)^{n-k} \\
&= M^{n+2}\sum_{k=0}^{n}(k+1)(k+2)2^{n-k} \\
&\le (3M)^{n+2}
\end{aligned}$$

By mathematical induction, we have $|D_n| \le (3M)^{n+2}$ for $n = 0, 1, 2, \dots$. ■

(4) Write the power series representation around the origin for $e^{Bx}/f(x)$ as $\sum_{n=0}^{\infty} E_n x^n$.

$$\frac{e^{Bx}}{f(x)} = \left(\sum_{n=0}^{\infty} \frac{B^n}{n!} x^n\right)\left(\sum_{n=0}^{\infty} \frac{B_n}{A_0} x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{B^k}{k!}\frac{B_{n-k}}{A_0}\right) x^n$$

$$E_n = \frac{1}{A_0}\sum_{k=0}^{n} \frac{B^k}{k!} B_{n-k}$$

$$\begin{aligned}
|E_n| &\le \frac{1}{|A_0|}\sum_{k=0}^{n} \frac{|B|^k}{k!}|B_{n-k}| \\
&\le \frac{1}{|A_0|}\sum_{k=0}^{n} \frac{|B|^k}{k!}(2M)^{n-k} \\
&= \frac{(2M)^n}{|A_0|}\sum_{k=0}^{n} \frac{1}{k!}\left(\frac{|B|}{2M}\right)^k \\
&\le \frac{e^{|B|/(2M)}}{|A_0|}(2M)^n.
\end{aligned}$$

(5) Write the power series representation around the origin for $e^{Bx} f'(x)/f(x)$ as $\sum_{n=0}^{\infty} F_n x^n$.

$$\frac{e^{Bx} f'(x)}{f(x)} = \left(\sum_{n=0}^{\infty} \frac{B^n}{n!} x^n\right)\left(\sum_{n=0}^{\infty} C_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{B^k}{k!} C_{n-k}\right) x^n$$

$$F_n = \sum_{r=0}^{n} \frac{B^k}{k!} C_{n-k}$$

$$\begin{aligned}
|F_n| &\le \sum_{k=0}^{n} \frac{|B|^k}{k!}|C_{n-k}| \\
&\le \sum_{k=0}^{n} \frac{|B|^k}{k!}(3M)^{n-k+1} \\
&= (3M)^{n+1}\sum_{k=0}^{n} \frac{1}{k!}\left(\frac{|B|}{3M}\right)^k \\
&= e^{|B|/(3M)}(3M)^{n+1}
\end{aligned}$$

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