Continuous time one-dimensional asset pricing models with analytic price-dividend functions*

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Abstract: A continuous time one-dimensional asset pricing model can be described by a second order linear ordinary differential equation which represents equilibrium or a no arbitrage condition within the economy. If the stochastic discount factor and dividend process are analytic, then the resulting differential equation has analytic coefficients. Under these circumstances, the one-dimensional Cauchy-Kovalevsky Theorem can be used to prove that the solution to such an asset pricing model is analytic. Also, this theorem allows for the development of a recursive rule, which speeds up the computation of an approximate solution. In addition, this theorem yields a uniform bound on the error in the numerical solution. Thus, the Cauchy-Kovalevsky Theorem yields a quick and accurate solution of many known asset pricing models.

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1 Introduction

Most applied research on asset pricing in continuous time assumes a linear structure for the stochastic discount factor (SDF) or risk free interest rate. Researchers make this assumption since there are closed form solutions for asset prices in this set up. However, it is known from the equity premium literature that non-linear SDF are necessary to capture the dynamic behavior of the equity premium.[1] In this paper we consider such asset pricing models in which the SDF and dividend process are analytic. Using analytic methods we show that their solutions are analytic and quickly compute polynomial approximations with precise error estimates.

An one-dimensional asset pricing model in continuous time is characterized by an ordinary differential equation (ODE) whose solution is the price or return of the asset under study. There are only a few examples of such models whose solution can be expressed in closed form. The most general such models are those whose conditional expected SDF is linear in the state variable and the conditional variance of this SDF is also linear in the state variable. They are called affine models and their solutions are log-linear in the state variable.[2] Observe for these simple asset pricing models the linearity of the ODE coefficients leads to log-linear solutions.

Similarly, for general non-affine models analytic characteristics (SDF and dividend process), which translate into analytic coefficients for the ODE of the model, lead to analytic solutions for the price-dividend functions. An analytic asset pricing function $f(x)$, defined on an open interval Ω in $\mathbb{R}$, has the desirable property that it can be represented by a Taylor series in some neighborhood of each point $x_0 \in \Omega$.[3] The radius of convergence is the largest number r such that the series converges

[1] Mehra and Prescott (1985). Constantinides (2002), Campbell and Viceira (2002), Mehra and Prescott (2003) and Cochrane (2005, Chapters 20 and 21) provide recent exposition of this work.

[2] See Duffie and Kan (1996), Duffie, Pan, and Singleton (2000), and Dai and Singleton (2000). Duffie (1996, Chapter 7) and Shreve (2003, Chapter 10) discuss higher dimensional versions of these models which are not dealt with here. The Heath, Jarrow and Morton (1992) model is based on the observed forward rates rather than the SDF. Their model allows for higher polynomial functions but they do not provide a solution. Rather they use numerical methods to solve the problem. Shreve (2003, Chapter 10) shows the relation between Heath, Jarrow and Morton and affine models. More recently Gabaix (2007) develops a linear price-dividend function by engineering the dividend process to cancel any non-linearity in the SDF.

[3] See CCCH and CCH (2008a) for a discussion of analyticity and how it applies to discrete time asset pricing models. Throughout this paper we use CCCH to refer to Calin, Chen, Cosimano and Himonas (2005). In addition, CCH (2008a), CCH (2008b), CCH (2008c) for Chen, Cosimano and Himonas (2008a), (2006b), and (2008c), respectively. These papers show how to use analytic methods to solve discrete time asset pricing models.

in the interval $(x_0 - r, x_0 + r)$. For a second order linear ODE with analytic coefficients near x_0 and non-zero coefficient of the second derivative term, the Cauchy-Kovalevsky Theorem states that its initial value problem at x_0 has a unique analytic solution in a neighborhood of x_0. We find here that most applied asset pricing models in one dimension yield an ODE with analytic coefficients, as long as the conditional mean and standard deviation of the stochastic processes for the SDF and the state variable are analytic. Thus, the Cauchy-Kovalevsky Theorem applies to most asset pricing models in one dimension so that the equilibrium price-dividend function is analytic.

In this paper we provide the complete derivation of the solution to the Campbell and Cochrane (1999) asset pricing model. We chose to solve this asset pricing model since its mathematical complexity and economic interest seems to make it the most appropriate model for demonstrating the analytic method. This model yields a second order linear ODE with analytic coefficients and forcing term whose radius of convergence is r near the stationary point x_0 of the stochastic process for its state variable. This radius of convergence is large enough so that the interval $(x_0 - r, x_0 + r)$ includes all values of interest to investors. Applying the Cauchy-Kovalevsky Theorem we conclude that the ODE for the Campbell and Cochrane model has a unique solution which is analytic about the point x_0. Furthermore, its radius of convergence r_0 is at least equal to the smallest radius of convergence of the two coefficients and the forcing term. The coefficients of the Taylor series for the price-dividend ratio are quickly calculated using a recursive rule. Our numerical solution is the n^{th} order polynomial approximation of the price-dividend function in the interval $(x_0 - r_0, x_0 + r_0)$. Having established a uniform bound on the coefficients and forcing term on a circle with radius $r < r_0$ in the complex plane, the Cauchy integral formula is used to determine a uniform bound on the error between the numerical solution and the true price-dividend function for $|x| < \mu r$, where $\mu \in (0, 1)$. The numerical solution can be made as accurate as one may desire by choosing sufficiently many coefficients from the Taylor series. Having developed a numerical scheme to quickly and accurately represent the price-dividend function for the Campbell and Cochrane model, below we catalog all the steps necessary to apply the analytic method to most continuous time one-dimensional asset pricing models.

The rest of the paper is structured as follows. Section 2 reviews the literature for one dimensional asset pricing models. Section 3 lays out the analytic method for solving one-dimensional asset pricing models. Section 4 provides a complete analysis of the Campbell and Cochrane (1999) asset pricing model. In addition, a menu is provided for accurately approximating most asset pricing models. Section 5 carries out the simulation of the Campbell and Cochrane model using the polynomial approximation method. Final comments are made in the last section.

2 Literature Review

One dimension asset pricing models are specified in either discrete or continuous time. The essential components of the model are the stochastic discount factor (SDF)[4], and the equation of motion for the state variable. In discrete time models the solution is the equilibrium price-dividend function which solves an integral equation. CCCH (2005) show how to use analytic methods to solve the discrete time version of the Mehra and Prescott (1985) model. In this model the utility function is assumed to be a constant relative risk averse utility, so that the SDF is an exponential function with base consumption growth. The state variable consumption and/or dividend growth is assumed to be a first order autoregressive (AR(1)) process. CCCH find that the analytic properties of these essential components transfer to the equilibrium price-dividend function. As a result, they can approximate the price-dividend function using an 9^{th} order polynomial. Using the analytic property of the price-dividend function, they establish a uniform bound on the price-dividend ratio for any level of dividend growth of interest to financial economist. Thus, they are able to accurately represent the price-dividend function for the Mehra and Prescott model with a higher order polynomial.

As is well known the Mehra and Prescott model leads to the equity premium puzzle in which the return on stocks relative to bonds is too low compared to the observed equity premium. CCH (2008a) use analytic methods to solve the discrete time version of Campbell and Cochrane's (1999) asset pricing model. To explain the equity premium Campbell and Cochrane introduce external habits to capture the time variation in the risk aversion of investors. This external habit is represented by their surplus consumption ratio which measures the investor's consumption relative to her habitual level. The logarithm of the surplus consumption ratio is the state variable for the

[4]This is often refer to as the pricing kernel in finance

model, and follows an AR(1) process. To vary the risk aversion of the investor the random shock to this surplus consumption ratio is hit by normal random shocks to consumption growth, which are amplified (dampen) by a sensitivity function, when consumption growth is low (high). CCH (2008a) demonstrate that the price-dividend function simulated by Campbell and Cochrane is highly sensitive to extreme negative levels of consumption growth. For example, the uniform bound on the price-dividend function for consumption growth per month in the interval $[x_0 - 25\%, x_0 + 25\%]$ is about 20% below the price-dividend ratio reported by Campbell and Cochrane in their numerical work. Here, x_0 is the logarithm of the stationary surplus consumption ratio. This conclusion arises because the amplification of the random shock for extreme negative consumption growth is unbounded, so that the integral equation places substantial weight on the value of the price-dividend ratio at very low levels of consumption growth. Finally, CCH (2008b) solve the discrete time model of Abel (1990) using the same methods. In Abel's model the utility function is a constant relative risk averse function in which utility is a function of consumption relative to a weighted average of internal and external habits. The stochastic process for consumption growth is an AR(1) process. For a coefficient of risk aversion of 3.25, and a fifty-fifty spilt between internal and external habits, CCH (2008b) are able to match the historic equity premium with a higher order polynomial.

In summary, the solution to many discrete time asset pricing models can be accurately represented using higher order polynomial approximations within a range for the state variable that includes any values of interest to investors.[5] However, most of the applied work uses a low order polynomial approximation in the neighborhood of the point x_0, where x_0 is usually the average value of the state variable observed in the data set. Campbell (1993) solves a discrete time model with the recursive utility of Epstein and Zin (1989, 1991).[6] He uses a first order polynomial approximation of the stochastic process for wealth, which is followed by a guess and verification that stock returns have a log-normal distribution. In this case the logarithm of the price-dividend ratio is linear in

[5] Analytic methods may also use orthogonal polynomials. Judd (1992, 1996, 1998) and Stoer and Bulirsch (2002) show how to use polynomial interpolation methods to represent such polynomials with orthogonal polynomials.

[6] CCH (2008c) show that an asset pricing model using recursive utility can be transformed into an integral equation similar to that found in the Mehra and Prescott, and Abel models. Consequently, one would also approximate the solution to these models with a higher order polynomial approximation.

the state variable, consumption growth.[7] This procedure for approximating the solution to asset pricing models is now standard as evidence by Bansal and Yaron (2004), who use a model similar to Campbell. Bansal and Yaron also consider a more general model in which the variance of the state variable is an AR(1) process. This adds a second state variable, the current variance of the state variable, to the linear function for the logarithm of the price-dividend ratio.[8] This procedure would be accurate for a small region of convergence around x_0. Yet, the accuracy of these approximations deteriorates as one considers the larger region of convergence around x_0, which is observed in the financial markets. Thus, empirical abnormalities obtained from these models could be the result of approximation errors.

One difficulty with discrete time models is that the coefficients of the polynomial approximation must be solved simultaneously, so that the computational cost may be quite large. In particular, the coefficients are found by substituting the hypothesized polynomial into the integral equation for the price-dividend function. One then manipulates the equation until it consists of the addition and subtraction of polynomials in the state variable. The final step is to equate the coefficients for each monomial. Each of these equations are linear in all the coefficients of the polynomial, so that the system must be solved simultaneously. A second difficulty with discrete time models is that the integral equation must be true over the whole range of the state variable allowed by its stochastic process. In both the Abel (1990), and Campbell and Cochrane (1999) model the assumed normality of the random shocks to dividend growth means that the price-dividend function must be evaluated over a range in which the price-dividend function is not well defined.[9] To overcome these difficulties this paper shows how to use analytic methods to solve continuous time asset pricing models in one dimension.

[7] Wang (1994), and Lo and Wang (2006) use a constant absolute risk aversion utility function with a normally distributed state variable, so that the price-dividend ratio is linear in the state variable.

[8] Bansal and Yaron also use Judd's projection method to represent the solution of the model, which they say is "quite close to" the first order approximation. Yet, they do not say the order of the polynomial approximation used in the projection method. In Croce, Lettau, and Ludvigson (2007), and Lettau, Ludvigson, and Wachter (2008), models similar to Bansal and Yaron are solved with alternative information structure. Croce, Lettau, and Ludvigson use a third order polynomial for their model with one state variable and second order for their model with two state variables. They do not provide an estimate for the error in their approximation.

[9] Samuelson (1970) first recognized this issue. See also Jin and Judd (2002) for a discussion of this issue when using the perturbation method. Geweke (2001) also encounters such problems.

Solving a continuous time asset pricing problem in one dimension boils down to finding the price-dividend function, $p(x)$, that solves the following initial value problem (IVP):

$$p''(x) + a(x)p'(x) + b(x)p(x) = g(x), \; p(x_0) = p_0, \; p'(x_0) = p_1, \tag{2.1}$$

where the coefficients $a(x)$, $b(x)$, and the forcing function, $g(x)$, are analytic near the point x_0. Comprehensive derivation of such IVP for the Campbell and Cochrane (1999) model is provided below. For now the important point is that the coefficients are determined by the assumed functional form for the SDF, and the instantaneous mean and standard deviation of the stochastic process for the state variable.

Up to now most of the asset pricing models in continuous time assume affine stochastic discount factors, so that the coefficients in the ODE (2.1) are affine as well.[10] Cochrane (2005, Chapter 19) demonstrates how affine models are generalizations of earlier work by Vasicek (1977), and Cox, Ingersoll and Ross (1985). Wang (1993) uses a constant absolute utility function with an Ohnstein-Uhlembech process for the state variable.[11] These assumptions lead to a linear price-dividend function. Menzly, Santos, and Veronesi (2004) modify a continuous time version of Campbell and Cochrane's (1999) external habit model by using logarithmic preferences, and a linear sensitivity function, so that the price-consumption ratio is linear in the state variable. In each of these models researchers have been able to guess and verify solutions to these models. Constantinides (1990, 1992) is one exception to this rule, yet he is also able to guess and verify a closed form solution. Cochrane, Longstaff, and Santa-Clara (2008) solve a two tree version of Lucas's (1979) asset pricing model with logarithmic preferences and stochastic process for the state variable (the relative size of shares in the two trees) with quadratic coefficients for the instantaneous mean and standard deviations. In this more general case, they are also able to guess and verify the functional form of the price-dividend ratio. Martin (2007) generalizes this model by allowing for constant relative risk averse utility, many assets, and dividend growth that is subject to a Poisson process. He uses a Fourier transformation to represent the price-dividend functions as an integral, which can be evaluated numerically. In each of these models the price-dividend function turns out to be an analytic function.

[10] See Duffie and Kan (1996), Duffie, Pan, and Singleton (2000), and Dai and Singleton (2000).

[11] Ohnstein-Uhlembech process is a continuous time version of an AR(1) stochastic process.

The state of the literature for continuous time asset pricing models begs the question as to whether asset pricing models can be solved using analytic methods. In this paper we answer this question in the affirmative. In particular, most asset pricing models can be represented as a second order linear ODE (2.1) with analytic coefficients, and forcing function. These differential equations can be represented as initial value problems in which the two initial conditions, $p(x_0) = p_0$, and $p'(x_0) = p_1$, are determined by the average price-dividend ratio, and the equity premium in the data. In this case one can represent the solution to these differential equations as a power series within the interval of convergence $(x_0 - r, x_0 + r)$, where r is at least as large as the smallest radius of convergence for the coefficients and the forcing function. As a result, the solution may be represented by a polynomial within a range for the state variable which includes all values of interest to investors. One can also calculate a uniform bound on the approximation error, when the state variable lies within the radius of convergence for the power series solution. In addition, the coefficients for this polynomial approximation are determined by a recursive rule starting with the first two coefficients determined by the two initial conditions. Consequently, only the local properties of the price-dividend ratio is used to determine the price-dividend ratio over the range of interest for the state variable. Thus, an accurate approximation of the price-dividend function can be calculated quickly relative to the discrete time models.[12] Finally, the extreme values of the state variable do not influence the accuracy of the numerical approximation. To demonstrate how the analytic method is used to solve continuous time asset pricing models all the details necessary to solve the continuous time version of Campbell and Cochrane's (1999) asset pricing model are provided.[13] Campbell and Cochrane's model is solved since it has proven to be the most challenging to solve in discrete time. Thus, the analytic method can be used to quickly and accurately solve most one dimensional continuous time asset pricing models.

[12] For example, a 20^{th} order polynomial approximation takes 20 minutes for the discrete time Campbell and Cochrane model, while a 110^{th} order polynomial takes 10 seconds for the continuous time version of the same model. Both programs were run in Maple on a PC with a duo core 2.66 GHz processor.

[13] The continuous time version of their model is contained in their 1994 working paper. They do not provide a solution to the continuous time model.

3 Analytic Properties of Asset Pricing Models

In this section, the analytic method for solving IVP problems (2.1) is explained. Recall that a function $f(x)$ is analytic near a point x_0 if it can be represented by its Taylor series, that is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \tag{3.1}$$

as long as $|x - x_0| < r$, where r is the radius of convergence.

The solution to the IVP (2.1) is analytic at x_0. This is a special case of the well known Cauchy-Kovalevsky Theorem. While this theorem holds for both linear and non-linear differential equations in one and several variables, here it is stated for second order linear differential equations of the form (2.1). For simplicity we shall also assume $x_0 = 0$, since otherwise it can be reduced to this case by a simply change of variable (translation).

Theorem 3.1. *The initial value problem (2.1) has a unique solution $p(x)$ near $x_0 = 0$, which is analytic with radius of convergence, r_0, equal to at least the smallest radius of convergence of the coefficients and the forcing term.*

The proof of this well-known theorem can be found in many ODE books.[14] However, for completeness sake, the proof is presented in the Appendix, including a useful error estimate.

This theorem qualifies the radius of convergence of the solution to be "at least" equal to the smallest radius of convergence of the coefficients and the forcing term. To see why consider the following example.

Example. The solution to the initial value problem

$$y'' - \frac{1}{x-1}y' = 0,\ y(0) = \frac{1}{2},\ y'(0) = -1,$$

is given by $y(x) = \frac{1}{2}(x-1)^2$. It is an analytic function with radius of convergence equal to infinity. However, Theorem 3.1 asserts only that its radius of convergence is greater or equal to 1.

There are two benefits of the proof of Theorem 3.1. First, it points to a procedure for solving the IVP (2.1). This procedure begins with a formal power series expansion for the solution to the

[14] See for example Coddington (1961) or Simmons (1991).

IVP of the following form

$$p(x) = \sum_{k=0}^{\infty} p_k x^k, \tag{3.2}$$

where p_k are to be determined. Substituting this together with the known Taylor series for the coefficients and the forcing function into the IVP, and manipulating the result using the operational rules for power series one obtains a recurrence relation for the coefficients of the solution, p_k. Then assuming that the Taylor series of the coefficients, and the forcing function has radius of convergence at least r (which is taken to be the optimal), and using the recurrence relation, one can show that the coefficients p_k satisfy appropriate estimates. Consequently, the radius of convergence of the power series (3.2) is at least r. Thus, the formal power series solution (3.2) provides an honest power series solution to the IVP (2.1).

The second benefit of the proof of Theorem 3.1 is that it yields an accurate estimate of the difference between the power series solution (3.2) and its Taylor's polynomial approximation. More precisely, if

$$p_n(x) = \sum_{k=0}^{n} p_k x^k \tag{3.3}$$

is the n^{th} order polynomial approximation of the power series solution (3.2), then the error is

$$R_n(x) = p(x) - p_n(x) = \sum_{k=n+1}^{\infty} p_k x^k. \tag{3.4}$$

This error, $R_n(x)$, can be estimated in terms of the coefficients $a(x)$ and $b(x)$, the forcing function, $g(x)$, and the initial data p_0 and p_1. For this write

$$a(x) = \sum_{k=0}^{\infty} a_k x^k, \quad b(x) = \sum_{k=0}^{\infty} b_k x^k, \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} d_k x^k, \tag{3.5}$$

and choose r such that $0 < r < r_0$, where r_0 is as in Theorem 3.1. Since r is smaller than the radius of convergence $a(x)$, $b(x)$, and $g(x)$, there exists non-negative constants M_a, M_b, and M_g such that

$$|a_k| \leq \frac{M_a}{r^k}, \quad |b_k| \leq \frac{M_b}{r^k}, \quad \text{and} \quad |d_k| \leq \frac{M_g}{r^k}, \quad k = 0, 1, 2, \ldots. \tag{3.6}$$

With this information in mind, the following corollary provides a uniform bound for the error $R_n(x)$.

Corollary 3.2. *The error $R_n(x)$ between the solution $p(x)$ and its n^{th} order Taylor approximation is estimated as follows*

$$|R_n(x)| \leq \frac{1}{2}\left[M_g + |p_1|(1+r)M + |p_0|M\right] \sum_{k=n+1}^{\infty} \prod_{l=2}^{k-1} \left[\frac{l-1}{r(l+1)} + M\frac{l+r}{(l+1)l}\right] (\mu r)^k, \quad |x| < \mu r,$$

where $M = \max\{M_a, M_b\}$ and $0 < \mu < 1$.

Thus, by adding a sufficient number of coefficients to the polynomial approximation (3.3), we obtain an accurate enough numerical solution.

Before considering how to use the analytic method to solve IVP like (2.1) its relationship with existing approximation methods is discussed. Judd (1998) shows how to use projection methods to approximate IVP problems using orthogonal polynomials.[15] In the projection method orthogonal polynomials are used since the collocation method chooses the coefficients such that the IVP problem holds exactly at the roots of the orthogonal polynomial. Orthogonal polynomials are useful since a uniform bound on approximation error between an orthogonal polynomial and the true solution within a given interval exists, when the solution is known to have continuous derivatives of order k, and there is a bound on the k^{th} order derivative within the same interval.[16] However, little guidance is provided concerning when and where the solution is k times differentiable, and when the conditions for a uniform bound are satisfied. As a result, Judd recommends using relative errors to judge the accuracy of the approximation error in the projection method.[17] The main benefit of the Cauchy-Kovalevsky Theorem 3.1 in one dimension is that analyticity can be used to establish a uniform bound on the approximation error (3.4) between a polynomial approximation and the actual solution for $|x| < \mu r$.

The other advantage of the analytic method is that the coefficients of the polynomial approximation (3.3) are calculated recursively. This speeds up the calculation of the polynomial approximation relative to the projection method, which typically simultaneously solves the residual equations for the coefficients of the orthogonal polynomial using a metric such as mean square error. This metric is

[15]Orthogonal polynomials could be used to represent the polynomial approximation (3.3). For details see Stoer and Bulirsch. However, this was not necessary for the Campbell and Cochrane model, since the coefficients of this polynomial approximation are solved using a recursive rule. As a result the numerical problem is not ill conditioned.

[16]See Judd (1998, p. 214).

[17]Santos (1991, 1992, 1993, 1999, 2000) provides bounds on this relative error in discrete time economic models.

measured over the whole range of the approximated function, so that the coefficients solve a system of equations. Consequently, the calculation of the coefficients is quicker under the analytic method. In addition, the whole system of equations must be solved when one adds coefficients, while only the additional coefficients have to be solved under the analytic method. Thus, the analytic method provides a uniform bound to determine the accuracy of this approximation, and the calculation of the solution is quicker.

4 Campbell and Cochrane's asset pricing model.

In this paper we consider continuous time asset pricing models with one state variable. The representative agent is assumed to choose equity so that the intertemporal Euler condition is

$$0 = \Lambda(t)D(t)dt + E_t \left[d \left[\Lambda(t)P(t) \right] \right].^{18} \tag{4.1}$$

Here, $\Lambda(t)$ is the stochastic discount factor (SDF) for the valuation of an investment, $D(t)$ is the dividend payment from the equity per unit of time, and $P(t)$ is the price of equity. This euler condition is the limit as the change in time tends to zero of the typical Euler condition in which the investor compares the marginal loss of utility today from purchasing the stock with the expected marginal gain from the future utility of consumption from the dividends and the possible sale of the security. The first term in (4.1) is the marginal value of the future dividend, while the second term is the expected change in the marginal value of the stock price.

In Campbell and Cochrane's (1999) asset pricing model the SDF is designed to capture the time variation in equity premium observed in the historical data.[19] It depends on the consumption of the investor, $C(t)$, and the surplus consumption ratio, $S(t) = \frac{C(t)-X(t)}{C(t)}$, which measures how close consumption is to past habits, $X(t)$. More precisely, it is of the following form

$$\Lambda = e^{-\beta t} \left[SC \right]^{-\gamma}. \tag{4.2}$$

[18] See Duffie (1996), Cochrane (2005) and Campbell and Viceira (2002). Strictly speaking a representative agent is not necessary. The absence of arbitrage opportunities is sufficient for the existence of a positive pricing kernel so that this condition is satisfied. See Cox, and Huang (1989).

[19] Cochrane (2005 Chapter 20) provides a recent analysis of the empirical facts, while Chapter 21 explains how the Campbell and Cochrane model captures these concepts.

Following Campbell and Cochrane, we use the new variables defined by

$$C = e^x, \quad \text{and} \quad S = e^s,$$

so that both consumption and the surplus consumption ratio are always positive. Then, the consumption growth, dx, is assumed to be a random walk with drift $\bar{x}$ of the form

$$dx = \bar{x}dt + \sigma d\omega, \tag{4.3}$$

where the random shock to consumption growth, $d\omega$, is a standard Brownian motion.[20] Consequently, consumption growth is not a state variable for the price-dividend function. The only state variable in the model is the surplus consumption ratio, which follows the stochastic process

$$ds = (\phi - 1)(s - \bar{s})dt + \lambda(s - \bar{s})\sigma d\omega. \tag{4.4}$$

Here $\bar{s}$ is the logarithm of the stationary surplus consumption ratio, and the sensitivity function $\lambda(s - \bar{s})$ is defined by

$$\lambda(s - \bar{s}) = \begin{cases} \left(\sqrt{1 - 2(s - \bar{s})}\right)/\bar{S} - 1 & \text{if } s < \bar{s} + \frac{1-(\bar{S})^2}{2}, \\ 0 & \text{if } s \geqslant \bar{s} + \frac{1-(\bar{S})^2}{2}, \end{cases} \tag{4.5}$$

where

$$\bar{S} = \sigma\sqrt{\frac{\phi\gamma}{1 - \phi - \frac{b}{\gamma}}}.$$ [21]

This sensitivity function is designed to increase the standard deviation of the surplus consumption ratio by multiplying the random shocks to consumption growth $\sigma d\omega$. Also, it is chosen so that the investor's habits are only dependent on the consumption level of others. Furthermore, it assures that random shocks are magnified during bad times and minimized during prosperous times.[22] Finally, the sensitivity function leads to a risk free rate, which is a linear function of the surplus consumption ratio.

[20] To conserve on space the discussion of the underlining Brownian motion is limited, since the focus of the paper is on solving the resulting IVP which represent these asset pricing models. Arnold (1993), Duffie (1996) or Shreve (2003) are good sources for the derivation of these differential equations as well as the vast literature on this subject.

[21] In the Campbell and Cochrane paper they set $b = 0$, while $b \neq 0$ in the Wachter (2002, 2006) models.

[22] The steady state distribution of the surplus consumption ratio is derived below.

The first step in the derivation of the IVP is to derive the stochastic process for the SDF. In this derivation use is made of Ito's lemma. First, let $x \in \mathbb{R}^n$ follow the stochastic process,

$$dx = f(x,u)dt + S(x,u)dW, \tag{4.6}$$

where $u \in \mathbb{R}^q$ are control variables. $f(x,u)$ is the instantaneous mean of the state variable $x \in \mathbb{R}^n$. dW is an $k \times 1$ vector of Brownian motion, so that the $n \times k$ matrix, $S(x,u)$, provides the instantaneous impact of these random shocks on the state variable. Consequently, $\Sigma(x,u) = S(x,u)E_t\left(dWdW^T\right)S(x,u)^T$ is the instantaneous variance-covariance matrix for the state variables. Since Ito's Lemma will be used repeatedly in the derivation of the ODE for the Campbell and Cochrane model, we recall it here for the convenience of the reader.

Lemma 4.1. *Suppose $F(x,t)$ is C^2 for $x \in X \subseteq \mathbb{R}^n$ and C^1 in t. Then*

$$dF = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}^T f(x,u) + \frac{1}{2}tr\left(\frac{\partial^2 F}{\partial x \partial x'}\Sigma(x,u)\right)\right)dt + \frac{\partial F}{\partial x}^T dW.^{23} \tag{4.7}$$

Also, the following multiplication rules are true: $dWdW^T = Idt$, $dWdt = 0$ and $dtdt = 0$.

The SDF (4.2) in the Campbell and Cochrane model is a C^2 function of two state variables consumption growth (4.3), and the surplus consumption ratio (4.4). In the appendix Ito's lemma is used to derive the stochastic process for the SDF in Campbell and Cochrane's model,

$$\frac{d\Lambda}{\Lambda} = \left[\gamma(1-\phi)s - \beta - \gamma\bar{x} + \frac{\gamma^2\sigma^2}{2}\left(1+\lambda(s)\right)^2\right]dt - \gamma\sigma\left(1+\lambda(s)\right)^2 d\omega\,. \tag{4.8}$$

Note that in (4.8) s stands for $s - \bar{s}$. Also, observe that the instantaneous mean and standard deviation of (4.8) are analytic whenever the sensitivity function $\lambda(s)$ is analytic. Following Cochrane (2005, p. 29), we use the basic pricing relation (4.1) together with (4.8), and the definition of $\lambda(s)$ (4.5) to obtain

$$R^b(s) = -E_t\left[\frac{d\Lambda}{\Lambda}\right] = \gamma(\phi-1)s + \beta + \gamma\bar{x} - \frac{\gamma^2\sigma^2}{2}\left(1+\lambda(s)\right)^2 = r^b - bs\,, \tag{4.9}$$

where

$$r^b = \beta + \gamma\bar{x} - \frac{1}{2}\left(\gamma(1-\phi) - b\right).$$

[23] See Chow (1997), and Shreve (2003). The superscript T mean the transpose of a column vector, and $tr(A)$ refers to the trace of the matrix A.

Thus, the risk free interest rate is a linear function of the surplus consumption ratio.

The second step is to write the Euler condition (4.1) in terms of the price-dividend ratio $p = \frac{P}{D}$ using Ito's Lemma 4.1.

$$\frac{1}{p}\,dt + E_t\left(\frac{d\Lambda}{\Lambda} + \frac{dp}{p} + \frac{dD}{D} + \frac{d\Lambda dp}{\Lambda p} + \frac{dDdp}{Dp} + \frac{d\Lambda dD}{\Lambda D}\right) = 0\,. \tag{4.10}$$

In equilibrium, it is assumed that $C = D$, so that the stochastic process for consumption growth (4.3) determines the stochastic process for dividend growth. The final stochastic process needed is the price-dividend function, which is assumed to be a C^2 function of the surplus consumption ratio, $p(s)$, in the interval $(-\infty, \frac{1-(\bar{S})^2}{2})$. As a result, Ito's Lemma 4.1 is used to find the stochastic process for the price-dividend ratio.

$$dp = \left(p'(s)(\phi - 1)s + \frac{1}{2}p''(s)\lambda(s)^2\sigma^2\right)dt + p'(s)\lambda(s)\sigma d\omega. \tag{4.11}$$

Once the solution to the price-dividend function $p(s)$ is found, this stochastic process will represent the behavior of the price-dividend ratio over time.

In the appendix the stochastic processes for consumption growth (4.3), the surplus consumption ratio (4.4), the SDF (4.8), and the price-dividend function (4.11) are substituted into (4.10). Applying the multiplication rules in Ito's Lemma 4.1 leads to the following second order linear ODE for the price-dividend function, $p(s)$,

$$c_2(s)p''(s) = c_1(s)p'(s) + c_0(s)p(s) - 1\,,\text{[24]} \tag{4.12}$$

where

$$c_2(s) = \frac{\sigma^2(1+\bar{S}^2)}{2\bar{S}^2} - \frac{\sigma^2}{\bar{S}^2}s - \frac{\sigma^2}{\bar{S}}r(s),$$
$$c_1(s) = \frac{\sigma^2(\bar{S}^2+\gamma)}{\bar{S}^2} + \frac{K_1\bar{S}^2 - 2\gamma\sigma^2}{\bar{S}^2}s - \frac{\sigma^2(1+\gamma)}{\bar{S}}r(s),$$

and

$$c_0(s) = \frac{2K_0\bar{S}^2 - \sigma^2\gamma^2 - \sigma^2\bar{S}^2}{2\bar{S}^2} + \frac{\sigma^2\gamma^2 - \gamma K_1\bar{S}^2}{\bar{S}^2}s + \frac{\sigma^2\gamma}{\bar{S}}r(s).$$

[24] Wachter (2005) derives this ODE for the Campbell and Cochrane model using no arbitrage techniques as in Duffie (Chapter 6 and 10) rather than equilibrium arguments as in Lucas (1978), which is used here.

Here, $K_0 = \beta + (\gamma - 1)\bar{x} > 0$, $K_1 = (1-\phi) > 0$, and

$$r(s) \doteq \bar{S}\Big(\lambda(s) + 1\Big) = \begin{cases} \sqrt{1-2s} & \text{if } s < \frac{1-\bar{S}^2}{2}, \\ \bar{S} & \text{if } s \geq \frac{1-\bar{S}^2}{2}. \end{cases} \tag{4.13}$$

The normal form of equation (4.12) is

$$p''(s) + a(s)p'(s) + b(s)p(s) = g(s)\,, \tag{4.14}$$

where

$$a(s) = -\frac{c_1(s)}{c_2(s}, \; b(s) = -\frac{c_0(s)}{c_2(s)}, \;\; \text{and} \;\; g(s) = -\frac{1}{c_2(s)}. \tag{4.15}$$

To apply the Cauchy-Kovalevsky Theorem for equation (4.14) the radius of convergence of the coefficients and forcing term needs to be determined. Looking at the definitions of these coefficients, two conditions must be imposed. First, $c_2(s)$ must be positive, which is true when

$$|s| < \frac{1-\bar{S}^2}{2}.$$

Second, $r(s)$ must be analytic. Since $1 - \bar{S}^2 < 1$ $r(s)$ is analytic if $s < (1-\bar{S}^2)/2$, and the radius of convergence of its power series about $s = 0$ is equal to $(1-\bar{S}^2)/2$. Since $r(s)$ and $\frac{1}{r(s)}$ have the same radius of convergence about 0, the radius of convergence of the coefficients $a(s)$, $b(s)$, and the forcing term $g(s)$ is

$$r_0 = \frac{1-\bar{S}^2}{2}. \tag{4.16}$$

Finally, applying Theorem 3.1, the solution for the price-dividend function $p(s)$ of the Campbell and Cochrane model is analytic near zero, and its Taylor series

$$p(s) = \sum_{j=0}^{\infty} p_j s^j, \tag{4.17}$$

has radius of convergence at least r_0 given by (4.16).

Recursive rule for coefficients of power series. Next, the recurrence relation for determining the coefficients p_j is derived. For this relation, write the Taylor series for the functions $c_0(s)$, $c_1(s)$ and $c_2(s)$. Observe that each of these coefficients has the functional form

$$c(a_0, a_1, a_2, s) = a_0 + a_1 s + a_2 r(s), \tag{4.18}$$

for some $(a_0, a_1, a_2) \in \mathbb{R}^3$, so that the derivatives of these coefficients are dependent on the derivatives of $r(s)$. These derivatives are given by

$$r^{(n)}(s) = \begin{cases} r(s) & \text{if } n = 0, \\ -\frac{1}{1-2s} r(s) & \text{if } n = 1, \\ -\frac{(2n-3)!!}{(1-2s)^n} r(s) & \text{if } n \geqslant 2 \end{cases} \quad \text{and} \quad r^{(n)}(0) = \begin{cases} 1 & \text{if } n = 0, \\ -1 & \text{if } n = 1, \\ -(2n-3)!! & \text{if } n \geqslant 2. \end{cases}$$ [25]

Therefore, the derivatives of any of the coefficients $c(a_0, a_1, a_2, s)$ are

$$c^{(n)}(a_0, a_1, a_2; s) = \begin{cases} a_0 + a_1 s + a_2 r(s) & \text{if } n = 0, \\ a_1 + a_2 r^{(1)}(s) & \text{if } n = 1, \\ a_2 r^{(n)}(s) & \text{if } n \geqslant 2. \end{cases}$$

As a result,

$$c^{(n)}(a_0, a_1, a_2; 0) = \begin{cases} a_0 + a_2 & \text{if } n = 0, \\ a_1 - a_2 & \text{if } n = 1, \\ -a_2(2n-3)!! & \text{if } n \geqslant 2. \end{cases}$$

So

$$c^{(n)}(a_0, a_1, a_2) = \begin{cases} a_0 + a_2 & \text{if } n = 0, \\ a_1 - a_2 & \text{if } n = 1, \\ -\frac{a_2(2n-3)!!}{n!} & \text{if } n \geqslant 2. \end{cases}$$

Here the abbreviation $c_j^{(n)}(a_0, a_1, a_2) = \frac{1}{n!} c_j^{(n)}(a_0, a_1, a_2; 0)$ is used for $j = 0, 1, 2$ and $n = 0, 1, 2, \ldots$.

In the appendix the power series (4.17) for $p(s)$, its first two derivatives, and the power series for the coefficients $c_0(s)$, $c_1(s)$, and $c_2(s)$ are substituted into the ODE (4.12) to find the following recurrence relation for the coefficients of this power series.

$$\begin{aligned} 2c_2^{(0)} p_2 &= c_1^{(0)} p_1 + c_0^{(0)} p_0 - 1, \text{ and} \\ (j+1)(j+2) c_2^{(0)} p_{j+2} &= \sum_{k=2}^{j} \left[c_0^{(j-k)} + k c_1^{(j-k+1)} - (k-1)k c_2^{(j-k+2)} \right] p_k \\ &\quad + (j+1)(c_1^{(0)} - j c_2^{(1)}) p_{j+1} + (c_0^{(j-1)} + c_1^{(j)}) p_1 + c_0^{(j)} p_0 \,. \end{aligned} \tag{4.19}$$

Initial conditions. To determine p_j recursively from the formulas (4.19) one needs to know the initial conditions $p(0) = p_0$ and $p'(0) = p_1$. The first initial condition is chosen to be $p_0 = 18.3 \times 12 = 219.60$, so that the price-dividend ratio would be the same as in Campbell and Cochrane (1999).

To choose the second initial condition the equity premium is related to the first derivative of the price-dividend function. The return on equity is given by

$$R^e(s)\, dt = \frac{dP}{P} + \frac{Ddt}{P} = \frac{dt}{p} + \frac{dp}{p} + \frac{dC}{C} + \frac{dCdp}{Cp}. \tag{4.20}$$

[25] The super script $^{(n)}$ refers to the n^{th} order derivative. The notation !! means $7!! = 7 \cdot 5 \cdot 3 \cdot 1$.

The second equality follows from Ito's Lemma 4.1, and the equality between consumption and dividends. In the appendix the stochastic processes for consumption growth (4.3), and the price-dividend (4.11) are substituted into the return on equity to yield

$$dR^e(s) = E_t\left[R^e(s)\right]\, dt + \Sigma(s)\, d\omega\,, \tag{4.21}$$

where the instantaneous expected return on equity is

$$E_t\left[R^e(s)\right] = \bar{x} + \frac{1}{2}\sigma^2 + \frac{(\phi-1)sp'(s) + \frac{\sigma^2}{2}\lambda(s)^2 p''(s) + \sigma^2\lambda(s)p'(s) + 1}{p(s)}, \tag{4.22}$$

and the instantaneous standard deviation for the return on equity is

$$\Sigma(s) = \left(\frac{\lambda(s)p'(s)}{p(s)} + 1\right)\sigma. \tag{4.23}$$

In addition, the risk free return on bonds is given by equation (4.9). Consequently, the Sharpe ratio is given by

$$S(s) = \frac{E_t\left[R^e(s)\right] - \left[r^b + bs\right]}{\Sigma(s)}. \tag{4.24}$$

In the appendix, the risk free interest rate (4.9), the return on equity (4.20) together with the Euler condition (4.10) are used to find a relation between the equity premium, and the first derivative of the price-dividend function which is given by

$$p'(s) = \frac{\{E_t[R^e(s)] - R^b(s) - \sigma^2\gamma(1+\lambda(s))\}p(s)}{\gamma\sigma^2\lambda(s)(1+\lambda(s))}\,. \tag{4.25}$$

Then evaluating (4.25) at $s = 0$ determines the second initial condition

$$p_1 = p'(0) = \frac{\left\{E_t[R^e(0)] - r^b - \frac{\gamma\sigma^2}{\bar{S}}\right\}p_0}{\frac{\gamma\sigma^2}{\bar{S}}\left(\frac{1}{\bar{S}} - 1\right)}\,. \tag{4.26}$$

The value of p_1 is found by replacing $E_t\left[R^e(0)\right] - r^b$ with the average equity premium in Campbell and Cochrane's data. In the simulation this initial condition is used to set $p_1 = 111.8$. Thus, the equity premium at a particular point can be used to determine the second initial condition in the IVP problem (2.1). Once the price-dividend function is found over the entire range of the surplus consumption ratio, equation (4.22) is used to determine the return on equity over the range of the surplus consumption ratio.

Thus, the historic average price-dividend ratio, and equity premium in the economy are used to establish the necessary conditions for the Cauchy-Kovalevsky Theorem 3.1 to hold. Consequently, the equilibrium price-dividend ratio for the Campbell and Cochrane model is the Taylor series around $s = 0$ with radius of convergence at least equal to $r = 0.4990$. In addition, the instantaneous mean and standard deviation for stock returns, given by (4.22) and (4.23), are analytic within the same interval of convergence.

Condition (4.25) is akin to the condition for the state-price beta model in the consumption CAPM developed by Duffie (1996, pp. 101-108 and pp. 227-230). This condition also satisfies the no arbitrage condition between stocks and bonds.[26] The no arbitrage condition (4.25), the standard deviation (4.23), and Sharpe ratio (4.24) can be combined to yield

$$S(s) = \gamma\sigma(1 + \lambda(s)),$$

so that the equity premium puzzle can be resolved in the Campbell and Cochrane model through the increased sensitivity of the random shock to consumption growth on the surplus consumption ratio. In particular, $\lambda(0) + 1 = \frac{1}{\bar{S}} = 22.31$ for the parameter values used in the simulation of the Campbell and Cochrane model, so that the Sharpe ratio, $\gamma\sigma(1 + \lambda(s))$, is close to its historic average.

Numerical solution and error analysis. The numerical solution of Campbell and Cochrane's model (4.12) is a n^{th} degree polynomial approximation, $p_n(s)$, to the power series expansion (4.17) of the price-dividend function, that is

$$p_n(s) = \sum_{j=0}^{n} p_j s^j. \tag{4.27}$$

The bigger the n the more accurate is the numerical solution $p_n(s)$. Corollary 3.2 is used to estimate the error $p(s) - p_n(s)$. However, to apply Corollary 3.2 one needs to establish a uniform bound on the coefficients, $a(s) = -c_1(s)/c_2(s)$, $b(s) = -c_0(s)/c_2(s)$, and the forcing function, $g(s) = -1/c_2(s)$ on a circle centered at 0, and of radius r in the complex plane. Note that $c_2(s) \to 0$ as $s \to r_0 = \frac{1-\bar{S}^2}{2} = 0.4990$. Choose r smaller than r_0, say $r = 0.4$, and restrict the domain of definition for the coefficients

[26] Wachter (2005) derives the continuous time ODE for the Campbell and Cochrane model by starting with this no arbitrage condition rather than the equilibrium approach used here.

and the forcing term to $|z| \leq r$. Then, the Cauchy integral formula yields the constants M_a, M_b, and M_g used in the estimates (3.6). For example, if $r = 0.4$, then $M \doteq \min\{M_a, M_b\} = 52.6373$, and $M_g = 1173.8511$.

Applying Corollary 3.2 with these values of M and M_g a uniform bound on the Taylor series remainder (numerical solution error) $p(s) - p_n(s)$ is found. For $\mu = 0.5$, and $n = 110$ this error is less than 10^{-9}, while for $\mu = 0.8$ the degree of the polynomial approximation must be increased to $n = 475$ to obtain the same degree of accuracy.[27] Thus, if an investor wants the support of the distribution of the surplus consumption ratio to be $S \doteq e^s \in [0.037, 0.054]$, then one chooses the polynomial approximation of degree greater than or equal to 110^{th} in order to keep the error to the Taylor remainder less than 10^{-9}. However, if the support is increased to $S \in [0.032, 0.061]$, then the degree of the polynomial approximation must increase to 475^{th} to maintain the same accuracy for the price-dividend ratio.[28] Using a standard PC and Maple, the 110^{th} degree polynomial approximation of the solution, as well as all the graphs related to the numerical solution in this paper are calculated in 10 seconds, while it takes 90 seconds for the 475^{th} order polynomial. Thus, the analytic method produces an accurate solution to Campbell and Cochrane's model in minimal time.

Stationary distribution of surplus consumption ratio. The price-dividend function in the Campbell and Cochrane model is analytic for $|s| \leq r$. Consequently, the steady state probability distribution for the surplus consumption ratio is restricted to a support that is a closed subset of the interval $[-r, r]$. This restriction assures that the price-dividend function is analytic for every possible realization of the surplus consumption ratio. Note in the original Campbell and Cochrane working paper this support was chosen to be $[0.17\bar{S}, 1.66\bar{S}]$ rather than $[-r, r]$, since determining the radius of convergence for the price-dividend function was not part of their considerations.

Merton (1990, Chapter 17), and Cox and Miller (1965) provide the mathematical argument for

[27] The formula in Corollary 3.2 contains a sum to ∞, however the computer cannot count this high. Consequently, the error is compared when the number of terms was 1500, and 3000. The change in error was only 0.00, so that this source of error is not significant enough to change the error bound at the level of accuracy of 10^{-16}.

[28] One concern with such a high order polynomial approximation is rounding error, since as n increases the coefficients get larger as x^n gets smaller. However, the approximations are not materially effected by this issue. For example, the sup-norm of $p_{60}(s) - p_{50}(s)$ for $|s| \leq .30$ is less than 10^{-9}, so that the solution is already accurate at a 50^{th} order polynomial approximation for all the circumstances considered in this paper. In addition, Maple allows for the increase in precision. As a result, the number of digits was set to 100 without changing any results reported here.

determining the probability distribution of a random variable which follows a stochastic process of the form

$$ds = b(s)dt + [a(s)]^{\frac{1}{2}} d\omega. \tag{4.28}$$

To find the stationary probability distribution of s in the Campbell and Cochrane case, that is when $b(s) = (\phi - 1)s$, and $a(s) = \lambda(s)^2\sigma^2$, let $L(s, t, s_0)$ be the conditional probability density for s at time t given initial s_0. This density function satisfies the Kolmogorov-Fokker-Planck forward equation

$$\frac{1}{2}\frac{\partial^2}{\partial s^2}\left[a(s)L(s,t,s_0)\right] - \frac{\partial}{\partial s}\left[b(s)L(s,t,s_0)\right] = \frac{\partial L}{\partial t}(s,t,s_0).$$ [29]

This equation measures the chance of a small change in s at any instant of time. It is derived by calculating the probability of a small change in s in a small change of time t, using a second order Taylor approximation for this change. To determine the steady state distribution, let

$$\lim_{t\to\infty} L(s,t,s_0) = \pi(s) \text{ so that } \lim_{t\to\infty}\frac{\partial L}{\partial t}(s,t,s_0) = 0.$$

As a result, the steady state distribution for s solves the second order ODE equation

$$\frac{1}{2}\frac{d^2}{ds^2}\left[a(s)\pi(s)\right] - \frac{d}{ds}\left[b(s)\pi(s)\right] = 0. \tag{4.29}$$

In the appendix this ODE is solved subject to a reflection boundary at $0 < s^* \leq r$, so that $\lambda(s) > 0$ for all $|s| \leq s^*$. As a result, the steady state distribution for s is

$$\pi(s) \quad = K \exp\left\{-\tfrac{2\sigma^2k^4 + (1-\phi)(3-k^2)}{\sigma^2k^4}\ln\lambda(s) - \tfrac{(1-\phi)}{\sigma^2k^4}\left[\tfrac{k^2-1}{\lambda(s)} + 3\lambda(s) + \tfrac{\lambda(s)^2}{2}\right]\right\} \tag{4.30}$$

for $s \in [-s^*, s^*]$ and zero otherwise. Here $K = \left[\int_{-s^*}^{s^*}\pi(v)dv\right]^{-1}$ and $k \doteq 1/\bar{S}$.[30]

For the parameter values in the simulation of the Campbell and Cochrane model Figure 1 plots this stationary probability distribution over the support $[-0.75r, 0.75r] = [-0.30, 0.30]$ for s which is skewed to the right.[31] Thus, Theorem 3.1 can be applied to Campbell and Cochrane's ODE (4.12) for all possible realizations of the surplus consumption ratio, since the coefficients and forcing term of this ODE are always analytic under the steady state distribution for the surplus consumption ratio.

[29] See Cox and Miller (1965, pp.208-209).

[30] The transitory distribution for s is still an open question.

[31] This graph corresponds to Figure 2 of Campbell and Cochrane (1999) although here the support for the distribution is smaller relative to theirs.

4.1 Overview of Analytic Method

This section has provided the complete details for solving the Campbell and Cochrane asset pricing model. Here, we discuss the generality of this method. First, the Cauchly-Kovalevsky Theorem 3.1 is presented for only IVP with a one-dimensional second order ODE. There is a more general Cauchly-Kovalevsky Theorem which is applicable in several variables, higher order, and includes both linear and non-linear differential equations. In future research we plan to investigate more complicated economic and finance models using this Theorem. We also want to point out the generality of the method presented here. The analysis of the Campbell and Cochrane model through the derivation of the differential equation (4.14) is independent of the Cauchly-Kovalevsky Theorem. For other economic and financial problems the underlining primitives such as the SDF need to be specified as analytic functions over the relevant region so that the coefficients of the ODE (4.14) are analytic in this same region. To drive this point home we first explain in this subsection how to derive the ODE for most popular asset pricing models. Once the differential equation is determined for the IVP the analytic method consists of six steps listed in the second part of this subsection. By applying this prescription one should be able to quickly and accurately characterize solutions to many one dimensional problems in economics and finance.

ODE derivation for asset pricing models. A similar to (4.14) ODE may be derived for most one dimensional asset pricing models. As in the Campbell and Cochrane model suppose the SDF (4.2) is an analytic function for a given radius of convergence r_1 around the stationary point of the state variable x_0. Also, let the instantaneous mean and standard deviation for stochastic process of the state variable (4.4) be analytic with the same radius of convergence. In this case the asset pricing model yields an ODE with analytic coefficients and forcing term with the same region of convergence around the stationary point x_0. The steps for deriving this ODE are as follows:

1. Use Ito's Lemma 4.1 to find the stochastic process for the SDF (4.8). In this circumstance, the instantaneous mean and standard deviation for the SDF is analytic in the same region of convergence.

2. Apply Ito's Lemma to rewrite the investor's Euler condition (4.1) for choosing stocks as a function (4.10) of the stochastic processes for the SDF, price-dividend ratio, and the dividend process. Also assume the price-dividend is a function of the state variable, so that Ito's Lemma yields a stochastic process for the price-dividend function (4.11).

3. Specify the equilibrium condition such that consumption is related to dividend growth so that by Ito's Lemma the stochastic process for dividend growth is a function of the stochastic process of consumption growth (4.3).

4. Substitute the stochastic processes for the SDF (4.8), the price-dividend function (4.11), and dividend growth (4.3) into the Euler condition (4.10) to find a second order linear ODE (4.12), which can be written in the normal form (4.14). In these circumstances the coefficients and forcing term of this ODE (4.15) are analytic in the same region of convergence as the instantaneous mean and standard deviation of the stochastic processes for the SDF (4.8), and the state variable (4.4).

One dimensional asset pricing models are generally distinguished by the stochastic process for the SDF and the state variable, such as (4.8) and (4.4), respectively. In applied asset pricing models the instantaneous means and standard deviations for the stochastic processes for the SDF and the state variable are usually analytic functions as in the Campbell and Cochrane model. Included in this class of SDF are the above mentioned affine models, as well as Epstein and Zin (1989, 1990, 1991), Abel (1990,1999), Constantinides (1990, 1992), Duffie and Epstein (1992a, b) using the Kreps-Porteus (1978) functional form, Campbell and Cochrane (1999), and Bansal and Yaron (2004). Thus, these asset pricing models will lead to an ODE like (4.14) with analytic coefficients and forcing term.

Summary of analytic method. Since most applied asset pricing models yield an ODE similar to (4.14) with analytic coefficients about x_0 with radius of convergence r_1, the Cauchy-Kovalevsky Theorem (3.1) yields an analytic price-dividend function of the state variable for a radius of convergence at least as large as r_1. To implement this method one would execute the following steps:

1. Use the Cauchly-Kovalevsky Theorem 3.1 to obtain an analytic price-dividend function (4.17)

near x_0 with radius of convergence r_0 which is equal to at least the smallest radius of convergence for the coefficients and forcing term for the ODE (4.14). In addition, determine the recurrence relation (4.19) for the coefficients of the power series for the price-dividend function.

2. Let the initial condition p_0 be equal to the observed average price-dividend ratio. The second initial condition can be determined by relating the equity premium to the first derivative of the price-dividend function (4.25) using the no arbitrage property between stocks and bonds. By evaluating the equity premium at its observed average value the second initial condition p_0 (4.26) is established.

3. Use a n^{th} order polynomial (4.27) to approximate the solution for the price-dividend ratio within the radius of convergence.

4. Choose an $r < r_0$ and determine bounds on the coefficients and forcing term for complex values z on a circle C_r of radius r on the complex plane. By Cauchy's integral formula there exist uniform bounds for the k^{th} order derivatives of the coefficients and forcing term (3.6).

5. Choosing the order of the polynomial approximation (4.27) high enough, one can achieve any desired degree of accuracy for the price-dividend ratio for the state variable $|x| < \mu r$ with $\mu \in (0, 1)$ following Corollary 3.2.

6. Once the solution to the price-dividend ratio is known, the stochastic process for the price-dividend ratio is given by an equation like (4.11), where the first and second derivatives of the power series solution (4.17) are substituted for p' and p''. In addition, the stochastic process for the return on equity is given by an equation like (4.21) with instantaneous mean (4.22) and standard deviation (4.23), so that the power series solution to the ODE can also be used to calculate the stochastic process for the return on equity.

Thus, the analytic method can be used to characterize the properties of most asset pricing models.

5 Simulation of Campbell and Cochrane model

After setting the initial conditions and the support of the distribution of the surplus consumption ratio, the solution of the ODE (4.12) for the Campbell and Cochrane model is unique and analytic over the entire support of the steady state probability distribution for the surplus consumption ratio. For concreteness let this support be $[-\mu r, \mu r]$. In the simulations the parameters are set using a monthly time frame following Campbell and Cochrane (1999) for their consumption claim model. The results of the simulations in Table 1 and Figures 2 − 6 are annualized. The parameters on a monthly basis are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9896$, $\gamma = 2$, $\sigma = 0.00323$, $b = 0$, $\bar{S} = 0.0448$, and $\mu r = 0.32$. The first initial condition is based on their historic average price-dividend ratio, $p_0 = 219.6$. The second initial condition is tied to their historic average equity premium, following equation (4.26), so that $p_1 = 111.76$.

Table 1 records in column 2 the moments from the solution of Campbell and Cochrane's model. Column 2 records the sample data from Campbell and Cochrane (1999) which is based on the U.S. stock market from 1947 to 1995. Following Campbell and Cochrane, the price-dividend ratio is 18.3 by construction. In Figure 2 the price-dividend function with 475 coefficients is drawn over the range $[\bar{S}e^{-0.49}, \bar{S}e^{0.49}]$ to demonstrate how the approximation deteriorates outside the range identified by the error analysis $[\bar{S}e^{-\mu r}, \bar{S}e^{\mu r}] = [0.032, 0.061]$. Figure 3 presents the price-dividend function over this smaller range. The price-dividend function in Figure 3 varies from 15.5 to 21.6 as the surplus consumption ratio varies in the interval $[\bar{S}e^{-\mu r}, \bar{S}e^{\mu r}] = [0.032, 0.061]$. Thus, there could be a change in the price-dividend function of 39% over the support of the steady state distribution of the surplus consumption ratio.

The graph in Figure 3 corresponds to Figure 3 of Campbell and Cochrane. The main difference from their graph is that the price-dividend function is portrayed over a smaller range. The smaller range was chosen based on the error analysis. The 110^{th} order polynomial approximation has an error close to zero in the interval $S \in [0.037, 0.054]$. By increasing μ to 0.75 so that $S \in [0.032, 0.061]$, the polynomial approximation must increase to 475^{th} order to keep error the same.[32] To increase

[32] As shown in Figure 2 the function becomes unstable for s within 0.001 of r, so that the graphs in the paper are a good representation of the range of the surplus consumption ratio in which the analytic method can be used.

the upper bound of the support to only 0.064, one would have to set $\mu = 0.9$. To reduce the error to 10^{-9} in this case, the order of the polynomial approximation must be increased to $n = 1160$, which would increase the computation time to 150 seconds. Thus, the behavior of the price-dividend ratio cannot be identified over as large a range considered in Campbell and Cochrane (1999), however dividend growth of $x_0 + 32\%$ per month is larger than any historic observation in the Campbell and Cochrane (1999) data sets.

To see the effect of additional coefficients compare the 475^{th} order polynomial approximation for the price-dividend ratio relative to its first order polynomial approximation. The dash line in Figure 4 shows that this error is small when the surplus consumption ratio is close to its steady state value of $\bar{s}$. However, the error is 3.0% for high surplus consumption ratio and 0.9% for low surplus consumption ratio. By moving to the fourth order polynomial approximation for the price-dividend ratio, the solid line is close to zero for almost all surplus consumption ratios but can still have 1% error for high surplus consumption ratios. This again is a reflection of the non-linear property of the true price-dividend function. By moving to the 475^{th} order polynomial approximation the change in the solution cannot be detected by the computer.[33]

The conditional expected return on equity given by (4.22) can also be calculated once the price-dividend function is known. The expected return on equity at $S = \bar{S}$ is 7.5% in Table 1. This value of the conditional expected return is close to the value in Campbell and Cochrane's data set.[34] By manipulating the parameter p_1 one can match the expected return on equity exactly. In Figure 5 the expected return on equity, given by the bottom (solid) line, changes from 12% to 1.4% over the possible range of the surplus consumption ratio. This graph corresponds to Figure 4 of Campbell and Cochrane (1999) except that the expected return declines faster for high surplus consumption ratios. This helps explain the ability of the price-dividend ratio to forecast future returns as demonstrated by Cochrane (2005, 2006). When the price-dividend ratio is above the normal value expected by

[33]This conclusion suggest that linear generated asset pricing model of Gabaix (2007), which leads to a linear price-dividend function, is inconsistent with non-linear asset pricing models such as Campbell and Cochrane's. This inconsistency becomes more pronounced as dividend growth moves further away from its steady state value.

[34]While the results in Table 1 are close to the moments found in Cambell and Cochrane's data set, the more systematic simulated method of moments of Christensen and Kiefer (2000) can be used to choose the optimal combination of parameters for the theory to match the data. This is feasible since the Maple program takes a few seconds to solve for 110 coefficients in the polynomial approximation for the price-dividend function.

individuals, the price-dividend ratio moves back toward normal times, so that expected returns are low during these time periods. These lower expected returns lead to lower realized returns as well, following (4.21). Thus, the solution captures the time variation in expected returns envisioned by Campbell and Cochrane.

The conditional standard deviation of stock returns is given by (4.23) for various values of the surplus consumption ratio. This standard deviation at $S = \bar{S}$ is about 13.3% in Table 1, and the dash line in Figure 5 varies between 2.3% and 18% as the surplus consumption ratio varies from 0.061 to 0.032 . This result corresponds to Figure 5 of Campbell and Cochrane (1999) for most values of the surplus consumption ratio. However, the decline in the standard deviation at higher levels of the surplus consumption ratio is faster for the true price-dividend function. In Campbell and Cochrane's model the volatility of stocks is lowest in good times while it is highest in bad times. This result is consistent with the direction of change in volatility of the stock market over time in that it is lower during expansions.[35]

Finally, the conditional Sharpe ratio can be calculated using (4.24). At the steady state surplus consumption ratio this Sharpe ratio is 0.56 in Table 1, which is close to the historic average found in Campbell and Cochrane's data set. Following the behavior of the mean and standard deviation of equity, the Sharpe ratio in Figure 6 varies between 0.68 and 0.49 as the economy moves from bad to good times, however it increases for surplus consumption ratios beyond 0.057. This corresponds to Figure 6 of Campbell and Cochrane (1999) except for this higher range for the surplus consumption ratio.

The replication of all the results in Campbell and Cochrane's (1999) paper using the analytic method to solve the continuous time version of their model is surprising given the results in CCH (2008a). Campbell and Cochrane (1999) simulate the discrete time version of their model. However, CCH (2008a) demonstrate that the solution to their integral equation in discrete time cannot generate the appropriate level of the price-dividend ratio and equity premium without considering extreme negative values for dividend growth. This problem does not arise in continuous time since the analytic solution of the ODE (4.12) is solved using local analytic methods. On the other hand,

[35] See Schwert (1989,1990).

the integral equation, which arises in discrete time, must be solved over the entire support of the probability distribution. In addition, the level of the price-dividend function, and equity premium is controlled by the initial conditions p_0 equal to the average price-dividend ratio in Campbell and Cochrane's data, and p_1 can be set based on the average equity premium (4.26). Thus, the continuous time version of Campbell and Cochrane's model more accurately represents the behavior of stock returns, which they wanted to capture.

6 Conclusion

Rather than summarizing the analytic method, which was done at the end section 4, we conclude by mentioning that the general Cauchy-Kovalevsky Theorem is applicable to many continuous time problems in finance. In finance, it is customary for continuous time problems, including option pricing, term structure, portfolio decisions, corporate finance, market microstructure and financial engineering, to have SDF which are analytic.[36] Thus, each of these problems can potentially benefit from using the analytic method discussed here. However, some of these problems have several state variables and are subject to boundary values rather than initial values. In future work we plan to explore to what extent the the Cauchy-Kovalevsky Theorem can be used to solve these alternative financial economic problems.

[36] See Sundaresan (2000) for a recent survey of the work in continuous time finance.

Table 1. Comparison of Model Relative to Data

Statistic	Campbell Cochrane	Campbell Cochrane Data
$E_t(R^e)$	0.075	0.076
$\sigma(R)$	0.133	0.157
$E_t(R^b)$	0.009	0.009
$E_t(R^e - R^b)$	0.066	0.067
$Sharpe$	0.56	0.34
P	18.3	24.7

$Notes$: R^e is the real return on stocks and R^b is the real return on bonds, and P is the price-dividend ratio. E_t is the conditional expectation operator and σ is the standard deviation. The statistics for the theoretical solutions are evaluated at the historic average for the state variable. The parameters for Campbell and Cochrane's model are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9896$, $\gamma = 2$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 111.76$, $\bar{S} = 0.0448$ and $\mu r = 0.32$. The data for Campbell and Cochrane is taken from their Table 4. We use the Postwar Sample from 1947 to 1995 for the U.S..

Figure 1 shows the steady state probability distribution in the Campbell and Cochrane model. The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9896$, $\gamma = 2$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 111.76$, $\bar{S} = 0.0448$ and $\mu r = 0.32$. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}] = [0.032, 0.061]$, The y-axis records the steady state probability distribution for the surplus consumption ratio.

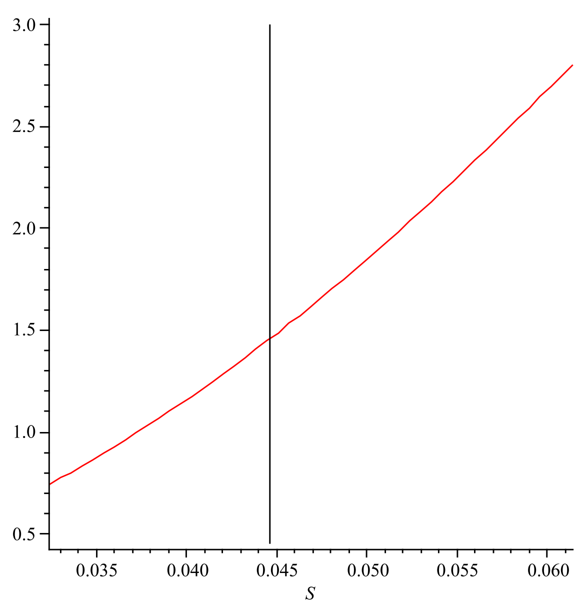

Figure 1

Figure 2 displays the price-dividend function in the Campbell and Cochrane model. The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9896$, $\gamma = 2$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 111.76$, $\bar{S} = 0.0448$. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.49}, \bar{S}e^{0.49}]$. The y-axis records the price-dividend ratio.

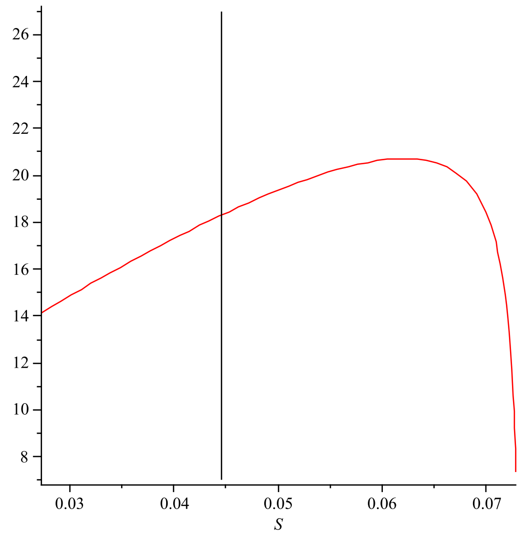

Figure 2

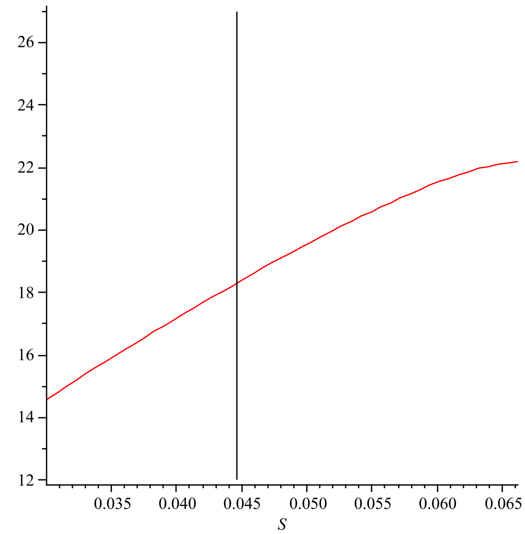

Figure 3

Figure 3 displays the price-dividend function in the Campbell and Cochrane model. The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9896$, $\gamma = 2$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 111.76$, $\bar{S} = 0.0448$ and $\mu r = 0.32$. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}] = [0.032, 0.061]$. The y-axis records the price-dividend ratio.

Figure 4 displays the error analysis for the Campbell and Cochrane model. The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9896$, $\gamma = 2$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 111.76$, $\bar{S} = 0.0448$ and $\mu r = 0.32$. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}] = [0.032, 0.061]$. The y-axis for the dotted line compares the 475^{th} order polynomial approximation for the price-dividend ratio with the first order polynomial approximation. In addition, the solid line compares the 475^{th} order polynomial approximation for the price-dividend ratio to it's fourth order polynomial approximation.

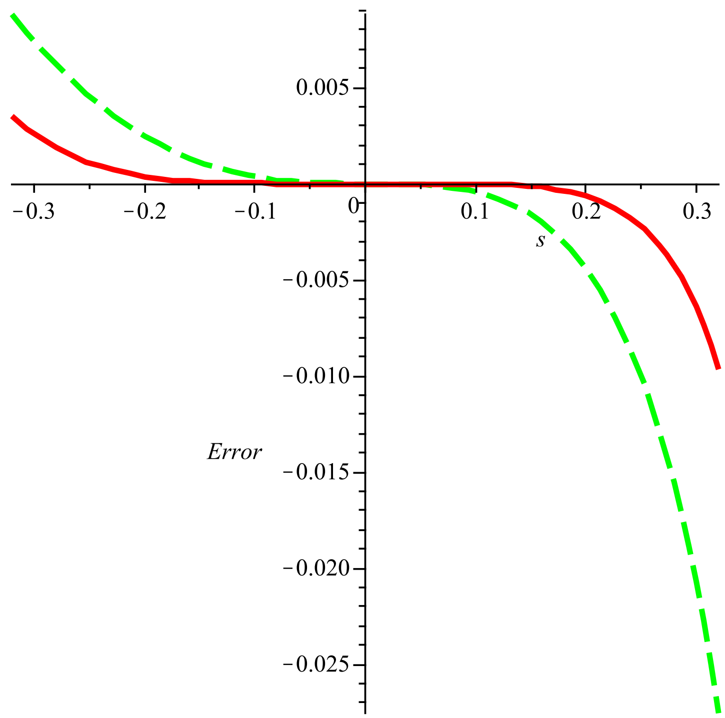

Figure 4

Figure 5 portrays the equity premium and standard deviation of equity in the continuous time model of Campbell and Cochrane. The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9896$, $\gamma = 2$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 111.76$, $\bar{S} = 0.0448$ and $\mu r = 0.32$. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}] = [0.032, 0.061]$. The y-axis records the equity premium and standard deviation. The equity premium line is the solid line, while the dotted line represents the standard deviation.

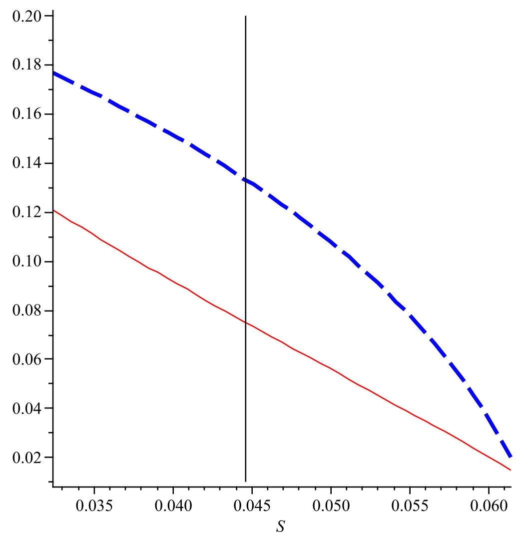

Figure 5

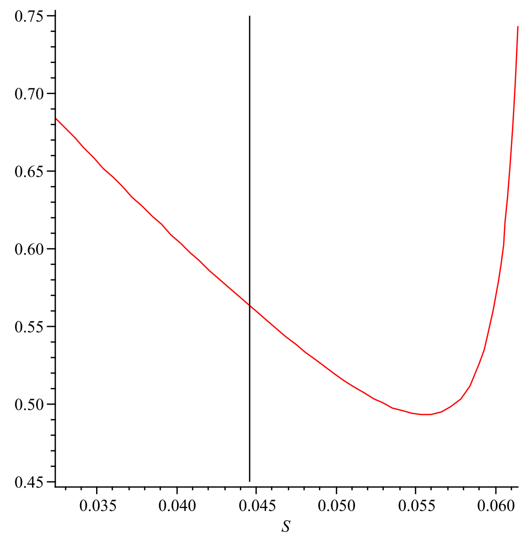

Figure 6

Figure 6 shows the Sharpe ratio in the model of Campbell and Cochrane. The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9896$, $\gamma = 2$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 111.76$, $\bar{S} = 0.0448$ and $\mu r = 0.32$. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}] = [0.032, 0.061]$. The y-axis records the Sharpe ratio.

7 Appendix

Proof of Theorem 3.1. We begin by recalling our initial value problem

$$y''(x) + a(x)y'(x) + b(x)y(x) = g(x),\ y(0) = y_0,\ y'(0) = y_1. \tag{7.1}$$

Since $a(x)$, $b(x)$ and $g(x)$ are analytic about $x = 0$ with radius of convergence r_0 we have

$$a(x) = \sum_{k=0}^{\infty} a_k x^k, \quad b(x) = \sum_{k=0}^{\infty} b_k x^k, \qquad g(x) = \sum_{k=0}^{\infty} d_k x^k, \tag{7.2}$$

and for any $0 < r < r_0$, there exist $M_a,\ M_b,\ M_g > 0$ such that

$$|a_k|r^k \le M_a, \quad |b_k|r^k \le M_b, \quad |d_k|r^k \le M_g, \quad k = 0, 1, 2, \cdots. \tag{7.3}$$

Now let us assume that the solution $y(x)$ can be written (at least formally) as a power series, that is

$$y(x) = \sum_{k=0}^{\infty} c_k x^k, \tag{7.4}$$

where $c_0 = y_0$, $c_1 = y_1$ and c_k, $k = 2, 3, \cdots$ are to be determined so that $y(x)$ is a solution. We have

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k, \tag{7.5}$$

and

$$y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k. \tag{7.6}$$

For y to be a solution we must have

$$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \Big(\sum_{k=0}^{\infty} a_k x^k\Big)\Big(\sum_{k=0}^{\infty} (k+1) c_{k+1} x^k\Big) + \Big(\sum_{k=0}^{\infty} b_k x^k\Big)\Big(\sum_{k=0}^{\infty} c_k x^k\Big) = \sum_{k=0}^{\infty} d_k x^k,$$

which, after multiplying the series, gives

$$\sum_{k=0}^{\infty} \Big[(k+2)(k+1) c_{k+2} + \sum_{j=0}^{k} a_{k-j}(j+1) c_{j+1} + \sum_{j=0}^{k} b_{k-j} c_j\Big] x^k = \sum_{k=0}^{\infty} d_k x^k. \tag{7.7}$$

From the last equation we obtain the recurrence relation

$$(k+2)(k+1) c_{k+2} = d_k - \sum_{j=0}^{k} \Big[a_{k-j}(j+1) c_{j+1} + b_{k-j} c_j\Big] \tag{7.8}$$

for computing the coefficients $c_2, c_3, \cdots$.

Taking absolute values in (7.8) and using the Cauchy estimates (7.3) gives

$$\begin{aligned}(k+2)(k+1)|c_{k+2}| &\le |d_k| + \sum_{j=0}^{k} \Big[|a_{k-j}|(j+1)|c_{j+1}| + |b_{k-j}||c_j|\Big] \\ &\le \frac{M_g}{r^k} + \sum_{j=0}^{k} \Big[\frac{M_a}{r^{k-j}}(j+1)|c_{j+1}| + \frac{M_b}{r^{k-j}}|c_j|\Big] \\ &\le \frac{M_g}{r^k} + \frac{M}{r^k}\sum_{j=0}^{k} \big[(j+1)|c_{j+1}| + |c_j|\big] r^j.\end{aligned}$$

Here $M \doteq \max\{M_a, M_b\}$. Adding the extra term $M|c_{k+1}|r$ (it will be helpful later) to the right-hand side of the last inequality gives

$$(k+2)(k+1)|c_{k+2}| \le \frac{M_g}{r^k} + \frac{M}{r^k}\sum_{j=0}^{k} \big[(j+1)|c_{j+1}| + |c_j|\big] r^j + M|c_{k+1}|r. \tag{7.9}$$

Letting $C_0 \doteq |c_0| = |y_0|$, $C_1 \doteq |c_1| = |y_1|$ and for $k \ge 2$ defining C_k by the recurrence relation

$$(k+2)(k+1)C_{k+2} = \frac{M_g}{r^k} + \frac{M}{r^k}\sum_{j=0}^{k}[(j+1)C_{j+1} + C_j]r^j + MC_{k+1}r\,, \tag{7.10}$$

we see that

$$|c_k| \le C_k,\ \ k = 0, 1, 2, \cdots. \tag{7.11}$$

Therefore, the series $\sum_{k=0}^{\infty} c_k x^k$ converges if $\sum_{k=0}^{\infty} C_k x^k$ does.

Next we shall show that the series $\sum_{k=0}^{\infty} C_k x^k$ converges for $|x| < r$. For this, by the ratio test, it suffices to show $\limsup_{k\to\infty} C_{k+1}/C_k \le 1/r$. In recurrence relation (7.10) replacing k with $k-1$ gives

$$(k+1)kC_{k+1} = \frac{M_g}{r^{k-1}} + \frac{M}{r^{k-1}}\sum_{j=0}^{k-1} \big[(j+1)C_{j+1} + C_j\big] r^j + MC_k r,\ \ k \ge 1, \tag{7.12}$$

and replacing k with $k-2$ gives

$$k(k-1)C_k = \frac{M_g}{r^{k-2}} + \frac{M}{r^{k-2}}\sum_{j=0}^{k-2} \big[(j+1)C_{j+1} + C_j\big] r^j + MC_{k-1} r,\ \ k \ge 2. \tag{7.13}$$

Multiplying (7.12) by r and using (7.13) gives

$$\begin{aligned} r(k+1)kC_{k+1} &\le \frac{M_g}{r^{k-2}} + \frac{M}{r^{k-2}}\Big\{\sum_{j=0}^{k-2}\big[(j+1)C_{j+1}+C_j\big]r^j + \big[kC_k + C_{k-1}\big]r^{k-1}\Big\} + MC_kr^2 \\ &\le \frac{M_g}{r^{k-2}} + \frac{M}{r^{k-2}}\sum_{j=0}^{k-2}\big[(j+1)C_{j+1}+C_j\big]r^j + MkC_kr + MC_{k-1}r + MC_kr^2 \\ &\le \frac{M_g}{r^{k-2}} + k(k-1)C_k - \frac{M_g}{r^{k-2}} - MC_{k-1}r + MkC_kr + MC_{k-1}r + MC_kr^2. \end{aligned}$$

From the last inequality we obtain

$$r(k+1)kC_{k+1} \le \big[k(k-1) + Mkr + Mr^2\big]C_k.$$

or

$$\frac{C_{k+1}}{C_k} \le \frac{(k-1)}{r(k+1)} + M\frac{k+r}{(k+1)k}. \tag{7.14}$$

Therefore $\limsup_{k\to\infty} C_{k+1}/C_k \le 1/r$. Thus, the function $y(x)$ defined by the power series (7.4), whose coefficients are defined by the recursion formula (7.8) has radius of convergence r_0. This justifies all operations performed above (multiplication and differentiation of series). Therefore, the solution $y(x)$ to the initial value problem (2.1) is analytic with radius r_0. $\Box$

Proof of Corollary 3.2. Iterating backwards using inequality (7.14) to obtain

$$\begin{aligned} C_k &\le C_{k-1}\left[\frac{k-2}{rk} + M\frac{k-1+r}{k(k-1)}\right] \\ &\le C_{k-2}\left[\frac{k-3}{r(k-1)} + M\frac{k-2+r}{(k-1)(k-2)}\right]\left[\frac{k-2}{rk} + M\frac{k-1+r}{k(k-1)}\right] \\ &\le C_2\prod_{l=2}^{k-1}\left[\frac{l-1}{r(l+1)} + M\frac{l+r}{(l+1)l}\right] \\ &\le \frac{1}{2}\left[M_g + |y_1|(1+r)M + |y_0|M\right]\prod_{l=2}^{k-1}\left[\frac{l-1}{r(l+1)} + M\frac{l+r}{(l+1)l}\right]. \end{aligned}$$

The last step uses the definition of C_2 in (7.12) when $k=1$.

Using this, the Taylor series remainder is estimated as follows

$$\begin{aligned}\left|y(x)-\sum_{k=0}^{n}c_kx^k\right| &= \sum_{k=n+1}^{\infty}|c_k||x|^k \le \sum_{k=n+1}^{\infty}C_k|x|^k \\ &\le \frac{1}{2}\left[M_g+|y_1|(1+r)M+|y_0|M\right]\sum_{k=n+1}^{\infty}\prod_{l=2}^{k-1}\left[\frac{l-1}{r(l+1)}+M\frac{l+r}{(l+1)l}\right]|x|^k \\ &\le \frac{1}{2}\left[M_g+|y_1|(1+r)M+|y_0|M\right]\sum_{k=n+1}^{\infty}\prod_{l=2}^{k-1}\left[\frac{l-1}{r(l+1)}+M\frac{l+r}{(l+1)l}\right]|\mu r|^k.\end{aligned}$$

Consequently, we have a uniform bound for the Taylor series remainder for $|x| \le |\mu r|$, where $0 \le \mu \le 1$. □

Derivation of stochastic process for the SDF (4.8). In the CC model the stochastic discount factor is given by (4.2), its partial derivatives are:

$$\frac{\partial\Lambda}{\partial S}=-\gamma\frac{\Lambda}{S},\ \frac{\partial\Lambda}{\partial C}=-\gamma\frac{\Lambda}{C},\ \frac{\partial\Lambda}{\partial t}=-\beta\Lambda, \tag{7.15}$$

$$\frac{\partial^2\Lambda}{\partial S^2}=\gamma(\gamma+1)\frac{\Lambda}{S^2},\ \frac{\partial^2\Lambda}{\partial C^2}=\gamma(\gamma+1)\frac{\Lambda}{C^2},\ \text{ and }\ \frac{\partial^2\Lambda}{\partial S\partial C}=\gamma^2\frac{\Lambda}{SC}.$$

The logarithm of consumption $x=\ln(C)$, and the surplus consumption ratio, $s=\ln(S)$, are assumed to follow the stochastic processes (4.3) and (4.4), respectively, so that by Ito's rule

$$\frac{dS}{S}=ds+\frac{1}{2}(ds)^2 \text{ so that } \left(\frac{dS}{S}\right)^2=(ds)^2, \tag{7.16}$$

and

$$\frac{dC}{C}=dx+\frac{1}{2}(dx)^2 \text{ so that } \left(\frac{dC}{C}\right)^2=(dx)^2. \tag{7.17}$$

As a result S and C are stochastic differential equations such that Ito's rule (4.7) may be applied to the function $\Lambda(t,S,C)$ to get

$$d\Lambda=\frac{\partial\Lambda}{\partial t}dt+\frac{\partial\Lambda}{\partial S}dS+\frac{\partial\Lambda}{\partial C}dC+\frac{1}{2}\left(\frac{\partial^2\Lambda}{\partial S^2}(dS)^2+2\frac{\partial^2\Lambda}{\partial S\partial C}dSdC+\frac{\partial^2\Lambda}{\partial C^2}(dC)^2\right).$$

Substitute in the partial derivatives of Λ (7.15) to find

$$d\Lambda=-\beta\Lambda dt-\gamma\frac{\Lambda}{S}dS-\gamma\frac{\Lambda}{C}dC+\frac{1}{2}\left(\gamma(\gamma+1)\frac{\Lambda}{S^2}(dS)^2+2\gamma^2\frac{\Lambda}{SC}dSdC+\gamma(\gamma+1)\frac{\Lambda}{C^2}(dC)^2\right).$$

Group common terms together, and use the rules for the stochastic process for S (7.16) and C (7.17) to yield

$$\frac{d\Lambda}{\Lambda} = -\beta dt - \gamma\left(ds + \frac{1}{2}(ds)^2\right) - \gamma\left(dx + \frac{1}{2}(dx)^2\right) + \frac{1}{2}\left(\gamma(\gamma+1)(ds)^2 + 2\gamma^2\,(dx)\,(ds) + \gamma(\gamma+1)(dx)^2\right).$$

Once again combine common terms, and substitute the stochastic processes for dx (4.3) and ds (4.4)

$$\begin{aligned}\frac{d\Lambda}{\Lambda} = &- \beta dt - \gamma\left((\phi-1)(s-\bar{s})dt + \lambda(s-\bar{s})\sigma d\omega\right) - \gamma\left(\bar{x}dt + \sigma d\omega\right) + \\ &\frac{1}{2}\left(\gamma^2((\phi-1)(s-\bar{s})dt + \lambda(s-\bar{s})\sigma d\omega)^2 + 2\gamma^2\left(\bar{x}dt + \sigma d\omega\right)((\phi-1)(s-\bar{s})dt + \right.\\ &\left.\lambda(s-\bar{s})\sigma d\omega) + \gamma^2(\bar{x}dt + \sigma d\omega)^2\right).\end{aligned}$$

Now use multiplication rules for Brownian motion $(d\omega)^2 = dt$, $d\omega dt = 0$, and $(dt)^2 = 0$, and combine the common terms so that

$$\frac{d\Lambda}{\Lambda} = -\beta dt - \gamma\left((\phi-1)(s-\bar{s})dt + \lambda(s-\bar{s})\sigma d\omega\right) - \gamma\left(\bar{x}dt + \sigma d\omega\right) + \frac{1}{2}\gamma^2\sigma^2\left(1 + \lambda(s-\bar{s})\right)^2 dt.$$

Finally, we have the stochastic process for the SDF (4.8) in the CC model

$$\frac{d\Lambda}{\Lambda} = \left(-\gamma(\phi-1)s - \beta - \gamma\bar{x} + \frac{1}{2}\gamma^2\sigma^2\left(1+\lambda(s)\right)^2\right)dt - \gamma\sigma\left(1 + \lambda(s-\bar{s})\right)d\omega,$$

where s is translated to $s - \bar{s}$ to simplify the subsequent algebra.

Derivation of the ODE (4.12) for the CC model. Cochrane (2005, p.28) shows that the equilibrium price of stocks satisfies the Euler equation:

$$\Lambda(t)D(t)\,dt + E_t\left[d(\Lambda(t)P(t))\right] = 0\,, \tag{7.18}$$

where $P(t)$ is the price of a stock at time t, and $D(t)$ is the dividend paid by this stock at time t. Define the price-dividend ratio to be $p(t) = P(t)/D(t)$ such that the Euler condition (7.18) is equivalent to

$$\Lambda(t)D(t)\,dt + E_t\left[d(\Lambda(t)p(t)D(t))\right] = 0\,.$$

By Ito's Lemma 4.1

$$\frac{d(\Lambda pD)}{\Lambda pD} = \frac{d\Lambda}{\Lambda} + \frac{dp}{p} + \frac{dD}{D} + \frac{d\Lambda dp}{\Lambda p} + \frac{dDdp}{Dp} + \frac{d\Lambda dD}{\Lambda D}\,,$$

so that the Euler condition (7.18) becomes

$$\frac{1}{p}\,dt + E_t\left(\frac{d\Lambda}{\Lambda} + \frac{dp}{p} + \frac{dD}{D} + \frac{d\Lambda dp}{\Lambda p} + \frac{dDdp}{Dp} + \frac{d\Lambda dD}{\Lambda D}\right) = 0\,, \tag{7.19}$$

which corresponds to (4.10). In equilibrium, we have $D = C$ so that the consumption is replaced by the dividends in (7.19). As a result, one can use the rule for the consumption level (7.17) to obtain

$$0 = \frac{1}{p}dt + E_t\left(\frac{d\Lambda}{\Lambda} + \frac{dp}{p} + dx + \frac{1}{2}(dx)^2 + \frac{d\Lambda}{\Lambda}\left(\frac{dp}{p} + dx + \frac{1}{2}(dx)^2\right) + \frac{dp}{p}\left(dx + \frac{1}{2}(dx)^2\right)\right). \tag{7.20}$$

From the stochastic process for consumption growth (4.3) it follows that

$$dx + \frac{1}{2}(dx)^2 = \bar{x}dt + \sigma d\omega + \frac{1}{2}(\bar{x}dt + \sigma d\omega)^2 = \left(\bar{x} + \frac{1}{2}\sigma^2\right)dt + \sigma d\omega. \tag{7.21}$$

Now substitute this result into the pricing equation (7.20), and use condition $E_t(d\omega) = 0$ to get

$$\begin{aligned}0 =& \frac{1}{p}dt + E_t\left(\frac{d\Lambda}{\Lambda} + \frac{dp}{p} + \left(\bar{x} + \frac{1}{2}\sigma^2\right)dt + \frac{d\Lambda}{\Lambda}\left(\frac{dp}{p} + \right.\right.\\ &\left.\left.\left(\bar{x} + \frac{1}{2}\sigma^2\right)dt + \sigma d\omega\right) + \frac{dp}{p}\left(\left(\bar{x} + \frac{1}{2}\sigma^2\right)dt + \sigma d\omega\right)\right).\end{aligned}$$

Substitute the expression for the stochastic discount factor (4.8), and again use $E_t(d\omega) = 0$ such that

$$\begin{aligned}0 =& \frac{1}{p}dt + E_t\left(\left(-\gamma(\phi - 1)s - \beta - \gamma\bar{x} + \frac{1}{2}\gamma^2\sigma^2\left(1 + \lambda(s)\right)^2\right)dt + \right.\\ &\left(\left(-\gamma(\phi - 1)s - \beta - \gamma\bar{x} + \frac{1}{2}\gamma^2\sigma^2\left(1 + \lambda(x)\right)^2\right)dt - \gamma\sigma\left(1 + \lambda(x)\right)d\omega\right)\\ &\times\left(\frac{dp}{p} + \left(\bar{x} + \frac{1}{2}\sigma^2\right)dt + \sigma d\omega\right) + \frac{dp}{p} + \left(\bar{x} + \frac{1}{2}\sigma^2\right)dt + \left.\frac{dp}{p}\left(\left(\bar{x} + \frac{1}{2}\sigma^2\right)dt + \sigma d\omega\right)\right).\end{aligned} \tag{7.22}$$

The solution for the price dividend function $p(s)$ is assumed to be a C^2 function of the surplus consumption ration, where ds is given by (4.4), so by Ito's rule (4.7) the stochastic process for the price-dividend ratio (4.11) is given by

$$dp = \left(p'(s)(\phi - 1)s + \frac{1}{2}p''(s)\lambda(s)^2\sigma^2\right)dt + p'(s)\lambda(s)\sigma d\omega. \tag{7.23}$$

By substituting this expression for the price dividend ratio into the asset pricing equation (7.22), and using $E_t(d\omega) = 0$ one has

$$
\begin{aligned}
0 =& \frac{1}{p(s)} dt + E_t \Bigg(\left(-\gamma(\phi-1)s - \beta - \gamma\bar{x} + \frac{1}{2}\gamma^2\sigma^2 \left(1+\lambda(s)\right)^2 \right) dt + \\
& \left(\left(-\gamma(\phi-1)s - \beta - \gamma\bar{x} + \frac{1}{2}\gamma^2\sigma_1^2 \left(1+\lambda(s)\right)^2 \right) dt - \gamma\sigma\left(1+\lambda(s)\right) d\omega \right) \\
& \times \left(\frac{\left(p'(s)(\phi-1)s + \frac{1}{2}p''(s)\lambda(s)^2\sigma^2\right) dt + p'(s)\lambda(s)\sigma d\omega}{p(s)} + \left(\bar{x} + \frac{1}{2}\sigma^2\right) dt + \sigma d\omega \right) \\
& + \frac{\left(p'(s)(\phi-1)s + \frac{1}{2}p''(s)\lambda(s)^2\sigma^2\right) dt}{p(s)} + \left(\bar{x} + \frac{1}{2}\sigma^2\right) dt + \\
& \left(\frac{\left(p'(s)(\phi-1)s + \frac{1}{2}p''(s)\lambda(s)^2\sigma^2\right) dt + p'(s)\lambda(s)\sigma d\omega}{p(s)} \right) \times \left(\left(\bar{x} + \frac{1}{2}\sigma^2\right) dt + \sigma d\omega \right) \Bigg).
\end{aligned}
$$

Finally, use Ito's multiplication rules $(d\omega)^2 = dt$, $d\omega dt = 0$, and $(dt)^2 = 0$ from Ito's Lemma 4.1, and multiply by $p(s)dt$, so that

$$
\begin{aligned}
0 =& 1 + \left(\gamma(1-\phi)s - \beta + (1-\gamma)\bar{x} + \frac{\sigma^2}{2}\left(\gamma(1+\lambda(s)) - 1\right)^2 \right) p(s) - \\
& \left((1-\phi)\, s + (\gamma-1)\,\sigma^2\lambda(s) + \gamma\sigma^2\lambda(s)^2 \right) p'(s) + \frac{\lambda(s)^2\sigma^2}{2} p''(x).
\end{aligned}
$$

Define the constants $K_0 = \beta + (\gamma-1)\bar{x} > 0$ and $K_1 = (1-\phi) > 0$, and use the definition for $r(x)$ to yield the ODE (4.12).

Derivation of initial condition (4.25). Following Cochrane (2005, p.26) the instantaneous total return on equity is

$$R^e(s)\, dt = \frac{dP}{P} + \frac{Ddt}{P}.$$

We now convert from stock price P to price-dividend ratio where $p = \frac{P}{D}$. so that by Ito's Lemma 4.1 the return on equity is given by the equation

$$R^e(s)\, dt = \frac{dt}{p} + \frac{dp}{p} + \frac{dC}{C} + \frac{dCdp}{Cp}\,. \tag{7.24}$$

Substituting in the stochastic process for consumption growth (7.17) and (7.21), as well as the price-dividend ratio (7.23) yields the stochastic process for the return on stocks (4.21).

Recall that $-R^b(s)\,dt = E_t[d\Lambda/\Lambda]$ and $D = C$. By (7.19) and (7.24), one finds

$$\begin{aligned}\left(E_t\left(R^e(s)\right) - R^b(s)\right)\,dt &= \frac{dt}{p} + E_t\left[\frac{d\Lambda}{\Lambda} + \frac{dp}{p} + \frac{dC}{C} + \frac{dC\,dp}{Cp}\right] \\ &= -E_t\left[\frac{d\Lambda dp}{\Lambda p} + \frac{d\Lambda dC}{\Lambda C}\right] \\ &= \sigma^2\gamma\left(1+\lambda(s)\right)\left(1+\lambda(s)\frac{p'(s)}{p(s)}\right)dt\,.\end{aligned}$$

In the last step the stochastic process for the SDF (4.8) is multiplied by the stochastic process for consumption growth (7.21), and the price-dividend ratio (7.23) taking account of the multiplication rules in stochastic calculus. The second initial condition (4.25) follows by solving this equation for $p'(s)$, and setting $s = 0$.

Recursive formula (4.19) for Campbell and Cochrane's model. By the Cauchy-Kovalevsky Theorem 3.1 we can write

$$p(s) = \sum_{n=0}^{\infty} p_n s^n \quad \text{near } s = 0.$$

Also, we write $c_i(s) = \sum_{n=0}^{\infty} c_i^{(n)} s^n$ near $s = 0$. We will derive the recurrence formula for p_n with $n \geqslant 2$. For this, we calculate

$$p'(s) = \sum_{n=0}^{\infty}(n+1)p_{n+1}s^n \quad \text{and} \quad p''(s) = \sum_{n=0}^{\infty}(n+1)(n+2)p_{n+2}s^n\,.$$

Then, using the product formula for convergent series we obtain

$$\begin{aligned}c_2(s)p''(s) &= \left[\sum_{n=0}^{\infty} c_2^{(n)}s^n\right]\left[\sum_{n=0}^{\infty}(n+1)(n+2)p_{n+2}s^n\right] = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(k+1)(k+2)c_2^{(n-k)}p_{k+2}\right]s^n, \\ c_1(s)p'(s) &= \left[\sum_{n=0}^{\infty} c_1^{(n)}s^n\right]\left[\sum_{n=0}^{\infty}(n+1)p_{n+1}s^n\right] = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(k+1)c_1^{(n-k)}p_{k+1}\right]s^n, \\ c_0(s)p(s) &= \left[\sum_{n=0}^{\infty} c_0^{(n)}s^n\right]\left[\sum_{n=0}^{\infty}p_n s^n\right] = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}c_0^{(n-k)}p_k\right]s^n.\end{aligned}$$

Substituting the formulas for $c_2(s)p''(s)$, $c_1(s)p'(s)$, and $c_0(s)p(s)$ into the differential equation gives

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(k+1)(k+2)c_2^{(n-k)}p_{k+2}\right]s^n = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(k+1)c_1^{(n-k)}p_{k+1} + \sum_{k=0}^{n}c_0^{(n-k)}p_k\right]s^n - 1.$$

Matching the coefficients of same powers we obtain

$$2c_2^{(0)}p_2 = c_1^{(0)}p_1 + c_0^{(0)}p_0 - 1,$$
$$\sum_{k=0}^{n}(k+1)(k+2)c_2^{(n-k)}p_{k+2} = \sum_{k=0}^{n}(k+1)c_1^{(n-k)}p_{k+1} + \sum_{k=0}^{n}c_0^{(n-k)}p_k \quad \text{for } n \geqslant 1.$$

Finally, solving the second equation for p_{n+2} ($n \geqslant 1$) gives

$$\begin{aligned}(n+1)(n+2)c_2^{(0)}p_{n+2} &= \sum_{k=1}^{n+1}kc_1^{(n-k+1)}p_k + \sum_{k=0}^{n}c_0^{(n-k)}p_k - \sum_{k=2}^{n+1}(k-1)kc_2^{(n-k+2)}p_k \\ &= (n+1)(c_1^{(0)} - nc_2^{(1)})p_{n+1} + \sum_{k=2}^{n}\left[c_0^{(n-k)} + kc_1^{(n-k+1)} - (k-1)kc_2^{(n-k+2)}\right]p_k \\ &\quad + (c_0^{(n-1)} + c_1^{(n)})p_1 + c_0^{(n)}p_0\,.\end{aligned}$$

If $c_2^{(0)} \neq 0$, then the above recurrence formula calculates all the p_n with $n \geqslant 2$. $\square$

Derivation of bounds on $a(x)$, $b(x)$, and $g(x)$. To estimate the bound on the coefficients it is necessary to find a bound on the sensitivity function $\lambda(s)$ within the complex plane following CCH (2006a). Let $z = x + yi$ be a point on the circle C_r, that is $x^2 + y^2 = r^2$. Also, we assume that $r < r_0 = \frac{1-\bar{S}^2}{2}$. We write

$$u + vi = \lambda(z) = \frac{1}{\bar{S}}\sqrt{1-2z} - 1 = \frac{1}{\bar{S}}\sqrt{1-2x-2yi} - 1 \quad \text{with } u+1 \geq 0,$$

which is equivalent to

$$(u+1)^2 - v^2 = \frac{1-2x}{\bar{S}^2} \quad \text{and} \quad (u+1)v = -\frac{y}{\bar{S}^2}\,.$$

These equations imply

$$(u+1)^4 - \frac{1-2x}{\bar{S}^2}(u+1)^2 - \frac{y^2}{\bar{S}^4} = 0 \quad \text{and} \quad v^4 + \frac{1-2x}{\bar{S}^2}v^2 - \frac{y^2}{\bar{S}^4} = 0\,.$$

If $y \neq 0$, then the quadratic formula yields

$$(u+1)^2 = \frac{1-2x+\sqrt{1-4x+4r^2}}{2\bar{S}^2} \quad \text{and} \quad v^2 = \frac{-1+2x+\sqrt{1-4x+4r^2}}{2\bar{S}^2}\,.$$

Applying

$$(1-2r)^2 = 1-4r+4r^2 \leq 1-4x+4r^2 \leq 1+4r+4r^2 = (1+2r)^2\,,$$

we get

$$\frac{1-2r}{\bar{S}^2} \le (u+1)^2 \le \frac{1+2r}{\bar{S}^2} \quad \text{and} \quad 0 \le v^2 \le \frac{2r}{\bar{S}^2}\,,$$

where the first inequality implies further that

$$0 < \lambda(r) = \frac{\sqrt{1-2r}}{\bar{S}} - 1 \le u \le \frac{\sqrt{1+2r}}{\bar{S}} - 1 = \lambda(-r).$$

So

$$0 < \lambda^2(r) \le u^2 + v^2 \le \lambda^2(-r) + \frac{2r}{\bar{S}^2}\,.$$

This set of inequalities can be used to establish bounds on the coefficients for z on C_r. Since

$$c_2(z) = \frac{\sigma^2}{2}\lambda^2(z)$$

we have that

$$|c_2(z)| = \frac{\sigma^2}{2}|\lambda^2(z)| \ge \frac{\sigma^2}{2}|\lambda^2(r)| \doteq m_2, \quad \text{for } |z| = r.$$

Also, since

$$c_1(z) = K_1 Z + (\gamma-1)\sigma^2\lambda(z) + \gamma\sigma^2\lambda^2(z),$$

we have that

$$|c_1(z)| \le K_1 r + (\gamma-1)\sigma^2\sqrt{\lambda^2(-r) + \frac{2r}{\bar{S}^2}} + \gamma\sigma^2\left[\lambda^2(-r) + \frac{2r}{\bar{S}^2}\right] \doteq m_1, \quad \text{for } |z| = r.$$

Finally, since

$$c_0(z) = K_0 - \gamma K_1 z - \frac{1}{2}\sigma^2\Big(\gamma\lambda(z) + \gamma - 1\Big)^2,$$

we have that

$$|c_0(z)| \le K_0 + \gamma K_1 r + \frac{\sigma^2(\gamma-1)^2}{2} + \gamma(\gamma-1)\sigma^2\sqrt{\lambda^2(-r) + \frac{2r}{\bar{S}^2}} + \frac{\gamma^2\sigma^2}{2}\left[\lambda^2(-r) + \frac{2r}{\bar{S}^2}\right] \doteq m_0,$$

for all $|z| = r$. Thus, the bounds on the coefficients and forcing term are

$$|a(z)| = \left|\frac{c_1(z)}{c_2(z)}\right| \le \frac{m_1}{m_2} \doteq M_a, \quad |b(z)| = \left|\frac{c_0(z)}{c_2(z)}\right| \le \frac{m_0}{m_2} \doteq M_b, \quad \text{and} \quad |g(z)| = \left|\frac{1}{c_2(z)}\right| \le \frac{1}{m_2} \doteq M_g,$$

for z on C_r.

Then, by Cauchy's integral formula we obtain

$$|a^{(k)}(0)| \leq \frac{k!}{2\pi} \oint_{C_r} \frac{|a(z)|}{r^{k+1}}\, dz = \frac{M_a k!}{2\pi} \cdot \frac{2\pi r}{r^{k+1}} = \frac{M_a k!}{r^k} \qquad \text{for } k = 0, 1, 2, \ldots .$$

which corresponds to the bounds in (7.3) with $|a_k| = k!|a^{(k)}(0)|$. Following the same argument for the coefficient $b(s)$ and $g(s)$, we get

$$|b^{(k)}(0)| \leq \frac{M_b k!}{r^k} \qquad \text{and } |g^{(k)}(0)| \leq \frac{M_g k!}{r^k} \qquad \text{for } k = 0, 1, 2, \ldots . \quad \square$$

Derivation of probability distribution (4.30). Assume that the steady state distribution $\pi(s)$ satisfies the reflection barrier condition at $s = s^*$ from Cox and Miller (1965, p. 223):

$$\left\{ \frac{1}{2} \frac{d}{ds}[a(s)\pi(s)] - b(s)\pi(s) \right\} \Big|_{s=s^*} = 0\,.$$

Then $\frac{1}{2}\frac{d^2}{ds^2}[a(s)\pi(s)] - \frac{d}{ds}[b(s)\pi(s)] = 0$ is equivalent to

$$\frac{1}{2} \frac{d}{ds}[a(s)\pi(s)] - b(s)\pi(s) = 0$$

or to the separable equation:

$$\frac{d\pi(s)}{ds} + \frac{a'(s) - 2b(s)}{a(s)} \pi(s) = 0\,.$$

So

$$\pi(s) = c_1 \exp\left\{ -\int^s \frac{a'(v) - 2b(v)}{a(v)}\, dv \right\} \qquad \text{for some } c_1 \in \mathbb{R}\,.$$

Recall that $a(s) = \sigma^2 \lambda(s)^2$, $b(s) = (\phi - 1)s$, and $\lambda(s) = \frac{1}{S}\sqrt{1 - 2s} - 1$.

$$\begin{aligned} \int \frac{a'(s) - 2b(s)}{a(s)}\, ds &= \int \frac{2\sigma^2 \lambda(s)\lambda'(s) - 2(\phi - 1)s}{\sigma^2 \lambda(s)^2}\, ds = 2 \int \frac{\lambda'(s)}{\lambda(s)}\, ds + \frac{2(\phi - 1)}{\sigma^2} \int \frac{s}{\lambda(s)^2}\, ds \\ &= 2 \ln \lambda(s) + \frac{2(1 - \phi)}{\sigma^2} \int \frac{s}{\lambda(s)^2}\, ds \end{aligned}$$

We will calculate $\int \frac{s}{\lambda(s)^2}\, ds$ via the change of variable: $y = \lambda(s) = \frac{1}{S}\sqrt{1 - 2s} - 1$. Then we get

$s = \frac{1}{2}[1-(\bar{S})^2(y+1)^2]$ and $ds = -(\bar{S})^2(y+1)\,dy$.

$$\begin{aligned}\int \frac{s}{\lambda(s)^2}\,ds &= \int \frac{1}{2y^2}\left[1-(\bar{S})^2(y+1)^2\right]\left[-(\bar{S})^2(y+1)\right]\,dy = \frac{(\bar{S})^2}{2}\int \frac{1}{y^2}\left[(\bar{S})^2(y+1)^3-(y+1)\right]\,dy \\ &= \frac{(\bar{S})^2}{2}\int \left[(\bar{S})^2 y + 3(\bar{S})^2 + \frac{3(\bar{S})^2-1}{y} + \frac{(\bar{S})^2-1}{y^2}\right]\,dy \\ &= \frac{(\bar{S})^2}{2}\left[\frac{(\bar{S})^2}{2}y^2 + 3(\bar{S})^2 y + (3(\bar{S})^2-1)\ln y + \frac{1-(\bar{S})^2}{y}\right] + C \\ &= \frac{(\bar{S})^2}{2}\left[\frac{(\bar{S})^2}{2}\lambda(s)^2 + 3(\bar{S})^2\lambda(s) + (3(\bar{S})^2-1)\ln\lambda(s) + \frac{1-(\bar{S})^2}{\lambda(s)}\right] + C\end{aligned}$$

As a result,

$$\begin{aligned}\int \frac{a'(s)-2b(s)}{a(s)}\,ds = {} & \frac{2\sigma^2 + (3(\bar{S})^2-1)(\bar{S})^2(1-\phi)}{\sigma^2}\ln\lambda(s) \\ & + \frac{(\bar{S})^2(1-\phi)}{\sigma^2}\left[\frac{(\bar{S})^2}{2}\lambda(s)^2 + 3(\bar{S})^2\lambda(s) + \frac{1-(\bar{S})^2}{\lambda(s)}\right] + C\,.\end{aligned}$$

Set

$$c_2 = \int_{-\infty}^{s^*}\left\{\frac{2\sigma^2 + (3(\bar{S})^2-1)(\bar{S})^2(1-\phi)}{\sigma^2}\ln\lambda(s) + \frac{(\bar{S})^2(1-\phi)}{\sigma^2}\left[\frac{(\bar{S})^2}{2}\lambda(s)^2 + 3(\bar{S})^2\lambda(s) + \frac{1-(\bar{S})^2}{\lambda(s)}\right]\right\}\,ds\,.$$

Then

$$\pi(s) = \frac{1}{c_2}\left\{\frac{2\sigma^2 + (3(\bar{S})^2-1)(\bar{S})^2(1-\phi)}{\sigma^2}\ln\lambda(s) + \frac{(\bar{S})^2(1-\phi)}{\sigma^2}\left[\frac{(\bar{S})^2}{2}\lambda(s)^2 + 3(\bar{S})^2\lambda(s) + \frac{1-(\bar{S})^2}{\lambda(s)}\right]\right\}$$

for $s < s^*$. Thus (4.30) is the steady state probability function for s. $\square$

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