Internet Appendix: What's Different about Bank Holding Companies?

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1 Trading Desk's Problem

The portfolio manager's problem (30) subject to (31) is solved by specifying the HJB equation when the change in the lifetime utility is found using (30) and (3) in the paper. After choosing the optimal portfolio the manager's problem boils down to the solution of a Partial Differential Equation (PDE) for the lifetime utility $h(\tau, X)$. Here $h(\tau, X)$ is $J(\tau, X)$ in equation (32) of our paper. This lifetime utility under the optimal behavior must be the solution to the PDE.

$$\begin{aligned}\frac{\partial h(\tau,X)}{\partial \tau} &= \frac{1}{2} Trace\left(A' \frac{\partial^2 h(\tau,X)}{\partial X \partial X} A\right) - 2\left(\frac{\partial h(\tau,X)}{\partial X}\right)' \Big[BX + C\Big] \\ &- h(\tau,X)\Big[X'DX + EX + F\Big] - \frac{1}{2h(\tau,X)} Trace\left(H \frac{\partial h(\tau,X)}{\partial X}\left(\frac{\partial h}{\partial X}\right)'\right) + G\end{aligned} \tag{1}$$

subject to

$$h(0, X) = h(X). \tag{2}$$

$h(X)$ is some given terminal lifetime utility of the investor.

The coefficients are given by

$$\begin{aligned} A &\equiv \Sigma_X \\ B &\equiv \frac{1}{2}\Big[A^{\mathcal{P}} - (\gamma^j - 1)\Sigma_X\Sigma_X'(b' - \iota b_n')\omega_1(b - \iota b_n)(A^{\mathcal{P}} - A^{\mathcal{Q}})\Big] \\ C &\equiv \frac{1}{2}\left[-\xi\Sigma_X\Sigma_X' b_n + (\gamma^j - 1)\Sigma_X\Sigma_X'\left(b - \iota b_n\right)'\omega_1 K - \gamma^{\mathcal{P}}\right] \\ D &\equiv \frac{\gamma^j - 1}{2\gamma^j}(A^{\mathcal{P}} - A^{\mathcal{Q}})'(b' - \iota b_n')\omega_1(b - b_n\iota')(A^{\mathcal{P}} - A^{\mathcal{Q}})\end{aligned} \tag{3}$$

$$\begin{aligned} E &\equiv \frac{\gamma^j - 1}{\gamma^j}\left(\delta_1 - \xi b_n(A^{\mathcal{P}} - A^{\mathcal{Q}})\right) - \frac{\gamma^j - 1}{\gamma^j}K'\omega_1(b - \iota b_n)(A^{\mathcal{P}} - A^{\mathcal{Q}}) \\ F &\equiv \frac{1 - \gamma^j}{2}\xi^2 b_n\Sigma_X\Sigma_X' b_n' + \frac{\gamma^j - 1}{2\gamma^j}K'\omega_1 K + \frac{\beta}{\gamma^j} + \frac{\gamma^j - 1}{\gamma^j}\Big[\delta_0 + \xi b_n(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}})\Big] \\ \text{with } K &\equiv (b - i b_n)(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}) - \gamma^j\left(b\Sigma_X\Sigma_X' b_n' - \iota b_n\Sigma_X\Sigma_X' b_n'\right)\xi \\ G &\equiv -\beta^{\frac{1}{\gamma^j}} \\ H &\equiv (\gamma^j - 1)\Big[\gamma^j\Sigma_X\Sigma_X'\left(b' - \iota b_n'\right)\omega_1\left(b - \iota b_n\right) - I_n\Big]\Sigma_X\Sigma_X'\end{aligned}$$

In the text we use four treasury securities so that $b_{4\tau}$ is used for the generic b_n.

$G = 0$ when the trading desk does not consider periodic withdrawals from the portfolio.

The coefficients in the PDE (1) are in Table 1 for the parameters from the term structure model and the preference parameters $\gamma^j = 10, \beta = 0.05, \xi = 1$.

Table 1: Estimates of Parameters for PDE (1).

A	B	C	D	E	F	H
0.0313	0.2716	0.0119	57.5221	3.2240	0.0828	0

Sangvinatsos and Wachter (2005, p. 192 JF) guess the solution when the trading desk does not make periodic withdrawals between ALC meetings.

$$h(\tau, X) = \exp\left\{-\frac{1}{2}X'\mathcal{B}_3(\tau)X + \mathcal{B}_2(\tau)'X + \mathcal{B}_1(\tau)\right\}, \tag{4}$$

where $\tau = T - t$.

This may be written as

$$\begin{aligned} h(\tau, X) = \exp\Bigg\{ & -\frac{1}{2}\left(X - (\mathcal{B}_3(\tau))^{-1}\mathcal{B}_2(\tau)\right)'\mathcal{B}_3(\tau)\left(X - (\mathcal{B}_3(\tau))^{-1}\mathcal{B}_2(\tau)\right) \\ & + \mathcal{B}_1(\tau) + \frac{1}{2}\mathcal{B}_2(\tau)'(\mathcal{B}_3(\tau))^{-1}\mathcal{B}_2(\tau)\Bigg\} \end{aligned} \tag{5}$$

so that $(\mathcal{B}_3(\tau))^{-1}\mathcal{B}_2(\tau)$ is the mean and $\mathcal{B}_3(\tau)^{-1}$ is the variance of the expected utility of terminal wealth. We call these terms $\mu_J(\tau)$ and $\sigma_J(\tau)$ in equation (32) of the paper.

If one takes the derivatives of the guess and substitute into the linear PDE, then one gets the Ricatti ordinary differential equations. The quadratic form matrix satisfies the ODE

$$\frac{\partial\mathcal{B}_3(\tau)}{\partial\tau} = -\mathcal{B}_3(\tau)A\left[I - H\right]A'\mathcal{B}_3(\tau) - 2\mathcal{B}_3(\tau)B + D \tag{6}$$

subject to

$$\mathcal{B}_3(0) = \mathcal{B}_3.$$

The first line uses the symmetry of A so that $A' = A$. In addition, the matrix $\mathcal{B}_3(\tau)$ must be positive definite, which is true when $A\left[I - H\right]A'$ and D are positive definite.

The ODE for the linear coefficients is

$$\frac{\partial\mathcal{B}_2(\tau)'}{\partial\tau} = -\mathcal{B}_2(\tau)'A\left[I - H\right]A'\mathcal{B}_3(\tau) + C'\mathcal{B}_3(\tau) - \mathcal{B}_2(\tau)'B - E \tag{7}$$

subject to

$$\mathcal{B}_2(0) = \mathcal{B}_2.$$

The ODE for the constant coefficients yields

$$\mathcal{B}_1(\tau) = \mathcal{B}_1(0) + \frac{1}{2}\int_0^\tau \left[\mathcal{B}_2(s)' A\left[I - H\right] A' \mathcal{B}_2(s) - Trace\left(AA'\mathcal{B}_3(s)\right) - 2\mathcal{B}_2(s)'C - 2F\right] ds \tag{8}$$

with

$$\mathcal{B}_1(0) = \mathcal{B}_1.$$

These expressions are similar to Sangvinatsos and Wachter (2005, p. 222 and 223). The solutions to these ODEs under the affine term structure estimates are given in Table 2. The results are reported in Table 2 for $\gamma = 10$ and $\beta = 0.05$ at a one year time horizon. Figure 1 provides a graph for this solution versus the level of the yield curve in the left hand graph. The domain is plus and minus 3 standard deviation of the lifetime utility relative to its mean. The right hand graph has the hedging demand for 5 year bonds, and the total demand for 3 Months and 5 Year government bonds following (8).

Table 2: . Solution to the ODEs (6), (7), and (8)

$\mathcal{B}_1(1)$	$\mathcal{B}_2(1)$	$\mathcal{B}_3(1)$	$\left(\mathcal{B}_3(\tau)\right)^{-1}\mathcal{B}_2(\tau)$	$\exp\left(\mathcal{B}_1(\tau) + \frac{1}{2}\mathcal{B}_2(\tau)'\left(\mathcal{B}_3(\tau)\right)^{-1}\mathcal{B}_2(\tau)\right)$
-0.0770	-2.5239	43.7308	-0.0593	0.9757

Figure 1: . The Expected Lifetime Utility of the trading desk and Portfolio Weights

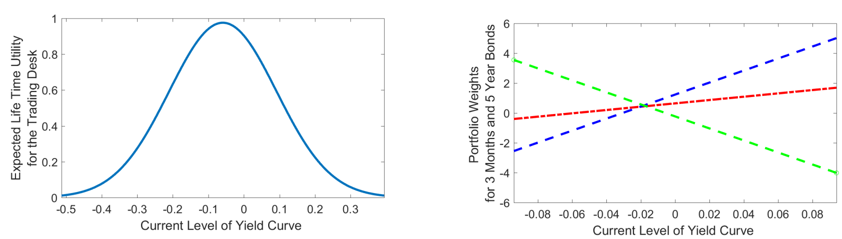


Table 11 in the paper comes from Table 2 and Figure 6 in the paper corresponds to 1.

We want to find the stochastic process for the lifetime utility given the solution to the PDE (1) for $h(X, t)$ and the optimal portfolio rule (33) in the paper.

First we find the stochastic process for the trading desk's utility from her bank capital, since the lifetime utility follows (32) in the paper.

$$J(\tau, K_M^j, X) = \frac{1}{1-\gamma^j} \left(K_M^j\right)^{1-\gamma^j} h^{\gamma^j}(\tau, X) \tag{9}$$

The stochastic process for the trading desk's utility from capital is

$$\frac{d\left(K_M^j\right)^{1-\gamma^j}}{\left(K_M^j\right)^{1-\gamma^j}} = \left(\mathcal{C}_1(\tau) - \frac{1}{2}\left(X(s)'\mathcal{C}_3(\tau)X(s) - 2\mathcal{C}_2(\tau)X(s)\right)\right)dt + \left(\mathcal{C}_4(\tau) + X(s)'\mathcal{C}_5(\tau)\right)d\epsilon_s. \tag{10}$$

$$\begin{aligned}
\mathcal{C}_1(\tau) \equiv& (1-\gamma^j)\Big\{\delta_0 + \xi b_n'(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}) - \frac{1}{2}\gamma^j\xi^2 b_{4\tau}\Sigma_X\Sigma_X' b_{4\tau}' + K'\omega_1 K_w + \frac{1}{2}\gamma^j K'\omega_1 K \\
&+ \gamma^j\left(K_w' - K'\right)\omega_1\left(b - \iota b_n\right)\Sigma_X\Sigma_X'\mathcal{B}_2(\tau) \\
&- \frac{1}{2}\left(\gamma^j\right)^2\mathcal{B}_2(\tau)'\Sigma_X\Sigma_X'\left(b' - \iota b_n'\right)\omega_1\left(b - \iota b_n\right)\Sigma_X\Sigma_X'\mathcal{B}_2(\tau)\Big\}, \\
\mathcal{C}_2(\tau) \equiv& (1-\gamma^j)\Big[\delta_1 - \xi b_n'(A^{\mathcal{P}} - A^{\mathcal{Q}}) - \left(K_w' + (1-\gamma^j)K'\right)\omega_1(b - \iota b_n)(A^{\mathcal{P}} - A^{\mathcal{Q}}) \\
&+ \gamma^j\left(K_w - K\right)\omega_1\left(b - \iota b_n\right)\Sigma_X\Sigma_X'\mathcal{B}_3(\tau) \\
&+ \gamma^j\mathcal{B}_2(\tau)'\Sigma_X\Sigma_X'(b' - \iota b_n')\omega_1\left(b - \iota b_n\right)\left(A^{\mathcal{P}} - A^{\mathcal{Q}}\right) \\
&- \left(\gamma^j\right)^2\mathcal{B}_2(\tau)'\Sigma_X\Sigma_X'\left(b' - \iota b_n'\right)\omega_1\left(b - \iota b_n\right)\Sigma_X\Sigma_X'\mathcal{B}_3(\tau)\Big], \\
\mathcal{C}_3(\tau) \equiv& (1-\gamma^j)\Big[\gamma^j(A^{\mathcal{P}} - A^{\mathcal{Q}})'(b' - \iota b_n')\omega_1(b - \iota b_n)(A^{\mathcal{P}} - A^{\mathcal{Q}}) \\
&- 2\gamma^j(A^{\mathcal{P}} - A^{\mathcal{Q}})(b' - \iota b_n')\omega_1\left(b - \iota b_n\right)\Sigma_X\Sigma_X'\mathcal{B}_3(\tau) \\
&+ \left(\gamma^j\right)^2\mathcal{B}_3(\tau)'\Sigma_X\Sigma_X'\left(b' - \iota b_n'\right)\omega_1\left(b - \iota b_n\right)\Sigma_X\Sigma_X'\mathcal{B}_3(\tau)\Big], \\
\mathcal{C}_4(\tau) \equiv& (1-\gamma^j)\Big[K + \gamma^j\mathcal{B}_2(\tau)'\Sigma_X\Sigma_X'\left(b' - \iota b_n'\right)\Big]\omega_1(b - \iota b_n)\Sigma_X + (1-\gamma^j)\xi b_n\Sigma_X \\
\mathcal{C}_5(\tau) \equiv& (1-\gamma^j)\Big[-(A^{\mathcal{P}} - A^{\mathcal{Q}})'(b' - \iota b_n') + \gamma^j\mathcal{B}_3(\tau)'\Sigma_X\Sigma_X'\left(b' - \iota b_n'\right)\Big]\omega_1(b - \iota b)\Sigma_X, \\
&K_w \equiv (b' - ib_n')(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}) - \gamma^j\left(b\Sigma_X\Sigma_X' b_n' - \iota b_n\Sigma_X\Sigma_X' b_n'\right)'\xi.
\end{aligned} \tag{11}$$

Table 3 gives the values of the coefficients $\mathcal{C}_i(\tau)$ for $\tau = 1$ given by (11). In this case the discounted future wealth is positively related to the future factor for low values of the level factor, i.e. $X_s < -\left(\mathcal{C}_3(\tau)\right)^{-1}\mathcal{C}_2(\tau) < 0$.

Table 3: Estimates of Parameters for Equation (11) with $\gamma^j = 0$.

$\mathcal{C}_1(\tau)$	$\mathcal{C}_2(\tau)$	$\mathcal{C}_3(\tau)$	$\mathcal{C}_4(\tau)$	$\mathcal{C}_5(\tau)$
0.0725	-11.5339	244.3535	-0.1512	-2.2074

By using Ito's Lemma again one finds:

$$
\begin{aligned}
J(w,X,\tau) = J(w,X,0)\exp\Bigg\{ &\int_0^\tau \Big(\mathcal{J}_1(0) - \frac{1}{2}\Big(X(s)'\mathcal{J}_3(0)X(s) - 2\mathcal{J}_2(0)\Big)\Big)ds \\
- &\int_0^\tau \Big(\mathcal{J}_4(0) + X(s)'\mathcal{J}_5(0)\Big)d\epsilon_s\Bigg\}.
\end{aligned}
\tag{12}
$$

The coefficients

$$
\begin{aligned}
\mathcal{J}_1(0) \equiv &\mathcal{C}_1(0) - \frac{\gamma^j(1-\gamma^j)}{2}\mathcal{C}_4(0)\mathcal{C}_4(0)' + \gamma^j F + \frac{\gamma^j}{2}\mathcal{B}_2(\tau)'\left[\left(\gamma^j+1\right)\Sigma_X\Sigma_X' + H\right]\mathcal{B}_2(\tau) \\
&- \gamma^j\mathcal{B}_2(\tau)'\left((1-\gamma^j)\Sigma_X\mathcal{C}_4(0)' + 2C - \gamma^{\mathcal{P}}\right) \\
\mathcal{J}_2(0) \equiv &\mathcal{C}_2(0) - \gamma^j(1-\gamma^j)\mathcal{C}_4(0)\mathcal{C}_5'(0) + \gamma^j E + \gamma^j\mathcal{B}_2(\tau)'\left[\left(\gamma^j+1\right)\Sigma_X\Sigma_X' + H\right]\mathcal{B}_3(\tau) \\
&- \gamma^j\mathcal{B}_2(\tau)'\left((1-\gamma^j)\Sigma_X\mathcal{C}_5(0)' - 2B + A^{\mathcal{P}}\right) \\
\mathcal{J}_3(0) \equiv &\mathcal{C}_3(0) + \gamma^j(1-\gamma^j)\mathcal{C}_5(0)\mathcal{C}_5'(0) - 2\gamma^j D + \gamma^j\mathcal{B}_3(\tau)'\left[\left(\gamma^j+1\right)\Sigma_X\Sigma_X' + H\right]\mathcal{B}_3(\tau) \\
&- 2\gamma^j\mathcal{B}_3(\tau)'\left((1-\gamma^j)\Sigma_X\mathcal{C}_5(0)' - 2B + A^{\mathcal{P}}\right), \\
\mathcal{J}_4(0) \equiv &\mathcal{C}_4(0) + \gamma^j\mathcal{B}_2(\tau)', \text{ and} \\
\mathcal{J}_5(0) \equiv &\mathcal{C}_5(0) + \gamma^j\mathcal{B}_3(\tau)'.
\end{aligned}
\tag{13}
$$

We want $J(w,X,\tau)$ to be a uniformly integrable martingale. We recognize that it is a stochastic exponential (Doléans-Dade exponential). See Protter (pp. 84-89). In our case, we have a continuous stochastic process for the factor. As a result, we have

$$\mathcal{E}(X_t) = exp\left\{X_t - \frac{1}{2}[X,X]_t\right\},$$

where $[X,X]_t$ is the quadratic variation of $J(w,X,\tau)$.

Theorem 45 of Protter (2005, p.141) demonstrates $J(w,X,\tau)$ to be a uniformly integrable martingale as long as

$$E\left[exp\left\{\frac{1}{2}[X,X]_t\right\}\right] < \infty.$$

In this case, the quadratic variation includes all the terms associated with the variance-covariance matrix $\Sigma_X \Sigma_X'$. In this case the quadratic variation is

$$E\left\{ \exp\left[\left(\mathcal{J}_4(0) + X(s)'\mathcal{J}_5(0) \right)' \left(\mathcal{J}_4(0) + X(s)'\mathcal{J}_5(0) \right) \right] \right\} < \infty. \tag{14}$$

This is called the Novikov's Criterion. Below we show these expectations are bounded for the investor's problem.

If this is true, then the lifetime utility of the investor is given by

$$J(w, X, \tau) = J(w, X, 0) E_t \left[\exp\left\{ \int_0^\tau \left(\mathcal{J}_1(0) - \frac{1}{2} \left(X(s)'\mathcal{J}_3(0)X(s) - 2\mathcal{J}_2(0) \right) \right) ds \right\} \right]. \tag{15}$$

To solve the COO's problem (42) in the paper, we need to find the probability distribution of expressions like (12). In the next section this is accomplished by finding the solution to the stochastic process (3) in the paper. This allows us to separate formulas like (12) between an expected and random part. We then use the Komogorov Forward equation to find the explicit formula for the probability distribution for the unanticipated component.

2 Probability Distribution for Exponential Functions of an Ornstein-Uhlembeck Process

In this section we find the probability distribution for terms like (12), so that we can evaluate the value at risk and the call option value of capital. As shown in the previous section the marginal value of capital is related to the interest rate factors. These factors follow the Ornstein-Uhlembeck process (3) in the paper.

$$dX(s) = \left(\gamma^{\mathcal{P}} - A^{\mathcal{P}} X(s) \right) ds + \Sigma_X d\epsilon_s. \tag{16}$$

Following Arnold (1974) Theorem 8.2.2, the fundamental solution is

$$\Phi(s) = e^{-A^{\mathcal{P}}(s-t)}.$$

The solution to (16) is

$$X(\tau) = e^{-A^{\mathcal{P}}(\tau-t)} X(t) + \left(I - e^{-A^{\mathcal{P}}(\tau-t)} \right) \left(A^{\mathcal{P}} \right)^{-1} \gamma^{\mathcal{P}} + \int_t^{t+\tau} e^{-A^{\mathcal{P}}(\tau-v)} \Sigma_X d\epsilon_v. \tag{17}$$

Here $\tau > t$.

Following Arnold (1974) Theorem 8.2.12 the integral

$$Y_\tau = \int_t^{t+\tau} e^{-A^{\mathcal{P}}(\tau-\upsilon)} \Sigma_X d\epsilon_\upsilon \sim N(Y; 0, K(\tau)). \tag{18}$$

Here, $N(Y; 0, K(\tau))$ represents a normal distribution with mean zero.

Its variance-covariance matrix is given by

$$K(\tau) = \int_t^{t+\tau} e^{-A^{\mathcal{P}}(\tau-\upsilon)} \Sigma_X \Sigma_X' e^{-A^{\mathcal{P}\prime}(\tau-\upsilon)} d\upsilon.$$

By exercise (1.2.11) of Hijab (1987)

$$K(\tau) = K_\infty - e^{-A^{\mathcal{P}}\tau} K_\infty e^{-A^{\mathcal{P}\prime}\tau}.$$

Here, the matrix K_∞ solves the Lyapunov equation

$$-A^{\mathcal{P}} K_\infty - K_\infty A^{\mathcal{P}\prime} = \Sigma_X \Sigma_X'.$$

As the time horizon tends to infinity, $K(\tau) \to K_\infty$. The solution to this equation is a positive definite symmetric matrix, which is easily calculated using lyap.m in Matlab.

We have encountered several stochastic processes for lifetime utility and the trading desk's capital stock. They all have the form

$$Z_t = \exp\left\{ -\frac{1}{2} \int_t^T \left[X_s' \mathcal{D}_3(s) X_s - 2\mathcal{D}_2(s) X_s \right] ds + \int_t^T \left(\mathcal{D}_4(s) + X_s' \mathcal{D}_5(s) \right) d\epsilon_s \right\}. \tag{19}$$

In particular, see (10), and (12) in which $\mathcal{D}_i(s)$ are replaced by $\mathcal{C}_i(s)$ and $\mathcal{J}_i(s)$ for $i = 1, 2, 3, 4, 5$, respectively. We use the notation X_s rather than $X(s)$, used in the text, to indicate that X is a stochastic process. In addition, the calculations are for a given terminal time T.

For this stochastic process to have a solution, the Novikov condition (14) must be satisfied. In this case, the quadratic variation is dependent on the convergence of the stochastic process for X_s. Its solution is given by (17). The deterministic part of this solution is convergent, as long as $A^{\mathcal{P}}$ has all positive roots. The stochastic part Y includes all the terms associated with the variance-covariance matrix which is bounded by

$$K(\tau) = K_\infty - e^{-A^{\mathcal{P}}\tau} K e^{-A^{\mathcal{P}\prime}\tau} \leq K \text{ with } \tau = T - t.$$

This together with the convergence of the solution X_s (17) assures the quadratic variation (14) exists.

We will now explain how the Backward and Forward Kolmogorov Equations apply to our problem. We then find the solution to these Kolmogorov equations.

2.1 The Backward Kolmogorov Equation

To solve for the expectation of the stochastic process (19) we use the backward Kolmogorov equation. We represent the transition probability from state X at time t to the state Y at time T by $p(t, X, T, Y)$. Subsequently, we will derive the transition probability using the forward Kolmogorov equation. In the text X is the vector of interest rate factors at the current time and Y is the random component of these factors at time T given by (18).

We now consider the conditional expectation of (19). As long as the Novikov's Criterion (14) holds, the conditional expectation of (19) is

$$\begin{aligned} f(t,X) = \int_{\mathbb{R}^N} \exp\left\{ -\frac{1}{2}\int_t^T \left[X_s'\mathcal{D}_3(s)X_s - 2\mathcal{D}_2(s)X_s \right] ds \right\} \\ \times f(T,Y)p(t,X,T,Y)dY. \end{aligned} \tag{20}$$

We will show $f(t, X)$ for any $t \in [0, T]$ is the solution to the backward Kolmogorov equation

$$\begin{aligned} &\frac{\partial f(t,X)}{\partial t} - \frac{1}{2}\left(X'\mathcal{D}_3(t)X - 2\mathcal{D}_2(t)X\right) f(t,X) \\ &+ \left(\frac{\partial f(t,X)}{\partial X}\right)' \left(\gamma^{\mathcal{P}} - A^{\mathcal{P}}X\right) + \frac{1}{2}Trace\left(\Sigma_X\Sigma_X' \frac{\partial^2 f(t,X)}{\partial X \partial X}\right) = 0 \end{aligned} \tag{21}$$

under the stochastic process (16).[1] We will be using in the subsequent argument the operator $\mathcal{K}_X$ defined by

$$\mathcal{K}_X \equiv \left(\frac{\partial}{\partial X}\right)' \left(\gamma^{\mathcal{P}} - A^{\mathcal{P}}X\right) + \frac{1}{2}Trace\left(\Sigma_X\Sigma_X' \frac{\partial^2}{\partial X \partial X}\right) \tag{22}$$

so that

$$\frac{\partial f(t,X)}{\partial t} - \frac{1}{2}\left(X'\mathcal{D}_3(t)X - 2\mathcal{D}_2(t)X\right) f(t,X) + \mathcal{K}_X f(t,X). \tag{23}$$

The Kolmogorov backward PDE is solved subject to the terminal condition

$$\lim_{t \uparrow T} f(t,X) = f(X), \quad X \in \mathbb{R}^N. \tag{24}$$

[1] This is a variation on the argument for Theorem 8.4.1 of Calin *et. al.* Also see Duffee (1992) Appendix E, and Karatzas and Shreve (1988, pp. 366-369).

Proof. Define the integrating factor

$$\phi(t,s) = \exp\left\{ -\frac{1}{2}\int_t^s \left[X_v'\mathcal{D}_3(v)X_v - 2\mathcal{D}_2(v)X_v \right] dv \right\}.$$

Let

$$Y_s = \phi(t,s)f(s,X_s) \quad s \in [t,T]$$

which is a function of the solution to the stochastic differential equation for X (17). As a result, we can apply Theorem 6.3.1 of Shreve (2006). For a Borel measurable function $h(y)$ on $t \in [0,T]$, we have

$$E\left[h(X(T)) \mid \mathcal{F}(t)\right] = g(t, X(t)).$$

Under these conditions, Lemma 6.4.2 of Shreve (2006), the stochastic process $g(t, X(t))$ is a martingale. Now introduce the discount process

$$D(t) = \phi(0,t).$$

Define

$$Y(t,X) - E\left[\phi(t,T)h(X(T)) \mid \mathcal{F}(t)\right],$$

then

$$Y(t,X) = \phi(0,t)f(t,X)$$

is a martingale and satisfies the PDE (23). However, $f(t,X)$ is not a martingale.

To see the reason for the PDE (23), apply Ito's lemma to Y_s under the stochastic process (16) to yield

$$\begin{aligned} dY_s = &-\frac{1}{2}\left[X_s'\mathcal{D}_3(s)X_s - 2\mathcal{D}_2(s)X_s \right]\phi(t,s)f(s,X_s)ds + \phi(t,s)\frac{\partial f(s,X_s)}{\partial s}ds \\ &+ \phi(t,s)\left(\frac{\partial f(s,X_s)}{\partial X}\right)'\left(\gamma^{\mathcal{P}} - A^{\mathcal{P}}X_S\right)ds + \frac{1}{2}\phi(t,s)Trace\left(\Sigma_X\Sigma_X'\frac{\partial^2 f(s,X_S)}{\partial X\partial X}\right)ds \\ &+ \phi(t,s)\left(\frac{\partial f(s,X_S)}{\partial X}\right)'\Sigma_X d\epsilon_s \end{aligned}$$

For Y_s to be a martingale the drift term must be zero. This property is satisfied by the PDE (23).

Since Y_s is a martingale we can integrate from t to T

$$\begin{aligned} \phi(t,T)f(T,X_T) - \phi(t,t)f(t,X_t) = &\int_t^T \phi(t,s)\left[\frac{\partial f(s,X_s)}{\partial s} - \frac{1}{2}\left(X_s'\mathcal{D}_3(s)X_s - 2\mathcal{D}_2(s)X_s\right)\right. \\ &\left. \times f(s,X_s) + \mathcal{K}_{X_s}f(s,X_s)\right]ds + \int_t^T \phi(t,s)\left(\frac{\partial f(s,X_s)}{\partial X_s}\right)'\Sigma_X d\epsilon_s \end{aligned}$$

We impose (21) subject to the terminal condition (24). In addition we can use the martingale property to take expectations, since Novikov's Criterion (14) is true.

$$f(t,X(t)) = E_t\Big[\phi(t,T)f(Y)\Big] + E_t\Big[\phi(t,s)\left(\frac{\partial f(s,X_S)}{\partial X}\right)'\Sigma_X d\epsilon_s\Big]$$

The second term is zero which leads to the result: Thus, solving the backward Kolmogorov equation (21) for $f(t,X)$ yields the expectation (20). ∎

2.2 Solving the Backward Kolmogorov Equation.

We set the terminal condition for the backward Kolmogorov equation

$$f(X) = \exp\left\{\frac{1}{2}X'\mathcal{D}_3 X + \mathcal{D}_2 X + \mathcal{D}_1\right\},$$

where $\mathcal{D}_i$ are constants for the terminal condition.

Guess the solution of (21) has the form

$$f(t,X) = \exp\left\{-\frac{1}{2}\Big[X'\mathcal{F}_3(t)X - 2\mathcal{F}_2(t)X + \mathcal{F}_1(t)\Big]\right\}, \tag{25}$$

$$\frac{\partial f(t,X)}{\partial X} = f(t,X)\left[-\mathcal{F}_3(t)X + \mathcal{F}_2(\tau)'\right].$$

$$\frac{\partial^2 f(t,X)}{\partial X \partial X} = f(t,X)\Big(\mathcal{F}_3(t)XX'\mathcal{F}_3(t) - 2\mathcal{F}_3(\tau)X\mathcal{F}_2(t) + \mathcal{F}_2(t)'\mathcal{F}_2(\tau) - \mathcal{F}_3(\tau)\Big).$$

$$\frac{\partial f(t,X)}{\partial t} = f(\tau,X)\left[-\frac{1}{2}X'\frac{\partial \mathcal{F}_3(t)}{\partial t}X + \frac{\partial \mathcal{F}_2(t)}{\partial t}X - \frac{1}{2}\frac{\partial \mathcal{F}_1(t)}{\partial t}\right].$$

Now substitute these results into the Kolmogorov backward equation (21).

$$\begin{aligned}
&\left[-\frac{1}{2}X'\frac{\partial \mathcal{F}_3(t)}{\partial t}X + \frac{\partial \mathcal{F}_2(t)}{\partial t}X - \frac{1}{2}\frac{\partial \mathcal{F}_1(t)}{\partial t}\right] - \frac{1}{2}\left(X'\mathcal{D}_3(t)X - 2\mathcal{D}_2(t)X\right) \\
&+ \left[-X'\mathcal{F}_3(t) + \mathcal{F}_2(\tau)\right]\left(\gamma^{\mathcal{P}} - A^{\mathcal{P}}X\right) \\
&+ \frac{1}{2}Trace\left(\Sigma_X\Sigma_X'\Big(\mathcal{F}_3(t)XX'\mathcal{F}_3(t) - 2\mathcal{F}_3(\tau)X\mathcal{F}_2(t) + \mathcal{F}_2(t)'\mathcal{F}_2(\tau) - \mathcal{F}_3(\tau)\Big)\right) = 0
\end{aligned}$$

$$\begin{aligned}
&\left[-\frac{1}{2}X'\frac{\partial \mathcal{F}_3(t)}{\partial t}X + \frac{\partial \mathcal{F}_2(t)}{\partial t}X - \frac{1}{2}\frac{\partial \mathcal{F}_1(t)}{\partial t}\right] - \frac{1}{2}X'\mathcal{D}_3(t)X + \mathcal{D}_2(t)X \\
&- X'\mathcal{F}_3(t)\gamma^{\mathcal{P}} + X'\mathcal{F}_3(t)A^{\mathcal{P}}X + \mathcal{F}_2(\tau)\gamma^{\mathcal{P}} - \mathcal{F}_2(\tau)A^{\mathcal{P}}X + \frac{1}{2}X'\mathcal{F}_3(t)\Sigma_X\Sigma_X'\mathcal{F}_3(t)X \\
&- \mathcal{F}_2(t)\Sigma_X\Sigma_X'\mathcal{F}_3(t)X + \frac{1}{2}\mathcal{F}_2(t)\Sigma_X\Sigma_X'\mathcal{F}_2(t)' - \frac{1}{2}Trace\left(\Sigma_X\Sigma_X'\mathcal{F}_3(t)\right) = 0
\end{aligned}$$

Now equate quadratic, linear, and constant terms to obtain three ODEs.

$$\frac{\partial \mathcal{F}_3(t)}{\partial t} = \mathcal{F}_3(t)\Sigma_X\Sigma_X'\mathcal{F}_3(t) - \mathcal{D}_3(t) + 2\mathcal{F}_3(t)A^{\mathcal{P}} \tag{26}$$

subject to

$$\mathcal{F}_3(0) = \mathcal{D}_3.$$

This is the Lyapunov equation.

$$\frac{\partial \mathcal{F}_2(t)}{\partial t} = \mathcal{F}_2(t)\left(\Sigma_X\Sigma_X'\mathcal{F}_3(t) + A^{\mathcal{P}}\right) - \mathcal{D}_2(t) + \gamma^{\mathcal{P}\prime}\mathcal{F}_3(t) \tag{27}$$

subject to

$$\mathcal{F}_2(0) = \mathcal{D}_2.$$

This ODE is linear so that we can use integrating factor to solve for $\mathcal{F}_2(t)$. The Final ODE is

$$\frac{\partial \mathcal{F}_1(t)}{\partial t} = 2\mathcal{F}_2(\tau)\gamma^{\mathcal{P}} + \mathcal{F}_2(t)\Sigma_X\Sigma_X'\mathcal{F}_2(t)' - Trace\left(\Sigma_X\Sigma_X'\mathcal{F}_3(t)\right) \tag{28}$$

subject to

$$\mathcal{F}_1(0) = \mathcal{D}_1.$$

This initial value problem is the simplest since everything on the right hand side of the ODE is known.

2.3 The Forward Kolmogorov Equation

Following Karatzas and Shreve (1988) the solution to the backward Kolmogorov equation (21) $f(t, X)$ for fixed (T, Y) is

$$f(t, X) \equiv p(t, X, T, Y). \tag{29}$$

In addition, for fixed (t, X) the function

$$g(\tau, Y) \equiv \phi(t, \tau)p(t, X, \tau, Y) \tag{30}$$

solves the forward Kolmogorov equation.[2]

$$\frac{\partial g(\tau, Y)}{\partial \tau} = \mathcal{K}_Y^* g(\tau, Y) - \frac{1}{2}\left(Y'\mathcal{D}_3(\tau)Y - 2\mathcal{D}_2(\tau)Y\right)g(\tau, Y). \tag{31}$$

[2]See Karatzas and Shreve (1988, p. 369) equation (7.24). Also see Theorem 8.7.1. of Calin *et. al* (2011), and Chirikjian (2009, p.118-121)

Here, the dual of $\mathcal{K}_X$ given by[3]

$$\begin{aligned}\mathcal{K}_X^* &= -\sum_{i=1}^{N}\frac{\partial}{\partial X_i}\left(\gamma^{\mathcal{P}} - A^{\mathcal{P}}X\right)_i + \frac{1}{2}\sum_{i,j=1}^{N}\frac{\partial^2}{\partial X_i \partial X_j}\Sigma_{ik}\Sigma'_{kj} \\ &= -\gamma^{\mathcal{P}\prime}\frac{\partial}{\partial X} + X'A^{\mathcal{P}\prime}\frac{\partial}{\partial X} + Trace(A^{\mathcal{P}}) + \frac{1}{2}Trace\left(\Sigma\Sigma'\frac{\partial^2}{\partial X\partial X}\right). \end{aligned} \tag{32}$$

Remark: Notice that only the distribution of the factors enters (32). The preferences of the investor only influences the discount factor $\phi(t,\tau)$ in (31).

To find the initial condition, let the Dirac distribution centered at $X \in \mathbb{R}^N$ be $f(X) = \delta_X$ such that

$$\delta_X(\theta) = \int_{\mathbb{R}^N}\delta_x(Y)\theta(Y)dY = \theta(X).$$

For a given $X_t = X \in \mathbb{R}^N$,

$$g(\tau, X) = \int_{\mathbb{R}^N}\delta_X(Y)\phi(t,\tau)p(t,X,\tau,Y)dY = \phi(t,\tau)p(t,X,\tau,X).$$

Consequently, if the initial condition for the Kolmogorov forward equation (31) is

$$\lim_{\tau\to 0^+} g(\tau, X(\tau)) = \delta_X, \tag{33}$$

then the solution to (31) is $\phi(t,\tau)p(t,X,\tau,Y) = g(\tau,Y)$.

Thus, we have

Theorem 2.1. *The discounted transition probability $\phi(t,\tau)p(t,X,\tau,Y)$ for a given $X_t = X \in \mathbb{R}^N$ is the solution to the Kolmogorov Forward equation (31) with (32) subject to the initial condition (33).*

Proof. We will use the property of the dual for the Kolmogorov operator, $\mathcal{K}_Y$ given by

$$\int_{\mathbb{R}^N}\mathcal{K}_Y f(Y)g(Y)dY = \int_{\mathbb{R}^N} f(Y)\mathcal{K}_Y^* g(Y)dY. \tag{34}$$

[3] See Øksendal (2005, p. 169). Also follow the derivation in Chirikjian (2009, p. 121)

We know from (20) that

$$
\begin{aligned}
f(t,X) =& \int_{\mathbb{R}^N} \exp\left\{ -\frac{1}{2}\int_t^T \left[X_s'\mathcal{D}_3(s)X_s - 2\mathcal{D}_2(s)X_s \right] ds \right\} \\
& \times f(Y)p(t,X,T,Y)dY \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \exp\left\{ -\frac{1}{2}\int_t^\tau \left[X_s'\mathcal{D}_3(s)X_s - 2\mathcal{D}_2(s)X_s \right] ds \right\} \\
& \exp\left\{ -\frac{1}{2}\int_\tau^T \left[X_s'\mathcal{D}_3(s)X_s - 2\mathcal{D}_2(s)X_s \right] ds \right\} \\
& f(Y)p(t,X,\tau,Z)p(\tau,Z,T,Y)dZdY \\
&= \int_{\mathbb{R}^N} \phi(t,\tau)f(\tau,Z)p(t,X,\tau,Z)dZ
\end{aligned}
$$

The next to last step uses the Chapman-Kolmogorov equation for a Markov process[4] and the last step uses the definition of $f(t,X)$. As a result, we know for any $t<\tau\leq T$

$$
f(t,X) = \int_{\mathbb{R}^N} f(\tau,Y)\phi(t,\tau)p(t,X,\tau,Y)dY. \tag{35}
$$

Next differentiate in τ

$$
\begin{aligned}
0 = \frac{\partial f(t,X)}{\partial \tau} &= \int_{\mathbb{R}^N} \frac{\partial f(\tau,Y)}{\partial \tau}\phi(t,\tau)p(t,X,\tau,Y)dY + \int_{\mathbb{R}^N} f(\tau,Y)\frac{\partial \phi(t,\tau)p(t,X,\tau,Y)}{\partial\tau}dY \\
&= \int_{\mathbb{R}^N} f(\tau,Y)\frac{\partial \phi(t,\tau)p(t,X,\tau,Y)}{\partial\tau}dY - \int_{\mathbb{R}^N} \mathcal{K}_Y f(\tau,Y)\phi(t,\tau)p(t,X,\tau,Y)dY \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} \left(Y'\mathcal{D}_3(\tau)Y - 2\mathcal{D}_2(\tau)Y\right) f(\tau,X)\phi(t,\tau)p(t,X,\tau,Y)dY
\end{aligned} \tag{36}
$$

The second step uses the backward Kolmogorov equation (21).

Now apply the property (34) to find

$$
\begin{aligned}
0 = \int_{\mathbb{R}^N} f(\tau,Y)\Bigg[& \frac{\partial \phi(t,\tau)p(t,X,\tau,Y)}{\partial\tau} - \mathcal{K}_Y^*(\phi(t,\tau)p(t,X,\tau,Y)) \\
& + \frac{1}{2}\left(Y'\mathcal{D}_3(\tau)Y - 2\mathcal{D}_2(\tau)Y\right)\phi(t,\tau)p(t,X,\tau,Y)\Bigg]dY
\end{aligned}
$$

This means we want to define $g(\tau,Y)=\phi(t,\tau)p(t,X,\tau,Y)$ for (31).to hold.

■

[4]See Chirikjian (2009, p. 108) equation (4.16).

2.4 Solving the Forward Kolmogorov Equation

It is difficult to impose the initial condition (33), since there is no explicit form for it. However, the Fourier transform of δ_X is 1. As a result, we will take the Fourier transform of the Kolmogorov equation (31) and find its solution. We will then apply the inverse Fourier transform to find the solution to the Kolmogorov forward equation given the initial condition.

Suppose that $f(X) \in \mathcal{S}(\mathbb{R}^N)$, on $\mathbb{R}^N$. This functional space refers to all functions which rapidly decrease, so that $f(X)$ is absolutely integrable over $\mathbb{R}^N$.[5] This allows one to move between Fourier transforms and its inverse. The Fourier transform of $f(X)$ is

$$F[f(X)] = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(X) e^{-i\xi' X}\, dX.$$

Here $\xi \in \mathbb{R}^N$ and $\xi \cdot X \equiv \xi' X = \xi_1 X_1 + \cdots + \xi_N X_N$.

The inverse Fourier transform of $\hat{f}(\xi)$ is

$$F^{-1}[\hat{f}(\xi)] = f(X) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi\cdot X}\, d\xi.$$

If the Fourier transforms of $f(X)$ exists, then

$$\begin{aligned} F_X\left[\frac{\partial f(X)}{\partial X_j}\right] =& i\xi_j F_X[f(X)] \Rightarrow F_X\left[\frac{\partial f(X)}{\partial X}\right] = i\xi F_X[f(X)]. \\ F_X\left[\frac{\partial^2 f(X)}{\partial X_j \partial X_k}\right] =& -\xi_j\xi_k F_X\left[f(X)\right] \Rightarrow F_X\left[\frac{\partial^2 f(X)}{\partial X \partial X}\right] = -\xi\xi' F_X[f(X)]. \end{aligned} \tag{37}$$

The subscript X is added to keep track of the integration over X not t.

$$F_X[-iXf(X)] = \frac{\partial \hat{f}(\xi)}{\partial \xi} \Rightarrow F_X[Xf(X)] = i\frac{\partial \hat{f}(\xi)}{\partial \xi}. \tag{38}$$

Proof: $\displaystyle \frac{\partial \hat{f}(\xi)}{\partial \xi_j} = \frac{\partial}{\partial \xi_j}\int_{-\infty}^{\infty} f(X)e^{-i\xi\cdot X}\, dX = \int_{-\infty}^{\infty} -iX_j f(X) e^{-i\xi\cdot X}\, dX = F_X[-iX_j f(X)].$

$\displaystyle \Rightarrow F_X[-iXf(X)] = \frac{\partial F_X\left[f(X)\right]}{\partial \xi}.$

$$\begin{aligned} F_X\left[\left(\frac{\partial f}{\partial X}\right)' A^{\mathcal{P}} X\right] &= Trace\left(A^{\mathcal{P}} F_X\left[X\left(\frac{\partial f}{\partial X}\right)'\right]\right) = iTrace\left(A^{\mathcal{P}} \frac{\partial F_X\left[\left(\frac{\partial f}{\partial X}\right)'\right]}{\partial \xi}\right) \\ &= i^2 Trace\left(A^{\mathcal{P}} \frac{\partial \xi' F_X[f(X)]}{\partial \xi}\right) = -Trace\left(A^{\mathcal{P}} \frac{\partial F_X[f(X)]}{\partial \xi}\xi' + A^{\mathcal{P}} F_X[f(X)]\right). \end{aligned}$$

[5] These results from Alex Himonas's Topics in PDE notes. Also see Evans (2002, pp. 182-186).

The first result applies the Trace to a quadratic form. The second step uses (38) for the function $\left(\frac{\partial f}{\partial X}\right)'$. In the third equality we use the first result in (37). Finally, we use the product rule of differentiation and $i^2 = -1$.

We also have to consider $F_X[X'Xf(X)]$.

$$F_X[X'Xf(X)] = \frac{\partial^2 \hat{f}(\xi)}{\partial\xi\partial\xi}$$

$$\text{Proof: } \frac{\partial \hat{f}(\xi)}{\partial\xi_j\partial\xi_k} = \frac{\partial}{\partial\xi_k}\int_{-\infty}^{\infty} -iX_j f(X)e^{-i\xi\cdot X}\,dX = \int_{-\infty}^{\infty} iX_k iX_j f(X)e^{-i\xi\cdot X}\,dX = F_X[-X_kX_jf(X)].$$

$$\Rightarrow F_X[-XX'f(X)] = \frac{\partial^2 F_X\left[f(X)\right]}{\partial\xi\partial\xi}.$$

Notice

$$\begin{aligned} F_X\left[X'\mathcal{D}_3(\tau)Xf(\tau,X)\right] =& F_X\left[Trace\left(X'\mathcal{D}_3(\tau)X\right)f(\tau,X)\right] = F_X\left[Trace\left(\mathcal{D}_3(\tau)XX'f(\tau,X)\right)\right] \\ &= Trace\left(F_X\left[\mathcal{D}_3(\tau)XX'f(\tau,X)\right]\right) = Trace\left(\mathcal{D}_3(\tau)\frac{\partial^2\hat{f}(\xi)}{\partial\xi\partial\xi}\right) \end{aligned}$$

The first step is true since $X'\mathcal{D}_3(\tau)X \in \mathbb{R}$. The second step uses the property $Trace\,(ABC) = Trace\,(BCA)$. The third step takes advantage of the trace being a linear operator so that the additive property of integrals can be used. Since $X'X$ is symmetric the last step uses the last property of Fourier transforms.

Recall the Kolmogorov forward equation

$$\begin{aligned} \frac{\partial g(\tau,Y)}{\partial t} =& -\gamma^{\mathcal{P}\prime}\frac{\partial g(\tau,Y)}{\partial Y} + \left(\frac{\partial g(\tau,Y)}{\partial Y}\right)' A^{\mathcal{P}}Y + Trace\left(A^{\mathcal{P}}\right)g(\tau,Y) \\ &+ \frac{1}{2}Trace\left(\Sigma\Sigma'\frac{\partial^2 g(\tau,Y)}{\partial Y\partial Y}\right) - \frac{1}{2}\left(Y'\mathcal{D}_3(\tau)Y - 2\mathcal{D}_2(\tau)Y\right)g(\tau,Y). \end{aligned} \tag{39}$$

subject to the initial condition

$$g(0,Y_0) = \delta_Y.$$

Apply the Fourier transform to the forward Kolmogorov equation.

$$\begin{aligned} \frac{\partial F_Y\left[g(\tau,Y)\right]}{\partial\tau} =& -\gamma^{\mathcal{P}\prime}F_Y\left[\frac{\partial g(\tau,Y)}{\partial Y}\right] + F_Y\left[\left(\frac{\partial g(\tau,Y)}{\partial Y}\right)' A^{\mathcal{P}}Y\right] \\ &+ Trace(A^{\mathcal{P}})F_Y\left[g(\tau,Y)\right] + \frac{1}{2}Trace\left(\Sigma\Sigma'F_Y\left[\frac{\partial^2 g(\tau,Y)}{\partial Y\partial Y}\right]\right) \\ &- \frac{1}{2}F_Y\left[\left(Y'\mathcal{D}_3(\tau)Y - 2\mathcal{D}_2(\tau)Y\right)g(\tau,Y)\right] \end{aligned} \tag{40}$$

subject to the initial condition

$$F_Y\left[g(0,Y_0)\right]=1.$$

Next use the rules for Fourier transform to obtain

$$\begin{aligned}\frac{\partial F_Y\left[g(\tau,Y)\right]}{\partial\tau} &= -i\gamma^{\mathcal{P}\prime}\xi F_Y\left[g(\tau,Y)\right] - Trace\left(A^{\mathcal{P}}\frac{\partial F_Y[g(\tau,Y)]}{\partial\xi}\xi' + A^{\mathcal{P}}F_Y[g(\tau,Y)]\right)\\ &+ Trace(A^{\mathcal{P}})F_Y\left[g(\tau,Y)\right] - \frac{1}{2}Trace\left(\Sigma\Sigma'\xi\xi' F_Y[g(\tau,Y)]\right) - \frac{1}{2}Trace\left(\mathcal{D}_3(\tau)\frac{\partial^2\hat{g}(\xi)}{\partial\xi\partial\xi}\right)\\ &+ i\left(\frac{\partial F_Y[g(\tau,Y)]}{\partial\xi}\right)'\mathcal{D}_2(t,\tau)'\end{aligned}$$

$$\begin{aligned}\Rightarrow&\frac{\partial F_Y\left[g(\tau,Y)\right]}{\partial\tau} + \frac{1}{2}\xi'\Sigma\Sigma'\xi F_Y[g(\tau,Y)] + i\gamma^{\mathcal{P}\prime}\xi F_Y\left[g(\tau,Y)\right]\\ &-\left(\frac{\partial F_Y[g(\tau,Y)]}{\partial\xi}\right)'\left(i\mathcal{D}_2(\tau)' - A^{\mathcal{P}\prime}\xi\right) + \frac{1}{2}Trace\left(\mathcal{D}_3(\tau)\frac{\partial^2\hat{g}(\xi)}{\partial\xi\partial\xi}\right) = 0\end{aligned} \tag{41}$$

subject to the initial condition

$$F_Y\left[g(0,Y_0)\right]=1.$$

Now that the initial value problem is defined we can use a guess and verify procedure to find its solution.

$$F_Y\left[g(\tau,Y)\right] = \exp\left\{-\frac{1}{2}\left[\xi'\mathcal{G}_3(\tau)\xi - 2i\mathcal{G}_2(\tau)'\xi + \mathcal{G}_1(\tau)\right]\right\}, \tag{42}$$

We do not assume the matrix is symmetric, since $\frac{1}{2}\xi'\left(\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right)\xi = \xi'\mathcal{G}_3(\tau)\xi$.

$$\frac{\partial F_Y\left[g(\tau,Y)\right]}{\partial\xi} = F_Y\left[g(\tau,Y)\right]\left[-\mathcal{G}_3(\tau)\xi - \mathcal{G}_3(\tau)'\xi + i\mathcal{G}_2(\tau)\right].$$

$$\begin{aligned}\frac{\partial^2 F_Y\left[g(\tau,Y)\right]}{\partial\xi\partial\xi} &= F_Y\left[g(\tau,Y)\right]\Big(-\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\xi\xi'\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\\ &\quad - 2i\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\xi\mathcal{G}_2(\tau)' - \mathcal{G}_2(\tau)\mathcal{G}_2(\tau)' - \left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\Big).\end{aligned}$$

$$\frac{\partial F_Y\left[g(\tau,Y)\right]}{\partial\tau} = F_Y\left[g(\tau,Y)\right]\left[-\frac{1}{2}\xi'\frac{\partial\mathcal{G}_3(\tau)}{\partial\tau}\xi + i\frac{\partial\mathcal{G}_2(\tau)}{\partial\tau}\xi - \frac{1}{2}\frac{\partial\mathcal{G}_1(\tau)}{\partial\tau}\right].$$

Now substitute these results into the Fourier transform (41) of the forward Kolmogorov equation (31).

$$\left[-\frac{1}{2}\xi'\frac{\partial\mathcal{G}_3(\tau)}{\partial\tau}\xi+i\frac{\partial\mathcal{G}_2(\tau)}{\partial\tau}\xi-\frac{1}{2}\frac{\partial\mathcal{G}_1(\tau)}{\partial\tau}\right]+\frac{1}{2}\xi'\Sigma_X\Sigma_X'\xi+i\xi'\gamma^{\mathcal{P}}$$
$$-\left(-\xi'\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]+i\mathcal{G}_2(\tau)\right)\left(i\mathcal{D}_2(\tau)'-A^{\mathcal{P}\prime}\xi\right)$$
$$+\frac{1}{2}Trace\Bigg(\mathcal{D}_3(\tau)\Big(\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\xi\xi'\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]$$
$$-2i\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\xi\mathcal{G}_2(\tau)-\mathcal{G}_2(\tau)'\mathcal{G}_2(\tau)-\left[\mathcal{G}_3(\tau)+\mathcal{G}_3(\tau)'\right]\Big)\Bigg)=0$$

$$\Rightarrow\left[-\frac{1}{2}\xi'\frac{\partial\mathcal{G}_3(\tau)}{\partial\tau}\xi+i\frac{\partial\mathcal{G}_2(\tau)}{\partial\tau}\xi-\frac{1}{2}\frac{\partial\mathcal{G}_1(\tau)}{\partial\tau}\right]+\frac{1}{2}\xi'\Sigma_X\Sigma_X'\xi+i\gamma^{\mathcal{P}\prime}\xi$$
$$+\mathcal{D}_2(\tau)\mathcal{G}_3(\tau)i\xi-\xi'\mathcal{G}_3(\tau)A^{\mathcal{P}\prime}\xi+\mathcal{G}_2(\tau)\mathcal{D}_2(\tau)'+\mathcal{G}_2(\tau)A^{\mathcal{P}\prime}i\xi+\frac{1}{2}\xi'\mathcal{G}_3(\tau)\mathcal{D}_3(\tau)\mathcal{G}_3(\tau)\xi$$
$$-\mathcal{G}_2(\tau)\mathcal{D}_3(\tau)\mathcal{G}_3(\tau)i\xi-\frac{1}{2}\mathcal{G}_2(\tau)\mathcal{D}_3(\tau)\mathcal{G}_2(\tau)'-\frac{1}{2}Trace\left(\mathcal{D}_3(\tau)\mathcal{G}_3(\tau)\right)=0.$$

Now equate quadratic, linear ($i\xi$), and constant terms to obtain three ODEs.

$$\frac{\partial\mathcal{G}_3(\tau)}{\partial\tau}=\mathcal{G}_3(\tau)\mathcal{D}_3(\tau)\mathcal{G}_3(\tau)-2\mathcal{G}_3(\tau)A^{\mathcal{P}\prime}+\Sigma_X\Sigma_X' \tag{43}$$

subject to

$$\mathcal{G}_3(0)=0_{N\times N}.$$

Again this is the Lyapunov equation.

$$\frac{\partial\mathcal{G}_2(\tau)}{\partial\tau}=\mathcal{G}_2(\tau)\left(\mathcal{D}_3(\tau)\mathcal{G}_3(\tau)-A^{\mathcal{P}\prime}\right)-\gamma^{\mathcal{P}\prime}-\mathcal{D}_2(\tau)\mathcal{G}_3(\tau) \tag{44}$$

subject to

$$\mathcal{G}_2(0)=0_N.$$

This ODE is linear so that we can use integrating factor to solve for $\mathcal{G}_2(\tau)$. The integrating factor is

$$int=e^{-\left(\mathcal{D}_3(\tau)\mathcal{G}_3(\tau)-A^{\mathcal{P}\prime}\right)\tau}.$$

Consequently,

$$\frac{\partial e^{-\left(\mathcal{D}_3(s)\mathcal{G}_3(s)-A^{\mathcal{P}\prime}\right)s}\mathcal{G}_2(s)}{\partial s}ds=-e^{-\left(\mathcal{D}_3(s)\mathcal{G}_3(s)-A^{\mathcal{P}\prime}\right)s}\left(\gamma^{\mathcal{P}\prime}-\mathcal{D}_2(s,X)\mathcal{G}_3(s)\right)ds.$$

Now integrate from τ to 0

$$\mathcal{G}_2(\tau, X) = e^{\left(\mathcal{D}_3(\tau)\mathcal{G}_3(\tau) - A^{\mathcal{P}\prime}\right)\tau}\mathcal{G}_2(0) - \int_0^\tau e^{-\left(\mathcal{D}_3(s)\mathcal{G}_3(s) - A^{\mathcal{P}\prime}\right)(s-\tau)} \left(\gamma^{\mathcal{P}\prime} - \mathcal{D}_2(s, X)\mathcal{G}_3(s)\right) ds.$$

The Final ODE is

$$\frac{\partial \mathcal{G}_1(\tau)}{\partial \tau} = 2\mathcal{G}_2(\tau)\mathcal{D}_2(\tau)' - \mathcal{G}_2(\tau)\mathcal{D}_3(\tau)\mathcal{G}_2(\tau)' - Trace\left(\mathcal{D}_3(\tau)\mathcal{G}_3(\tau)\right) \tag{45}$$

subject to

$$\mathcal{G}_1(0) = 0.$$

This initial value problem is the simplest since everything on the right hand side of the ODE is known.

Solving these three ODEs leads to the solution (42) to the Fourier transform of the Kolmogorov equation (41). The final step is to take the inverse Fourier transform to (42)

$$g(\tau, Y) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2}\left[\xi'\mathcal{G}_3(\tau)\xi - 2\left(\mathcal{G}_2(\tau) - Y'\right) i\xi + \mathcal{G}_1(\tau)\right]\right\} d\xi. \tag{46}$$

To calculate this integral we use the following Lemma following Strauss (2008, p. 345) and Strichartz (2008, pp. 41-43).

Lemma 2.2. *Let α be a positive number and let x_0 and y_0 be real numbers.*

$$\int_{-\infty}^{\infty} e^{-\alpha(x+x_0+iy_0)^2}\, dx = \sqrt{\frac{\pi}{\alpha}} \tag{47}$$

We also need the multiple dimension version of Lemma 2.2.

Lemma 2.3. *Let A be a $N \times N$ symmetric matrix with all positive eigenvalues and let $Z \in \mathbb{C}^N$.*

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}\left(X+A^{-1}Z\right)' A\left(X+A^{-1}Z\right)} dX = \sqrt{\frac{(2\pi)^N}{detA}}. \tag{48}$$

To apply the Lemma 2.3 to the inverse Fourier transform (46) we have to multiply out the quadratic exponent

$$\left(X + A^{-1}Z\right)' A\left(X + A^{-1}Z\right) = X'AX + 2Z'X + Z'(A^{-1})Z \tag{49}$$

Now match up the coefficients in (46) to yield

$$A = \mathcal{G}_3(\tau) \text{ and } Z = (\mathcal{G}_2(\tau)' - X)\, i \tag{50}$$

As a result, we can complete the square in the exponent of (46) to find

$$g(\tau, Y) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2}\left[\xi' \mathcal{G}_3(\tau)\xi - 2\left(\mathcal{G}_2(\tau) - Y'\right) i\xi + \mathcal{G}_1(\tau)\right]\right\} d\xi$$

$$= \exp\left\{ -\frac{1}{2}\mathcal{G}_1(\tau) - \frac{1}{2}\left(\mathcal{G}_2(\tau)' - Y\right)' \mathcal{G}_3(\tau)^{-1}\left(\mathcal{G}_2(\tau)' - Y\right)\right\}$$

$$\times \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2}\left(Y + \mathcal{G}_3(\tau)^{-1}\left(\mathcal{G}_2(\tau)' - Y\right) i\right)' \mathcal{G}_3(\tau)\left(Y + \mathcal{G}_3(\tau)^{-1}\left(\mathcal{G}_2(\tau)' - Y\right) i\right)\right\} d\xi$$

$$= \frac{1}{\sqrt{(2\pi)^N\, det(\mathcal{G}_3(\tau))}} \exp\left\{ -\frac{1}{2}\mathcal{G}_1(\tau) - \frac{1}{2}\left(\mathcal{G}_2(\tau)' - Y\right)' \mathcal{G}_3(\tau)^{-1}\left(\mathcal{G}_2(\tau)' - Y\right)\right\}. \tag{51}$$

By applying this solution to the forward Kolmogorov equation for the stochastic process (19), we can find the probability distribution for the trading desk's utility from bank capital (10) and her overall utility (12).

These random terms are probability densities of a normal distribution. We denote these probabilities densities by

$$\mathcal{N}(x; \mu, \Sigma) \equiv \frac{1}{\sqrt{(2\pi)^N\, det(\Sigma)}} \exp\left\{ -\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right\} \tag{52}$$

for $x \in \mathbf{R}^n$.

By (51) the discounted transition probability can be written as

$$\phi(t,\tau)p(t, X, \tau, Y) = \exp\left\{ -\frac{1}{2}\mathcal{G}_1(\tau)\right\} \mathcal{N}\left(Y; \mathcal{G}_2(\tau)', \mathcal{G}_3(\tau)\right). \tag{53}$$

Note that

$$\phi(t,s) = \exp\left\{ -\frac{1}{2}\int_t^s \left[X_v' \mathcal{D}_3(v) X_v - 2\mathcal{D}_2(v) X_v\right] dv\right\}$$

does not include the constant term

$$\mathcal{D}_0(\tau) = \exp\left\{ -\frac{1}{2}\mathcal{D}_1(\tau)\tau\right\}$$

so it has to be added back in. The same is true for the backward Kolmogorov equation (21).

In the analysis of option values and VaR we will use various rules for Gaussian probability distributions which we recall from Petersen and Pedersen (2008). First we use the rule for the product of two normal distributions.

$$\begin{aligned} &\mathcal{N}\left(x;\mu_1,\Sigma_1\right)\times\mathcal{N}\left(x;\mu_1,\Sigma_1\right)=\vartheta\mathcal{N}\left(x;\mu_c,\Sigma_c\right)\\ \text{where } \vartheta\equiv{}&\frac{1}{\sqrt{\left(2\pi\right)^N det\left(\Sigma_1+\Sigma_2\right)}}\exp\left\{-\frac{1}{2}(\mu_1-\mu_2)'\left(\Sigma_1+\Sigma_2\right)^{-1}\left(\mu_1-\mu_2\right)\right\},\\ &\mu_c\equiv\left(\Sigma_1^{-1}+\Sigma_2^{-1}\right)^{-1}\left(\Sigma_1^{-1}\mu_1+\Sigma_2^{-1}\mu_2\right),\\ \text{and } &\Sigma_c=\left(\Sigma_1^{-1}+\Sigma_2^{-1}\right)^{-1}. \end{aligned} \tag{54}$$

We also use the linear rule[6]

$$Ax\sim\mathcal{N}(x,A\mu,\Sigma A'), \tag{55}$$

Finally, we convert to a standard normal using the rule

$$x=\sigma Z+\mu \text{ such that } Z\sim\mathcal{N}\left(0_N,I_N\right). \tag{56}$$

Here, $\Sigma=\sigma\sigma'$ is the Cholesky decomposition of the variance covariance matrix. By following these basic rules for a normal distribution we are able to represent the probability distribution for the trading desk's bank capital and her lifetime utility.

2.5 Stochastic Discount Factor

We now have all the tools necessary to break a stochastic process like (10) and (12) into expected and random components. First, we apply the argument to the stochastic discount factor. The other stochastic processes will be solved using the same technique.

[6]See Petersen and Pedersen (2008) 8.1.4, p. 41.

$$
\begin{aligned}
\frac{M_{\tau,t}}{M_{t,t}} =& \exp\left\{-\int_t^{t+\tau}\left[r\left(X(s)\right)+\frac{1}{2}\Lambda\left(X(s)\right)'\Lambda\left(X(s)\right)\right]ds+\int_t^{t+\tau}\Lambda\left(X(s)\right)'d\epsilon_s\right\}\\
=& \exp\Bigg\{-\int_t^{t+\tau}\Bigg[r\left(X(s)\right)+\frac{1}{2}\Big(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}-\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)X(s)\Big)'\left(\Sigma_X'\Sigma_X\right)^{-1}\\
&\Big(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}-\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)X(s)\Big)\Bigg]ds+\int_t^{t+\tau}\Big(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}-\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)X(s)\Big)'\left(\Sigma_X'\right)^{-1}d\epsilon_s\Bigg\}\\
=& \exp\Bigg\{-\int_t^{t+\tau}\Bigg[\delta_0+\frac{1}{2}\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right)'\left(\Sigma_X'\Sigma_X\right)^{-1}\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right)\\
&+\Big(\delta_1-\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right)'\left(\Sigma_X'\Sigma_X\right)^{-1}\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)\Big)X(s)\\
&+\frac{1}{2}X(s)'\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)'\left(\Sigma_X'\Sigma_X\right)^{-1}\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)X(s)\Bigg]ds\\
&+\int_t^{t+\tau}\Big(\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right)'\left(\Sigma_X'\right)^{-1}-X(s)'\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)'\left(\Sigma_X'\right)^{-1}\Big)d\epsilon_s\Bigg\}\\
=& \exp\left\{\int_0^{\tau}\Bigg(-\mathcal{M}_1(0)-\frac{1}{2}\Big(X_s'\mathcal{M}_3(0)X_s-2\mathcal{M}_2(0)X_s\Big)\Bigg\}\Bigg]ds+\int_t^{T}\left(\mathcal{M}_4+\mathcal{M}_5X_s\right)d\epsilon_s\right\}.
\end{aligned}
$$

We use the risk free rate, the risk premium and the risk neutral coefficients in this derivation.

The constants are given by

$$
\begin{aligned}
\mathcal{M}_1 \equiv& \delta_0+\frac{1}{2}\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right)'\left(\Sigma_X'\Sigma_X\right)^{-1}\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right),\\
\mathcal{M}_2 \equiv& -\left[\delta_1-\left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right)'\left(\Sigma_X'\Sigma_X\right)^{-1}\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)\right],\\
\mathcal{M}_3 \equiv& \left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)'\left(\Sigma_X'\Sigma_X\right)^{-1}\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right),\\
\mathcal{M}_4 \equiv& \left(\gamma^{\mathcal{P}}-\gamma^{\mathcal{Q}}\right)'\left(\Sigma_X'\right)^{-1} \text{ and } \mathcal{M}_5 \equiv -\left(A^{\mathcal{P}}-A^{\mathcal{Q}}\right)'\left(\Sigma_X'\right)^{-1}.
\end{aligned}
\tag{57}
$$

As a result, the stochastic process for the pricing kernel is

$$
\frac{M_{\tau,t}}{M_{t,t}} = \exp\left\{\int_0^{\tau}\Bigg(-\mathcal{M}_1(0)-\frac{1}{2}\Big(X_s'\mathcal{M}_3(0)X_s-2\mathcal{M}_2(0)X_s\Big)\Bigg)ds+\int_t^{t+\tau}\left(\mathcal{M}_4+X_s'\mathcal{M}_5\right)\Sigma_X'd\epsilon_s\right\}.
\tag{58}
$$

These coefficients for (57) are provided in Table 4. $\mathcal{M}_3$ is positive so that the stochastic discount factor has the Gaussian shape. For $X_s > \left(\mathcal{M}_3\right)^{-1}\mathcal{M}_2 = -0.0210$ an increase in the factor leads to a decrease in the stochastic discount rate, but it reverses sign for lower values

Table 4: Estimates of Parameters in (57).

$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$
0.3212	-26.8228	1278	0.7223	35.7529

of the factor. $\mathcal{M}_5$ is negative so that shocks to the interest rate factors lowers the stochastic discount factor.

We need the probability distribution for the pricing kernel in solving the loan desk's problem. Before applying the forward Kolmogorov results, we factor out all the deterministic terms from (58). We have from (17)

$$X(\tau) = A_0(\tau) + e^{-A^{\mathcal{P}}(\tau-t)}X(t) + Y_\tau, \tag{59}$$

where

$$A_0(\tau) = \left(I - e^{-A^{\mathcal{P}}(\tau-t)}\right)\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}.$$

We also will use

$$\int_t^{t+\tau} e^{-A^{\mathcal{P}}(s-t)}ds = \left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}\tau}\right].$$

Now factor the square term to find

$$\begin{aligned}
X(\tau)'\mathcal{M}_3X(\tau) &= \left(A_0(\tau) + e^{-A^{\mathcal{P}}(\tau-t)}X(t) + Y_\tau\right)'\mathcal{M}_3\left(A_0(\tau) + e^{-A^{\mathcal{P}}(\tau-t)}X(t) + Y_\tau\right)\\
&= \left(\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(I - e^{-A^{\mathcal{P}\prime}(\tau-t)}\right) + X(t)'e^{-A^{\mathcal{P}\prime}}\right)\mathcal{M}_3\\
&\left(\left(I - e^{-A^{\mathcal{P}}(\tau-t)}\right)\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}} + e^{-A^{\mathcal{P}}(\tau-t)}X(t)\right)\\
&+ 2\left(\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(I - e^{-A^{\mathcal{P}\prime}(\tau-t)}\right) + X(t)'e^{-A^{\mathcal{P}\prime}(\tau-t)}\right)\mathcal{M}_3Y_\tau + Y_\tau'\mathcal{M}_3Y_\tau\\
&= \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(I - e^{-A^{\mathcal{P}\prime}(\tau-t)}\right)\mathcal{M}_3\left(I - e^{-A^{\mathcal{P}}(\tau-t)}\right)\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\\
&+ 2\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(I - e^{-A^{\mathcal{P}\prime}(\tau-t)}\right)\mathcal{M}_3e^{-A^{\mathcal{P}}(\tau-t)}X(t) + X(t)'e^{-A^{\mathcal{P}\prime}}\mathcal{M}_3e^{-A^{\mathcal{P}}(\tau-t)}X(t)\\
&+ 2\left(\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(I - e^{-A^{\mathcal{P}\prime}(\tau-t)}\right) + X(t)'e^{-A^{\mathcal{P}\prime}(\tau-t)}\right)\mathcal{M}_3Y_\tau + Y_\tau'\mathcal{M}_3Y_\tau.
\end{aligned}$$

Now integrate the first term over the time horizon τ given $X(t) = X$.

$$
\begin{aligned}
&-\frac{1}{2}\int_t^{t+\tau} X(s)'\mathcal{M}_3 X(s)ds = \\
&-\frac{1}{2}\int_t^{t+\tau} \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(I - e^{-A^{\mathcal{P}\prime}(s-t)}\right)\mathcal{M}_3\left(I - e^{-A^{\mathcal{P}}(s-t)}\right)ds\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}} \\
&-\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau}\left(I - e^{-A^{\mathcal{P}\prime}(s-t)}\right)\mathcal{M}_3 e^{-A^{\mathcal{P}}(s-t)}dsX(t) - \frac{1}{2}X(t)'\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 e^{-A^{\mathcal{P}}(s-t)}dsX(t) \\
&-\left(\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau}\left(I - e^{-A^{\mathcal{P}\prime}(s-t)}\right)\mathcal{M}_3 Y_s ds + X(t)'\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 Y_s ds\right) - \frac{1}{2}\int_t^{t+\tau} Y_s'\mathcal{M}_3 Y_s \\
&= -\frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau \\
&+\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}ds\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}} \\
&-\frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 e^{-A^{\mathcal{P}}(s-t)}ds\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}} \\
&-\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\int_t^{t+\tau} e^{-A^{\mathcal{P}}(s-t)}dsX(t) + \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 e^{-A^{\mathcal{P}}(s-t)}dsX(t) \\
&-\frac{1}{2}X(t)'\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 e^{-A^{\mathcal{P}}(s-t)}dsX(t) - \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\int_t^{t+\tau} Y_s ds \\
&+\left(\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 Y_s ds - X(t)'\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 Y_s ds\right) - \frac{1}{2}\int_t^{t+\tau} Y_s'\mathcal{M}_3 Y_s.
\end{aligned}
$$

If we use the definition of Y_s, we have

$$
\begin{aligned}
&-\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\int_t^{t+\tau} Y_s ds + \left(\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3\int_t^{s} e^{-A^{\mathcal{P}}(s-v)}\Sigma_X d\epsilon_v ds\right. \\
&\left.-X(t)'\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3\int_t^{s} e^{-A^{\mathcal{P}}(s-v)}\Sigma_X d\epsilon_v ds\right) = 0,
\end{aligned} \tag{60}
$$

since $d\epsilon_v ds = 0$ by Ito's Rule.

We need the result

$$
\int_t^{t+\tau} e^{-A^{\mathcal{P}\prime}(s-t)}\mathcal{M}_3 e^{-A^{\mathcal{P}}(s-t)}ds = \left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right],
$$

where the matrix $\mathcal{M}$ solves the Lyapunov equation

$$
-A^{\mathcal{P}}\mathcal{M} - \mathcal{M}A^{\mathcal{P}\prime} = \mathcal{M}_3. \tag{61}
$$

The solution to this equation is a positive definite symmetric matrix, which is easily calculated using lyap.m in Matlab.

$$-\frac{1}{2}\int_t^{t+\tau} X(\tau)'\mathcal{M}_3 X(\tau)ds = -\frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau$$
$$+\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[I-e^{-A^{\mathcal{P}\prime}\tau}\right]\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$-\frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M}-e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$-\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\left[I-e^{-A^{\mathcal{P}}\tau}\right]X(t)+\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M}-e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]X(t)$$
$$-\frac{1}{2}X(t)'\left[\mathcal{M}-e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]X(t)-\frac{1}{2}\int_t^{t+\tau} Y_s\mathcal{M}_3 Y_s$$

We also need

$$\begin{aligned}
\int_t^{t+\tau}\mathcal{M}_2 X_s ds =& \mathcal{M}_2\int_t^{t+\tau}\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}ds-\mathcal{M}_2\int_t^{t+\tau}e^{-A^{\mathcal{P}}(\tau-t)}ds\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}ds\\
&+\mathcal{M}_2\int_t^{t+\tau}e^{-A^{\mathcal{P}}(s-t)}dsX(t)+\int_t^{t+\tau}\mathcal{M}_2 Y_s ds\\
=&\mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau-\mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\left[I-e^{-A^{\mathcal{P}}(\tau)}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\\
&+\mathcal{M}_2\left(A^{\mathcal{P}\prime}\right)^{-1}\left[I-e^{-A^{\mathcal{P}}(\tau)}\right]X(t)+\int_t^{t+\tau}\mathcal{M}_2\int_t^s e^{-A^{\mathcal{P}}(s-\upsilon)}\Sigma_X d\epsilon_\upsilon ds\\
=&\mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau-\mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\left[I-e^{-A^{\mathcal{P}}(\tau)}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\\
&+\mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\left[I-e^{-A^{\mathcal{P}}(\tau)}\right]X(t).
\end{aligned}$$

The last step uses the rule $d\epsilon_\upsilon dt = 0$

We also need

$$\begin{aligned}
\int_t^{t+\tau} d\epsilon_s'\Sigma_X\mathcal{M}_5' X_s =& \int_t^{t+\tau} d\epsilon_s'\Sigma_X\mathcal{M}_5'\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}-\int_t^{t+\tau} d\epsilon_s'\Sigma_X\mathcal{M}_5' e^{-A^{\mathcal{P}}(\tau-t)}\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\\
&+\int_t^{t+\tau} d\epsilon_s'\Sigma_X\mathcal{M}_5' e^{-A^{\mathcal{P}}(s-t)}X(t)+\int_t^{t+\tau} d\epsilon_s'\Sigma_X\mathcal{M}_5' Y_s\\
=&\int_t^{t+\tau} d\epsilon_s'\Sigma_X\Big[\mathcal{M}_5'\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}-\mathcal{M}_5' e^{-A^{\mathcal{P}}(\tau-t)}\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\\
&+\mathcal{M}_5' e^{-A^{\mathcal{P}}(s-t)}X(t)\Big]+\int_t^{t+\tau} d\epsilon_s'\Sigma_X\mathcal{M}_5' Y_s\\
=&\int_t^{t+\tau}\left(\mathbb{M}_4+X_t'\mathbb{M}_5+Y_s'\mathcal{M}_5\Sigma_X'\right)d\epsilon_s
\end{aligned}$$

We now bring all these calculations into the stochastic process for the pricing kernel.

$$\frac{M_{\tau,t}}{M_{t,t}} = \exp\Bigg\{ -\mathcal{M}_1(\tau)\tau - \frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau$$
$$+\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[I - e^{-A^{\mathcal{P}\prime}\tau}\right]\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$-\frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$-\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}\tau}\right]X(t) + \gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]X(t)$$
$$-\frac{1}{2}X(t)'\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]X(t) + \mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau - \mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}(\tau)}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$+\mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}(\tau)}\right]X(t) - \frac{1}{2}\int_t^{t+\tau} Y_s'\mathcal{M}_3 Y_s + \int_t^{t+\tau}\left(\mathbb{M}_4 + X_t'\mathbb{M}_5 + Y_s'\mathcal{M}_5\Sigma_X'\right)d\epsilon_s\Bigg\}. \tag{62}$$

Define

$$\mathcal{M}(\tau, X) \equiv \exp\Bigg\{ -\mathcal{M}_1(\tau)\tau - \frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau$$
$$+\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[I - e^{-A^{\mathcal{P}\prime}\tau}\right]\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$-\frac{1}{2}\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$+\mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\tau - \mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}(\tau)}\right]\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}$$
$$+\Bigg[\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right] + \mathcal{M}_2\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}(\tau)}\right]$$
$$-\gamma^{\mathcal{P}\prime}\left(A^{\mathcal{P}\prime}\right)^{-1}\mathcal{M}_3\left(A^{\mathcal{P}}\right)^{-1}\left[I - e^{-A^{\mathcal{P}}\tau}\right]\Bigg]X(t) - \frac{1}{2}X(t)'\left[\mathcal{M} - e^{-A^{\mathcal{P}\prime}\tau}\mathcal{M}e^{-A^{\mathcal{P}}\tau}\right]X(t)\Bigg\}$$
$$= \exp\left\{ -\frac{1}{2}\left(X - \mathfrak{M}_3^{-1}\mathfrak{M}_2\right)'\mathfrak{M}_3\left(X - \mathfrak{M}_3^{-1}\mathfrak{M}_2\right) + \frac{1}{2}\mathfrak{M}_2'\mathfrak{M}_3^{-1}\mathfrak{M}_2 + \mathfrak{M}_1\right\} \tag{63}$$

This result can be used to separate the portion of the pricing kernel dependent on the current factors X from future random changes in these factors Y_s for $s > t$. We substitute the known part (63) into the pricing kernel (62) so that

$$\frac{1}{\mathcal{M}(\tau, X)}\frac{M_{\tau,t}}{M_{t,t}} = \exp\left\{ -\frac{1}{2}\int_t^{t+\tau} Y_s'\mathcal{M}_3 Y_s ds + \int_t^{t+\tau}\left(\mathbb{M}_4 + X_t'\mathbb{M}_5 + Y_s'\mathcal{M}_5\Sigma_X'\right)d\epsilon_s\right\} \tag{64}$$

This relation is an example of the stochastic process (19) so that its probability distribution is the solution to the forward Kolmogorov equation (31). Notice (64) is dependent on the current X through $\mathbb{M}_5$. This means that $\mathcal{D}_4 \equiv \mathbb{M}_4 + X_t'\mathbb{M}_5$ and $\mathcal{D}_5 = \mathcal{M}_5\Sigma_X'$. These terms do not influence the forward Kolmogorov equation, since this error term has mean zero.

The solution to the forward Kolmogorov equation yields the probability distribution for the pricing kernel.

$$\frac{1}{\mathcal{M}(\tau,X)}\frac{M_{\tau,t}}{M_{t,t}} \sim \frac{1}{\sqrt{(2\pi)^N\, det(\mathcal{A}_3(\tau,X))}} \exp\left\{-\frac{1}{2}\mathcal{A}_1(\tau,X) - \frac{1}{2}Y'\mathcal{A}_3(\tau,X)^{-1}Y\right\}$$

which has the same form as (19) with the appropriate definitions of the coefficients $\mathcal{D}'s$.

The coefficients for the discounted probability distribution for the pricing kernel (51) are given in Table 5 which has a normal distribution, since $\mathcal{A}_3(\tau) > 0$. These coefficients are reported at one year time horizon. $\mathcal{A}_2$ and $\mathcal{A}_3(\tau)$ quickly converges to the steady state. Also, recall from (51) that the maximum of the pricing kernel is $\mathcal{A}_2(\tau)$ with variance $\mathcal{A}_3$.

Table 5: Solution to Forward Kolmogorov Equation (40) for coefficients in (57) .

$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	σ_M
-0.0691	0.0000	1.1158 10^{-4}	0.0106

Thus the probability distribution for the pricing kernel is given by

$$\frac{M_{\tau,t}}{M_{t,t}} \sim \exp\left\{-\frac{1}{2}\left(X-\mathfrak{M}_3^{-1}\mathfrak{M}_2\right)'\mathfrak{M}_3\left(X-\mathfrak{M}_3^{-1}\mathfrak{M}_2\right) + \frac{1}{2}\mathfrak{M}_2'\mathfrak{M}_3^{-1}\mathfrak{M}_2 + \mathfrak{M}_1 - \frac{1}{2}\mathcal{A}_1(\tau)\right\}$$
$$\times \frac{1}{\sqrt{(2\pi)^N\, det(\mathcal{A}_3(\tau))}} \exp\left\{-\frac{1}{2}Y'\mathcal{A}_3(\tau)^{-1}Y\right\}$$

This leads to equation (10) in the text with $\sigma_M \equiv \mathcal{A}_3(\tau)$.

$$E_t\left[\frac{M_{\tau,t}}{M_{t,t}}\right] = \exp\left\{-\frac{1}{2}\left(X-\mathfrak{M}_3^{-1}\mathfrak{M}_2\right)'\mathfrak{M}_3\left(X-\mathfrak{M}_3^{-1}\mathfrak{M}_2\right) + \frac{1}{2}\mathfrak{M}_2'\mathfrak{M}_3^{-1}\mathfrak{M}_2 + \mathfrak{M}_1 - \frac{1}{2}\mathcal{A}_1(\tau)\right\}$$
$$\times \frac{1}{\sqrt{(2\pi)^N\, det(\mathcal{A}_3(\tau))}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}Y'\mathcal{A}_3(\tau)^{-1}Y\right\}dY$$
$$= \exp\left\{-\frac{1}{2}\left(X-\mathfrak{M}_3^{-1}\mathfrak{M}_2\right)'\mathfrak{M}_3\left(X-\mathfrak{M}_3^{-1}\mathfrak{M}_2\right) + \frac{1}{2}\mathfrak{M}_2'\mathfrak{M}_3^{-1}\mathfrak{M}_2 + \mathfrak{M}_1 - \frac{1}{2}\mathcal{A}_1(\tau)\right\} \tag{65}$$

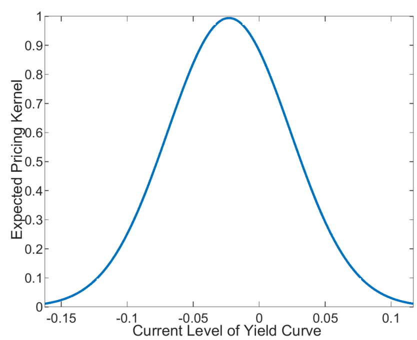
This corresponds to equation (9) in the text with

$$\begin{aligned}(\sigma_{\mathcal{M}}(\tau))^{-1} &\equiv \mathfrak{M}_3 \\ \mathcal{M}(\tau) &\equiv \exp\left\{\frac{1}{2}\mathfrak{M}_2'\mathfrak{M}_3^{-1}\mathfrak{M}_2 + \mathfrak{M}_1 - \frac{1}{2}\mathcal{A}_1(\tau)\right\}.\end{aligned} \tag{66}$$

Table 6: Coefficients in (65) for Pricing Kernel.

$\mathfrak{M}_1$	$\mathfrak{M}_2$	$\mathfrak{M}_3$	$\mathfrak{M}_3^{-1}\mathfrak{M}_2$	$\exp\left\{\frac{1}{2}\mathfrak{M}_2'\mathfrak{M}_3^{-1}\mathfrak{M}_2 + \mathfrak{M}_1 - \frac{1}{2}\mathcal{A}_1(\tau)\right\}$
-0.1610	-10.5396	461.52543	-0.0228	0.9940

Figure 2: The Expected Pricing Kernel (65).



The expected pricing kernel is less than one. For example, a level of the term structure of 0 leads to $E_t\left[\frac{M_{\tau,t}}{M_{t,t}}\right] = 0.8812$ and 0.9879 for the stationary point of the level, -0.0177. At the maximum level of the yield curve 0.0256 we have $E_t\left[\frac{M_{\tau,t}}{M_{t,t}}\right] = 0.5785$.

All the other distributions have the deterministic term (63) with the appropriate changes in the constants. For example the capital stock replaces $\mathcal{M}'s$ with $\mathcal{K}'s$.

2.6 Gross Growth Rate of the Trading Desk's Capital

In Table 7 we apply the forward Kolmogorov equation to the gross rate of growth of the trading desk's capital using the coefficients in (11). This corresponds to equation (36) in the paper. We can also express in Table 8 the conditional probability of this growth rate of capital as in the case of the pricing kernel (63). This leads to equation (35) in the paper. Tables 7 and 8 are used to construct Table 5 in the paper.

Table 7: Solution to Forward Kolmogorov Equation (40) for coefficients in (11).

$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_3$	σ_K
-0.0130	0.0000	1.0895 10^{-4}	0.0104

Table 8: Coefficients in (63) for Trading Desk's Capital.

$\mathfrak{K}_1$	$\mathfrak{K}_2$	$\mathfrak{K}_3$	$\mathfrak{K}_3^{-1}\mathfrak{K}_2$	$\exp\left\{-\frac{1}{2}\mathfrak{K}_2'\mathfrak{K}_3^{-1}\mathfrak{K}_2+\mathfrak{K}_1-\frac{1}{2}\mathcal{K}_1(\tau)\right\}$
0.0074	-5.6309	88.1733	-0.0639	1.2138

The expected gross growth rate of the capital for the trading desk is a Gaussian distribution in Figure 8 in the paper. At the stationary point for the level of the yield curve -0.177 we have $E_t\left[\frac{K_{\tau,t}}{K_{t,t}}\right] = 1.1048$. With the highest level of the yield curve at 0.0256 we have $E_t\left[\frac{K_{\tau,t}}{K_{t,t}}\right] = 0.8529$, since the higher level leads to a decrease in the price of bonds. In addition, a higher level should revert to the lower mean, resulting in an expected loss on the portfolio.

We will also deal with the product of the pricing kernel with the gross growth rate of the trading desk's capital. Its coefficients are given by

$$\begin{aligned}
\mathcal{D}_1(\tau) \equiv& \mathcal{C}_1(\tau) + \mathcal{M}_1, \\
\mathcal{D}_2(\tau) \equiv& \mathcal{C}_2(\tau) + \mathcal{M}_2, \\
\mathcal{D}_3(\tau) \equiv& \mathcal{C}_3(\tau) + \mathcal{M}_3, \\
\mathcal{D}_4(\tau) \equiv& \mathcal{C}_4(\tau) + \mathcal{M}_4, \text{ and} \\
\mathcal{D}_5(\tau) \equiv& \mathcal{C}_5(\tau) + \mathcal{M}_5.
\end{aligned} \tag{67}$$

Tables 10 and 11 apply the forward Kolmogorov equation to find the probability distributions for this product and its conditional expectation which corresponds to equation 38 in the paper.

Figure 3: The Unexpected Gross Growth Rate for The Trading Desk's Capital, (65).

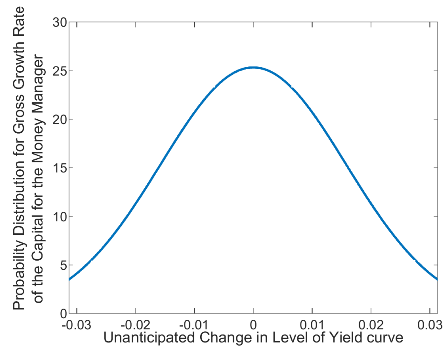


Table 9: Estimates of Parameters for Equation (67).

$\mathcal{D}_1(\tau)$	$\mathcal{D}_2(\tau)$	$\mathcal{D}_3(\tau)$	$\mathcal{D}_4(\tau)$	$\mathcal{D}_5(\tau)$
0.3937	-38.3567	1,522.60	0.5711	33.5455

Table 10: Solution to Forward Kolmogorov Equation (40) for coefficients in (67).

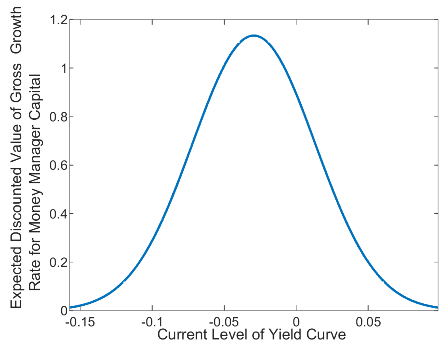
$\mathcal{KM}_1$	$\mathcal{KM}_2$	$\mathcal{KM}_3$	$\sigma_{\mathcal{KM}}$
-0.0825	0.0000	1.1222×10^{-4}	0.0106

Table 11: Coefficients in (63) for Product of Pricing Kernel with Trading Desk's Capital.

$\mathfrak{MK}_1$	$\mathfrak{MK}_2$	$\mathfrak{MK}_3$	$\mathfrak{MK}_3^{-1}\mathfrak{MK}_2$	$\exp\left\{-\frac{1}{2}\mathfrak{KK}_2'\mathfrak{MK}_3^{-1}\mathfrak{MK}_2 + \mathfrak{MK}_1 - \frac{1}{2}\mathcal{MK}_1(\tau)\right\}$
-0.1536	-16.1705	549.4276	-0.0294	1.1339

The expected discounted value for the gross growth rate of the capital for the trading desk is a Gaussian distribution in Figure 4. At the stationary point for the level of the yield curve -0.177 we have $E_t\left[\frac{MK_{\tau,t}}{MK_{t,t}}\right] = 1.0917$. With the highest level of the yield curve observed at 0.0256 we have $E_t\left[\frac{MK_{\tau,t}}{MK_{t,t}}\right] = 0.4935$. Finally, the maximum value is $E_t\left[\frac{MK_{\tau,t}}{MK_{t,t}}\right] = 1.1339$.

Figure 4: The Expected Discounted Value of Gross Growth Rate for The Trading Desk's Capital, (65).



Therefore we have expressed all the stochastic processes for solving the loan desk's problem as standard normal random variables with mean zero. This can be done by adding

$$\exp\left\{-\frac{1}{2}\mathcal{A}_1(\tau, X)\right\}$$

to the constant term in $\mathcal{MK}(\tau, X)$.

We write these probabilities as

$$\frac{1}{\mathcal{M}(\tau,X)}\frac{M_{\tau,t}}{M_{t,t}} = p_M(t,X,\tau,Y), \qquad \frac{1}{\mathcal{K}(\tau,X)}\frac{K_M^j(t+\tau)}{K_M^j(t)} = p_K(t,X,\tau,Y), \tag{68}$$

$$\frac{1}{\mathcal{KM}(\tau,X)}\frac{M_{\tau,t}}{M_{t,t}}\frac{K_M^j(t+\tau)}{K_M^j(t)} = p_{MK}(t,X,\tau,Y), \qquad \frac{1}{\mathcal{MP}(\tau,X)}\frac{M_{\tau,t}}{M_{t,t}}\frac{\bar{P}_{3\tau,t}}{P_{3\tau,t}} = p_{MP}(t,X,\tau,Y),$$

$$\frac{1}{\mathcal{MY}(\tau,X)}\frac{M_{\tau,t}}{M_{t,t}}Y = p_{MY}(t,X,\tau,Y) \text{ and } \frac{1}{\mathcal{KMY}(\tau,X)}\frac{M_{\tau,t}}{M_{t,t}}\frac{K_M^j(t+i\tau)}{K_M^j(t)}Y = p_{MKY}(t,X,\tau,Y).$$

3 Loan Desk's Optimization Problem

This section derive additional results for the model in section 5 of the paper. The Lagrangian function for the COO's problem after using the balance sheet constraint for the loan desk to remove the quantity of deposits is given by

$$
\begin{aligned}
&V\left(t, K_M^j(t), K_L^j(t), r_{2\tau,t-\tau}^j, X(t)\right) = K_M^j(t)\mathcal{M}(\tau,X)\mathcal{K}(\tau,X) - \frac{r^p}{\bar{D}}\mathcal{M}(\tau,X)\left[\bar{D} - K_M^j(t)\right]^2 \\
&+ \max E_t\Bigg\{\mathcal{M}(\tau,X)\Bigg[(r_{\tau,t}^j - c^j - r_{\tau,t}^D)\left\{\gamma_{0,\tau}^j - \gamma_{1,\tau}^j r_{\tau,t}^j + \sigma(r_{\tau,t}^j)\varepsilon_{\tau,t}^j\right\} \\
&+ (r_{2\tau,t}^j - c^j - r_{\tau,t}^D)\left\{\gamma_{0,2\tau}^j - \gamma_{1,2\tau}^j r_{2\tau,t}^j + \sigma(r_{2\tau,t}^j)\varepsilon_{2\tau,t}^j\right\} + r_{\tau,t}^D\left(K_L^j(t) + (1-\xi)K_M^j(t) - R_t^j\right) \\
&+ (r_{2\tau,t-\tau}^j - c^j - r_{\tau,t}^D)\left\{\gamma_{0,2\tau}^j - \gamma_{1,2\tau}^j r_{2\tau,t-\tau}^j + \sigma(r_{2\tau,t-\tau}^j)\varepsilon_{2\tau,t-\tau}^j\right\} - (1-\chi)r_{\tau,t}^{jK} + (1-\eta)q^j\Bigg]\tau \\
&+ \lambda_1\mathcal{M}(\tau,X)\left[K_M^j(t) + K_L^j(t) - \kappa_L\left(L_{\tau,t}^j + L_{2\tau,t}^j + L_{2\tau,t-\tau}^j\right) - \kappa_T\xi K_M^j(t) - c_b\left(\frac{\bar{P}_{3\tau,t}}{P_{3\tau,t}} - 1\right)^+\right] \\
&+ \lambda_2\mathcal{M}(\tau,X)\left[K_t^j - \alpha_\tau L_{\tau,t}^j - \alpha_{2\tau}\left(L_{2\tau,t}^j + L_{2\tau,t-\tau}^j\right) - \alpha_T\xi K_M^j(t) + \alpha_R R_t^j\right] \\
&+ E_t\left[\frac{M_{2\tau,t}}{M_{t,t}}V\left(t+\tau, K_M^j(t+\tau), K_L^j(t) + \left[\pi_L^j(t) - r_{\tau,t}^{jK} + q^j\right]\tau, r_{2\tau,t}^j, X(t+\tau)\right)\right]\Bigg\}.
\end{aligned}
\tag{69}
$$

For the two period loans we have

$$
\begin{aligned}
&\mathcal{M}(\tau,X)\Bigg[2\left(-\gamma_{1,2\tau}^j + \sigma_1\varepsilon_{2\tau,t}^j\right)r_{2\tau,t}^j - \left(c^j + r_{2\tau,t}^D\right)\left(-\gamma_{1,2\tau}^j + \sigma_1\varepsilon_{2\tau,t}^j\right) \\
&+ \gamma_{0,2\tau}^j + \sigma_0\varepsilon_{2\tau,t}^j\Bigg]\tau - \mathcal{M}(\tau,X)\left[\lambda_1\kappa_L + \lambda_2\alpha_{2\tau}\right]\left(-\gamma_{1,2\tau}^j + \sigma_1\varepsilon_{2\tau,t}^j\right) \\
&+ E_t\left[\frac{M_{2\tau,t}}{M_{t,t}}\frac{\partial V}{\partial m_{2\tau,t}^j}\right] + E_t\left[\frac{M_{2\tau,t}}{M_{t,t}}\frac{\partial V}{\partial K_L^j(t+\tau)}\right]\frac{\partial \pi_L^j}{\partial m_{2\tau,t}^j} = 0.
\end{aligned}
$$

Here,

$$
\frac{\partial \pi_L^j}{\partial m_{2\tau,t}^j} = \left[2\left(-\gamma_{1,2\tau}^j + \sigma_1\varepsilon_{2\tau,t}^j\right)r_{2\tau,t}^j - \left(c^j + r_{2\tau,t}^D\right)\left(-\gamma_{1,2\tau}^j + \sigma_1\varepsilon_{2\tau,t}^j\right) + \gamma_{0,2\tau}^j + \sigma_0\varepsilon_{2\tau,t}^j\right]\tau,
$$

and

$$
\frac{\partial V}{\partial m_{2\tau,t}^j} = \frac{\partial \pi_L^j}{\partial m_{2\tau,t}^j} - \left[\lambda_1\kappa_L + \lambda_2\alpha_{2\tau}\right]\left(-\gamma_{1,2\tau}^j + \sigma_1\varepsilon_{2\tau,t}^j\right).
$$

Using the optimal condition for paying dividends (44) in the paper, the optimal condition for the two period loan margin becomes

$$
\left[2r_{2\tau,t}^j - \left(c^j + r_{2\tau,t}^D\right) + \frac{\gamma_{0,2\tau}^j + \sigma_0\varepsilon_{2\tau,t}^j}{\left(-\gamma_{1,2\tau}^j + \sigma_1\varepsilon_{2\tau,t}^j\right)}\right]\tau = \frac{\left[\mathcal{M}(\tau,X) + \mathcal{M}(2\tau,X)\right]}{\left[\chi\mathcal{M}(\tau,X) + \mathcal{M}(2\tau,X)\right]}\left(\lambda_1\kappa_L + \lambda_2\alpha_{2\tau}\right).
\tag{70}
$$

Here, the stochastic discount factor is given by (10) in the paper.[7]

We can also examine the behavior of the two period loans. We focus on the capital constraint (21) in the paper. First we solve (70), when these constraints do not bind, then

$$r^{j*}_{2\tau,t} = \frac{1}{2}\left(c^j + r^D_{2\tau,t}\right) - \frac{\gamma^j_{0,2\tau} + \sigma_0 \varepsilon^j_{2\tau,t}}{2\left(-\gamma^j_{1,2\tau} + \sigma_1 \varepsilon^j_{2\tau,t}\right)}. \tag{71}$$

Consequently, the two period loan follows the same rule as one period loans using the demand for two period loans.

When the liquidity and capital constraints are binding, the analysis follows the same argument as for the one period loan in the paper. First the loan rate is set on the demand for two period loans.

$$r^{jK}_{2\tau,t} = \frac{1}{\left(-\gamma^j_{1,2\tau} + \sigma_1 \varepsilon^j_{2\tau,t}\right)}\left[L^j_{2\tau\kappa,t} - \left(\gamma^j_{0,2\tau} + \sigma_0 \varepsilon^j_{2\tau,t}\right)\right]. \tag{72}$$

The subscript $'2\tau\kappa, t'$ in the two period loan rate refers to the loan rate when the two period loans just satisfy the capital constraint (21) at time t in the paper.

The lagrange multiplier for the capital constraint (21) at time t in the paper is found by solving (70) for λ_1 when $\lambda_2 = 0$. In addition, we use (72).

$$\lambda_1 = \frac{2\tau\left[\chi\mathcal{M}(\tau,X) + \mathcal{M}(2\tau,X)\right]}{\kappa_L\left[\mathcal{M}(\tau,X) + \mathcal{M}(2\tau,X)\right]}\left(r^{jK}_{2\tau,t} - \frac{1}{2}\left(c^j + r^D_{2\tau,t}\right) + \frac{\gamma^j_{0,2\tau} + \sigma_0 \varepsilon^j_{2\tau,t}}{2\left(-\gamma^j_{1,2\tau} + \sigma_1 \varepsilon^j_{2\tau,t}\right)}\right). \tag{73}$$

By using (71), the lagrange multiplier for the capital constraint (21) in the paper also has a payoff similar to a European call option

$$\lambda^*_1(t) = \begin{cases} \frac{2\tau[\chi\mathcal{M}(\tau,X)+\mathcal{M}(2\tau,X)]}{\kappa_L[\mathcal{M}(\tau,X)+\mathcal{M}(2\tau,X)]}\left[r^{jK}_{2\tau,t} - r^{j*}_{2\tau,t}\right] & \text{for } r^{jK}_{\tau,t} > r^{j*}_{\tau,t} \\ 0 & \text{for } r^{jK}_{\tau,t} \leq r^{j*}_{\tau,t}. \end{cases} \tag{74}$$

The essential difference from (57) in the paper is that the slope of the payoff is now influenced by the expectation of the marginal investor's intertemporal rate of substitution, $\frac{[\chi\mathcal{M}(\tau,X)+\mathcal{M}(2\tau,X)]}{[\mathcal{M}(\tau,X)+\mathcal{M}(2\tau,X)]}$, as well as the weights on two period loans in the capital constraint. This result can be seen by comparing this with (57) in the paper. As a result, the relative payoff between the two and one period loans satisfies

$$\frac{\left[r^{jK}_{2\tau,t} - r^{j*}_{2\tau,t}\right]}{\left[r^{jK}_{\tau,t} - r^{j*}_{\tau,t}\right]} = \frac{\left[\chi\mathcal{M}(\tau,X) + \mathcal{M}(2\tau,X)\right]}{\left[\mathcal{M}(\tau,X) + \mathcal{M}(2\tau,X)\right]\chi}. \tag{75}$$

[7] If the bank chooses to issue equity, then 1 is replaced by $1+\eta$ and χ becomes 2 .

Thus, the relative payoffs on two and one period loans is dependent on the stochastic discount factor for the marginal investor, the response of the demand for loans to the two period interest rate, and the weights on two and one period loans in the capital constraint.

4 The Capital Option Value under Capital Constraint

Now that we have the probability distributions (68) we can evaluate the expected marginal value of the loan desk's capital (65) in the paper. With the solution to this expected marginal value of the loan desk's capital, we can then determine the optimal amount of the loan desk's capital using (23) in the paper. To find this option value we use the logic for solving the Black-Scholes option pricing formula.

To illustrate these calculations we start with the counter cyclical buffer. Using (11) in the paper the counter cyclical buffer is dependent on

$$\frac{P_{\tau,\tau}}{\bar{P}_{\tau,\tau}} = exp\Bigg\{ b_\tau \Bigg[e^{-A^{\mathcal{P}}(\tau-t)}(X - \bar{X}) + Y \Bigg] \Bigg\} = \mathcal{P}(\tau, X) \exp \Bigg\{ b_\tau Y \Bigg\} \tag{76}$$

The last step uses the fact that Y has a normal distribution with mean 0 and variance $b_\tau K(\tau) b'_\tau$ so that $\frac{P_{\tau,\tau}}{\bar{P}_{\tau,\tau}}$ has a standard normal distribution after adjusting for the variance.

As a result, the counter cyclical buffer is positive when

$$Y < e^{-A^{\mathcal{P}}(\tau-t)}(\bar{X} - X),$$

since $b_\tau < 0$.

The counter cyclical buffer applies whether or not the liquidity or the capital constraint binds. As a result, we can calculate the cost of the counter cyclical buffer. We need the probability distribution for $\frac{M_{2\tau,t}}{M_{t,t}}$. Following the derivation of the forward Kolmogorov equation the two terms have a normal distribution in (68).

$$\exp \Bigg\{ -\frac{1}{2} Y' \mathcal{A}_3(\tau)^{-1} Y + b_\tau Y \Bigg\} = exp\Bigg\{ -\frac{1}{2} \left(Y - \mathcal{A}_3(\tau) b'_\tau \right)' \mathcal{A}_3(\tau)^{-1} \left(Y - \mathcal{A}_3(\tau) b'_\tau \right) + \frac{1}{2} b_\tau \mathcal{A}_3(\tau) b'_\tau \Bigg\}$$

As a result we have

$$\frac{M_{2\tau,t}}{M_{t,t}}\left(\frac{\bar{P}_{\tau,s}}{P_{\tau,s}}-1\right)^{+} = \Bigg(\Bigg(\mathcal{P}(\tau,X)\mathcal{M}(2\tau,X)p_M(t,X,2\tau,Y)exp\Bigg\{b_\tau Y\Bigg\} \tag{77}$$

$$- \mathcal{M}(2\tau,X)p_M(t,X,2\tau,Y)\Bigg)\Bigg)^{+}$$

$$= \frac{1}{\sqrt{(2\pi)^N\, det(\mathcal{A}_3(\tau))}}\Bigg(\Bigg(\mathcal{P}(\tau,X)\mathcal{M}(2\tau,X)exp\Bigg\{\frac{1}{2}b_\tau\mathcal{A}_3(\tau)^{-1}b'_\tau\Bigg\}$$

$$\times\, exp\Bigg\{-\frac{1}{2}\left(Y-\mathcal{A}_3(\tau)b'_\tau\right)'\mathcal{A}_3(\tau)^{-1}\left(Y-\mathcal{A}_3(\tau)b'_\tau\right)\Bigg\} \tag{78}$$

$$- \mathcal{M}(2\tau,X)\exp\Bigg\{-\frac{1}{2}Y'\mathcal{A}_3(\tau)^{-1}Y\Bigg\}\Bigg)\Bigg)^{+}.$$

Let ρ_b for given X be defined by

$$\mathcal{P}(\tau,X)\mathcal{M}(2\tau,X)\exp\left\{\frac{1}{2}b_\tau\mathcal{A}_3(\tau)b'_\tau\right\}\exp\left\{-\frac{1}{2}\left(\rho_b-\mathcal{A}_3(\tau)b'_\tau\right)'\mathcal{A}_3(\tau)^{-1}\left(\rho_b-\mathcal{A}_3(\tau)b'_\tau\right)\right\}$$

$$- \mathcal{M}(2\tau,X)\exp\left\{-\frac{1}{2}\rho'_b\mathcal{A}_3(\tau)^{-1}\rho_b\right\} = 0 \Rightarrow \exp\left\{b_\tau e^{-A^{\mathcal{P}}(\tau-t)}(X-\bar{X})+b_\tau\rho_b\right\}-1=0$$

$$\Rightarrow \rho_b = e^{-A^{\mathcal{P}}(\tau-t)}(\bar{X}-X). \tag{79}$$

Let $\mathcal{A}_3(\tau)=\Sigma_M\Sigma'_M$. Thus, the option value of the counter cyclical buffer is

$$c_bE_t\left(\frac{M_{2\tau,t}}{M_{t,t}}\left(\frac{\bar{P}_{\tau,s}}{P_{\tau,s}}-1\right)^{+}\right)$$

$$= c_b\mathcal{M}(2\tau,X)\Bigg(\mathcal{P}(\tau,X)\exp\left\{\frac{1}{2}b'_\tau\mathcal{A}_3(\tau)^{-1}b_\tau\right\}\left(1-\Phi\left(\Sigma_M^{-1}\left(\rho_b-\left(\Sigma_M\Sigma'_M\right)b_\tau\right)\right)\right)$$

$$- \left(1-\Phi\left(\Sigma_M^{-1}\rho_b\right)\right)\Bigg). \tag{80}$$

This corresponds to (25) in the paper.

Here, define the probability

$$Pr\left\{Z>\rho\right\} \equiv \frac{1}{\sqrt{(2\pi)^N}}\int_\rho^\infty e^{-\frac{1}{2}Z'Z}dZ \equiv \Phi\left(\rho\right),$$

so that

$$\frac{\partial \Phi\left(\rho\right)}{\partial \rho} = -\frac{1}{\sqrt{\left(2\pi\right)^{N}}} e^{-\frac{1}{2}\rho'\rho}.$$

In the rest of the derivation we take the counter cyclical buffer as given by CCB, since its marginal cost is known given current information.

4.1 Option value of capital constraint

The option value of capital for the loan desk is dependent on the marginal value of the capital of the loan desk in the future. This marginal value of capital is

$$\begin{aligned} \frac{\partial V}{\partial K_L^j(t)} =& \mathcal{M}(\tau, X)\left[r^D(t) + \lambda_1^*(t) + \lambda_2^*(t)\right]\tau + E_t\left[\frac{M_{2\tau,t}}{M_{t,t}}\frac{\partial V}{\partial K_L^j(t)}\right] \\ =& \mathcal{M}(\tau, X)\left[r^D(t) + \lambda_1^*(t) + \lambda_2^*(t)\right]\tau + \mathcal{M}(2\tau, X)E_t\left[p_M(2\tau, Y)\frac{\partial V}{\partial K_L^j(t)}\right] \\ =& \mathcal{M}(\tau, X)\left[r^D(t) + \lambda_1^*(t) + \lambda_2^*(t) + (\chi - 1)\right]\tau \end{aligned} \tag{81}$$

The first step uses the property of the stochastic discount factor which divides it into an expected and random component. The second step uses (44) from the paper. As a result, the expected marginal value of capital for the loan desk is

$$\begin{aligned} \mathcal{M}(2\tau, X)p_M(2\tau, Y)\frac{\partial V}{\partial K_L^j(t+1)} =& \mathcal{M}(2\tau, X)p_M(2\tau, Y) \\ & \times\left[r^D(t+1) + \lambda_1^*(t+1) + \lambda_2^*(t+1) + (\chi - 1)\right]\tau, \end{aligned} \tag{82}$$

where we have used the properties of the stochastic discount factor from t to $t + 2\tau$.

We start with the expression for the expected marginal value of the loan desk's capital under the capital constraint in (65) of the paper. This term is dependent on the term

$$E_t\left\{p_M(2\tau, Y)\left[\ \frac{1}{\kappa_L}\left(r_{\tau,t+\tau}^{j\kappa} - r_{\tau,t+\tau}^{j*}\right)^{+}\right]\right\}. \tag{83}$$

We will bring back the constant $\frac{2\chi}{\kappa_L}\mathcal{M}(2\tau, X)$ after the derivation.

Recall from (55) in the paper

$$r_{\tau,t}^{j*} = \frac{1}{2}\left(c^j + r_{\tau,t}^D\right) + \frac{\gamma_{0,\tau}^j + \sigma_0\varepsilon_{\tau,t}^j}{2\left(\gamma_{1,\tau}^j - \sigma_1\varepsilon_{\tau,t}^j\right)}, \tag{84}$$

For the capital constraint (21) in the paper we also have (56) in the paper replaced by

$$r^{j\kappa}_{\tau,t} = \frac{1}{\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)} \left[\left(\gamma^j_{0,\tau} + \sigma_0 \varepsilon^j_{\tau,t}\right) - L^j_{\kappa,t}\right], \tag{85}$$

and the loans subject to the capital constraint (23) in the paper yields

$$\begin{aligned} L^j_{\kappa,t+\tau} =& \frac{1}{\kappa_L}\Bigg[K^j(t+\tau) - \kappa_T \xi K^j_M(t+\tau) - \kappa_L \left(L^j_{2\tau,t+\tau} + L^j_{2\tau,t}\right) \\ & - c_b \left(\frac{\bar{P}_{3\tau,t}}{P_{3\tau,t}} - 1\right)^+\Bigg]. \end{aligned} \tag{86}$$

As a result, the Lagrange multiplier (57) in the paper is

$$\begin{aligned} \lambda^*_1(t) =& 2\frac{\chi}{\kappa_L}\left[\frac{1}{\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)}\left[\left(\gamma^j_{0,\tau} + \sigma_0 \varepsilon^j_{\tau,t}\right) - L^j_{\kappa,t}\right] - \frac{1}{2}\left(c^j + r^D_{\tau,t}\right) - \frac{\gamma^j_{0,\tau} + \sigma_0 \varepsilon^j_{\tau,t}}{2\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)}\right] \\ =& \frac{\chi}{\kappa_L}\left[-\frac{2}{\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)}\left[L^j_{\kappa,t}\right] - \left(c^j + r^D_{\tau,t}\right) + \frac{\gamma^j_{0,\tau} + \sigma_0 \varepsilon^j_{\tau,t}}{\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)}\right] \\ =& \frac{2\chi}{\kappa_L\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)}\Bigg[\frac{\gamma^j_{0,\tau} + \sigma_0 \varepsilon^j_{\tau,t}}{2} - \frac{1}{\kappa_L}\Bigg[K^j(t+\tau) - \kappa_T \xi K^j_M(t+\tau) - \kappa_L\left(L^j_{2\tau,t+\tau} + L^j_{2\tau,t}\right) \\ & - c_b\left(\frac{\bar{P}_{3\tau,t}}{P_{3\tau,t}} - 1\right)^+\Bigg] - \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)\left(c^j + r^D_{\tau,t}\right)\Bigg] \end{aligned}$$

The option payoff for the capital constraint using (55) and (56) in the paper is

$$\begin{aligned} & \frac{2\chi}{\kappa_L} p_M(2\tau, Y)\left(r^{j\kappa}_{\tau,t+\tau} - r^{j*}_{\tau,t+\tau}\right)^+ \\ & = \frac{2\chi}{\kappa_L\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)} p_M(2\tau, Y)\left(\frac{1}{2}\left(\gamma^j_{0,\tau} + \sigma_0 \varepsilon^j_{\tau,t}\right) - L^j_{\kappa,t+\tau} - \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)\left(c^j + r^D_{\tau,t+\tau}\right)\right)^+. \end{aligned} \tag{87}$$

This imposes a bound on the constant which determines whether or not the constraint binds.

$$\frac{1}{2}\left(\gamma^j_{0,\tau} + \sigma_0 \varepsilon^j_{\tau,t}\right) > L^j_{\kappa,t+\tau} + \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)\left(c^j + r^D_{\tau,t+\tau}\right). \tag{88}$$

The conditional expectation of the counter cyclical buffer in the capital constraint (23) in the paper was found in (80). This is the simplest term in (87) so that it illustrates how to calculate the whole expression (87).

We continue with the calculation of the probability distribution for the option payoff (87). We now use the rules for future capital of the trading desk (36) in the paper and the factors (17). We also use the linear rules for the deposit rate (51) and bank reserves (52) in the paper.

$$
\begin{aligned}
&p_M(2\tau, Y)\left[\frac{1}{2}\left(\gamma^j_{0,\tau} + \sigma_0\varepsilon^j_{\tau,t}\right) - L^j_{\kappa,t} - \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)\left(c^j + r^D_{\tau,t}\right)\right] \\
&= \frac{CCB}{\mathcal{M}(2\tau, X)} + p_M(2\tau, Y)\Bigg[-\frac{1}{\kappa_L}K^j_L(t+\tau) - \frac{1}{\kappa_L}(1-\kappa_T\xi)K^j_M(t+\tau) + \left(L^j_{2\tau,t+\tau} + L^j_{2\tau,t}\right) \\
&+ \frac{1}{2}\left(\gamma^j_{0,\tau} + \sigma_0\varepsilon^j_{\tau,t}\right) - \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)c^j - \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)\left(d_0 + d_1X(t+\tau)\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{CCB}{\mathcal{M}(2\tau, X)} + \Bigg\{\frac{1}{2}\left(\gamma^j_{0,\tau} + \sigma_0\varepsilon^j_{\tau,t}\right) - \frac{1}{\kappa_L}K^j_L(t+\tau) + \left(L^j_{2\tau,t} + L^j_{2\tau,t}\right) - \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)\left(c^j + d_0\right) \\
&- \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)d_1\left[e^{-A^{\mathcal{P}}\tau}X(t) + \left(I - e^{-A^{\mathcal{P}}\tau}\right)\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}\right]\Bigg\}p_M(2\tau, Y) \\
&- \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)d_1 p_M(2\tau, Y)Y \\
&- \frac{1}{\kappa_L}(1-\kappa_T\xi)K^j_M(t)p_M(2\tau, Y)\exp\left\{\int_t^{t+\tau}\left[\mathcal{C}_1(\tau) + \mathcal{C}_2(\tau)X(s) + \frac{1}{2}X(s)'\mathcal{C}_3(\tau)X(s)\right]ds\right. \\
&\left. + \int_t^{t+i\tau_i}\left[\mathcal{C}_4(\tau) + \mathcal{C}_5(\tau)X(s)\right]d\epsilon_s\right\} \\
&= \frac{CCB}{\mathcal{M}(2\tau, X)} + \Bigg\{\frac{1}{2}\left(\gamma^j_{0,\tau} + \sigma_0\varepsilon^j_{\tau,t}\right) - \frac{1}{\kappa_L}K^j_L(t+\tau) + \left(L^j_{2\tau,t+\tau} + L^j_{2\tau,t}\right) \\
&- \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)\left(c^j + d_0 + d_1\mu(\tau, X)\right)\Bigg\}\frac{\exp\left\{-\frac{1}{2}Y'\sigma_M(2\tau)^{-1}Y\right\}}{\sqrt{(2\pi)^N det(\sigma_M(2\tau))}} \\
&- \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)d_1\frac{\exp\left\{-\frac{1}{2}Y'\left(\sigma_M(2\tau)^{-1} + \sigma_Y(\tau)^{-1}\right)Y\right\}}{\sqrt{(2\pi)^N det(\sigma_M(2\tau) + \sigma_Y(\tau))}} \\
&- \frac{(1-\kappa_T\xi)}{\kappa_L}K^j_M(t)\mathcal{K}(\tau, X)\frac{\exp\left\{-\frac{1}{2}Y'\left(\sigma_M(2\tau)^{-1} + \sigma_K(\tau)^{-1}\right)Y\right\}}{\sqrt{(2\pi)^N det(\sigma_M(2\tau) + \sigma_K(\tau))}}.
\end{aligned}
$$

In the first equality we substitute CCB for the counter cyclical buffer, the critical loans (86), the deposit rate (51) and bank reserves (52) in the paper. In the second step we substitute in the rules for future capital of the trading desk (36) in the paper and the factors (17). In the final step we use the appropriate probability rules (68).

We also used the definition

$$\mu(\tau, X) \equiv e^{-A^{\mathcal{P}}\tau} X + \left(I - e^{-A^{\mathcal{P}}\tau}\right) \left(A^{\mathcal{P}}\right)^{-1} \gamma^{\mathcal{P}}$$

We want to evaluate the option value from equation (65) in the paper.

$$\begin{aligned}
&2\frac{\chi}{\kappa_L} E_t \left[p_M(2\tau, Y) \left(r^{j\kappa}_{\tau,t+\tau} - r^{j*}_{\tau,t+\tau}\right)^+\right] = \frac{2\chi}{\kappa_L \left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)} \frac{CCB}{\mathcal{M}(2\tau, X)} \\
&+ \frac{2\chi}{\kappa_L \left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)} \int_{-\infty}^{\infty} \left\{ \left\{ \frac{1}{2}\left(\gamma^j_{0,\tau} + \sigma_0 \varepsilon^j_{\tau,t}\right) - \frac{1}{\kappa_L} K^j_L(t+\tau) + \left(L^j_{2\tau,t+\tau} + L^j_{2\tau,t}\right) \right.\right. \\
&- \left. \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right) \left(c^j + d_0 + d_1 \mu(\tau, X)\right) \right\} \frac{\exp\left\{ -\frac{1}{2} Y' \sigma_M(2\tau)^{-1} Y \right\}}{\sqrt{(2\pi)^N det(\sigma_M(2\tau))}} \\
&- \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right) d_1 \frac{\exp\left\{ -\frac{1}{2} Y' \left(\sigma_M(2\tau)^{-1} + \sigma_Y(\tau)^{-1}\right) Y \right\}}{\sqrt{(2\pi)^N det(\sigma_M(2\tau) + \sigma_Y(\tau))}} \\
&- \left. \frac{(1-\kappa_T \xi)}{\kappa_L} K^j_M(t) \mathcal{K}(\tau, X) \frac{\exp\left\{ -\frac{1}{2} Y' \left(\sigma_M(2\tau)^{-1} + \sigma_K(\tau)^{-1}\right) Y \right\}}{\sqrt{(2\pi)^N det(\sigma_M(2\tau) + \sigma_K(\tau))}} \right\}^+ dY.
\end{aligned}$$

Find ρ_κ for each $\left\{K^j_L(t+\tau), K^j_M(t), X, \varepsilon^j_i\right\}$ such that

$$\begin{aligned}
&F\left(\rho_k, K^j_L(t+\tau), K^j_M(t), X, \varepsilon^j_i\right) = \left\{ \frac{1}{2}\left(\gamma^j_{0,\tau} + \sigma_0 \varepsilon^j_{\tau,t}\right) - \frac{1}{\kappa_L} K^j_L(t+\tau) + \left(L^j_{2\tau,t+\tau} + L^j_{2\tau,t}\right) \right. \\
&- \left. \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right) \left(c^j + d_0 + d_1 \mu(\tau, X)\right) \right\} \frac{\exp\left\{ -\frac{1}{2} \rho'_\kappa \sigma_M(2\tau)^{-1} \rho_\kappa \right\}}{\sqrt{(2\pi)^N det(\sigma_M(2\tau))}} \\
&- \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right) d_1 \frac{\exp\left\{ -\frac{1}{2} \rho'_\kappa \left(\sigma_M(2\tau)^{-1} + \sigma_Y(\tau)^{-1}\right) \rho_\kappa \right\}}{\sqrt{(2\pi)^N det(\sigma_M(2\tau) + \sigma_Y(\tau))}} \\
&- \frac{(1-\kappa_T \xi)}{\kappa_L} K^j_M(t) \mathcal{K}(\tau, X) \frac{\exp\left\{ -\frac{1}{2} \rho'_\kappa \left(\sigma_M(2\tau)^{-1} + \sigma_K(\tau)^{-1}\right) \rho_\kappa \right\}}{\sqrt{(2\pi)^N det(\sigma_M(2\tau) + \sigma_K(\tau))}} = 0. \qquad (89)
\end{aligned}$$

Cancel the common terms to find

$$F\left(\rho_\kappa, K_L^j(t+\tau), K_M^j(t), X, \varepsilon_i^j\right) = \Bigg\{\frac{1}{2}\left(\gamma_{0,\tau}^j + \sigma_0\varepsilon_{\tau,t}^j\right) - \frac{1}{\kappa_L}K_L^j(t+\tau) + \left(L_{2\tau,t+\tau}^j + L_{2\tau,t}^j\right)$$

$$- \frac{1}{2}\left(\gamma_{1,\tau}^j - \sigma_1\varepsilon_{\tau,t}^j\right)\left(c^j + d_0 + d_1\mu(\tau, X)\right)\Bigg\}\frac{1}{\sqrt{(2\pi)^N\, det(\sigma_M(2\tau))}}$$

$$- \frac{1}{2}\left(\gamma_{1,\tau}^j - \sigma_1\varepsilon_{\tau,t}^j\right) d_1 \frac{\exp\Big\{ - \frac{1}{2}\rho_\kappa'\sigma_Y(\tau)^{-1}\rho_\kappa\Big\}}{\sqrt{(2\pi)^N\, det(\sigma_M(2\tau) + \sigma_Y(\tau))}}$$

$$- \frac{(1-\kappa_T\xi)}{\kappa_L}K_M^j(t)\mathcal{K}(\tau, X)\frac{\exp\Big\{ - \frac{1}{2}\rho_\kappa'\sigma_K(\tau)^{-1}\rho_\kappa\Big\}}{\sqrt{(2\pi)^N\, det(\sigma_M(2\tau) + \sigma_K(\tau))}} = 0. \tag{90}$$

This result leads to the equation (67) in the paper.

This relation is nonlinear in ρ_k, because of the terms like

$$f(\rho_\kappa, \mathcal{KM}) = \exp\Big\{ - \frac{1}{2}\rho_\kappa'\sigma_K(\tau)^{-1}\rho_\kappa\Big\}. \tag{91}$$

However, we do know

$$\frac{\partial F\left(\rho_\kappa, K_L^j(t+\tau), K_M^j(t), X, \varepsilon_i^j\right)}{\partial\rho_\kappa} = \frac{1}{2}\left(\gamma_{1,\tau}^j - \sigma_1\varepsilon_{\tau,t}^j\right) d_1 \frac{\exp\Big\{ - \frac{1}{2}\rho_\kappa'\sigma_Y(\tau)^{-1}\rho_\kappa\Big\}}{\sqrt{(2\pi)^N\, det(\sigma_M(2\tau) + \sigma_Y(\tau))}} 2\rho_\kappa'\sigma_Y(\tau)^{-1}$$

$$+ \frac{(1-\kappa_T\xi)}{\kappa_L}K_M^j(t)\mathcal{K}(\tau, X)\frac{\exp\Big\{ - \frac{1}{2}\rho_\kappa'\sigma_K(\tau)^{-1}\rho_\kappa\Big\}}{\sqrt{(2\pi)^N\, det(\sigma_M(2\tau) + \sigma_K(\tau))}}\frac{1}{2}\rho_\kappa'\sigma_K(\tau)^{-1} \geq 0, \tag{92}$$

so the function is always increasing in the cutoff. In addition, it reaches it minimum at $\rho_\kappa = 0$.

If

$$F\left(0, K_L^j(t+\tau), K_M^j(t), X, \varepsilon_i^j\right) < 0, \tag{93}$$

then there exists exactly two solution ρ_κ. Otherwise the capital constraint is always binding, so that you just take the expected value without a cutoff.

We have the partial derivative for loan desk manager's future capital

$$\frac{\partial F\left(\rho_\kappa, K_L^j(t+\tau), K_M^j(t), X, \varepsilon_i^j\right)}{\partial K_L^j} = -\frac{1}{\kappa_L}\frac{1}{\sqrt{(2\pi)^N\, det(\sigma_M(2\tau))}} < 0. \tag{94}$$

We have the partial derivative for the trading desk's capital

$$\frac{\partial F\left(\rho_\kappa, K_L^j(t+\tau), K_M^j(t), X, \varepsilon_i^j\right)}{\partial K_M^j(t)} = -\frac{(1-\kappa_T\xi)}{\kappa_L}\mathcal{K}(\tau, X)\frac{\exp\left\{-\frac{1}{2}\rho_\kappa'\sigma_K(\tau)^{-1}\rho_\kappa\right\}}{\sqrt{(2\pi)^N det(\sigma_M(2\tau)+\sigma_K(\tau))}} < 0. \tag{95}$$

We have the partial derivative for the yield curve factors.

$$\frac{\partial F\left(\rho_\kappa, K_L^j(t+\tau), K_M^j(t), X, \varepsilon_i^j\right)}{\partial X} = -\frac{1}{2}\left(\gamma_{1,\tau}^j - \sigma_1\varepsilon_{\tau,t}^j\right) d_1 e^{-A^{\mathcal{P}}\tau}\frac{1}{\sqrt{(2\pi)^N det(\sigma_M(2\tau))}}$$
$$+\frac{(1-\kappa_T\xi)}{\kappa_L}K_M^j(t)\mathcal{K}(\tau, X)\left(\sigma_{\mathcal{K}}(\tau)\right)^{-1}\Big(X-\mu_{\mathcal{K}}(\tau)\Big)\frac{\exp\left\{-\frac{1}{2}\rho_\kappa'\sigma_K(\tau)^{-1}\rho_\kappa\right\}}{\sqrt{(2\pi)^N det(\sigma_M(2\tau)+\sigma_K(\tau))}}. \tag{96}$$

This partial derivative is used to find conditions 1 and 2 on page 37 of the paper.

We have the partial derivative for the loan shocks.

$$\frac{\partial F\left(\rho_\kappa, K_L^j(t+\tau), K_M^j(t), X, \varepsilon_i^j\right)}{\partial \varepsilon_i^j} = \left\{\frac{1}{2}\sigma_0 + \sigma_1\left(c^j + d_0 + d_1\mu(\tau, X)\right)\right\}\frac{1}{\sqrt{(2\pi)^N det(\sigma_M(2\tau))}}$$
$$+\frac{1}{2}\sigma_1 d_1\frac{\exp\left\{-\frac{1}{2}\rho_\kappa'\sigma_Y(\tau)^{-1}\rho_\kappa\right\}}{\sqrt{(2\pi)^N det(\sigma_M(2\tau)+\sigma_Y(\tau))}} > 0. \tag{97}$$

The total differential is

$$\begin{aligned} dF\left(\rho_\kappa, K_L^j(t+\tau), K_M^j(t), X, \varepsilon_i^j\right) =& F_{\rho_\kappa}d\rho_\kappa + F_{K_L^j(t+\tau)}dK_L^j(t+\tau) \\ &+ F_{K_M^j(t)}dK_M^j(t) + F_X dX + F_{\varepsilon_i^j}d\varepsilon_i^j \end{aligned} \tag{98}$$

It is straight forward to calculate all these derivatives using the Chebfun add on for Matlab. As a result, we have

$$\begin{aligned} &\frac{\partial\rho_\kappa}{\partial K_L^j(t+\tau)} = -\frac{F_{K_L^j(t+\tau)}}{F_{\rho_\kappa}} > 0, \quad \frac{\partial\rho_\kappa}{\partial K_M^j(t)} = -\frac{F_{K_M^j(t)}}{F_{\rho_\kappa}} > 0, \\ &\frac{\partial\rho_\kappa}{\partial X} = -\frac{F_X}{F_{\rho_\kappa}}, \text{ and } \quad \frac{\partial\rho_\kappa}{\partial\varepsilon_i^j} = -\frac{F_{\varepsilon_i^j}}{F_{\rho_\kappa}} < 0. \end{aligned} \tag{99}$$

These results are the bases for Proposition 5.3 in the text.

In choosing the optimal capital for the loan desk we need to know how this critical value is affected by changes in the loan desk's capital stock.

$$\frac{\partial \rho_\kappa(K_L^j(t+\tau), \Omega_{t,\tau})}{\partial K_L^j(t+\tau)} > 0, \text{ and } \frac{\partial^2 \rho_\kappa(K_L^j(t+\tau), \Omega_{t,\tau})}{\partial^2 K_L^j(t+\tau)} < 0.$$

Thus, the critical function $\rho_\kappa(K_L^j(t+\tau), \Omega_{t,\tau})$ must be concave in $K_L^j(t+\tau)$.

We also want to find the Cholesky decompositions for $\sigma_M(2\tau) = \Sigma_M \Sigma_M'$, $\sigma_M(2\tau) + \sigma_Y(\tau) = \Sigma_{MY}\Sigma_{MY}'$ and $\sigma_M(2\tau) + \sigma_K(\tau) = \Sigma_{MK}\Sigma_{MK}'$. In this case the option value using (55) and (56) in the paper is

$$\begin{aligned}
&\frac{2\chi}{\kappa_L}\mathcal{M}(2\tau, X) E_t\left[p_M(2\tau, Y)\left(r^{j\kappa}_{\tau,t+\tau} - r^{j*}_{\tau,t+\tau}\right)^+\right] = \frac{2\chi\mathcal{M}(2\tau, X)}{\kappa_L\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)}\Bigg\{\frac{CCB(X)}{\mathcal{M}(2\tau, X)} \\
&+\Bigg[\frac{1}{2}\left(\gamma^j_{0,\tau} + \sigma_0\varepsilon^j_{\tau,t}\right) - \frac{1}{\kappa_L}K_L^j(t+\tau) + \left(L^j_{2\tau,t+\tau} + L^j_{2\tau,t}\right) \\
&-\frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)\left(c^j + d_0 + d_1\mu(\tau, X)\right) \\
&-\frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right) d_1\Phi\left(\Sigma_Y(\tau)^{-1}\rho_\kappa\right)\frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau) + \sigma_Y(\tau))}} \\
&-\frac{(1-\kappa_T\xi)}{\kappa_L}K_M^j(t)\mathcal{K}(\tau, X)\Phi\left(\Sigma_K(\tau)^{-1}\rho_\kappa\right)\frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau) + \sigma_K(\tau))}}\Bigg]\Phi\left(\Sigma_M^{-1}\rho_\kappa\right)\Bigg\}. \quad (100)
\end{aligned}$$

As a result, we can calculate the Delta for the option value of the loan desk's capital.

$$
\begin{aligned}
\Delta_\kappa = &\frac{\partial \mathcal{M}(2\tau,X) E_t\left[p_M(2\tau,Y)\left[r^{j\kappa}_{\tau,t+\tau} - r^{j*}_{\tau,t+\tau}\right]^+\right]}{\partial K^j_L(t+\tau)} = \frac{2\chi \mathcal{M}(2\tau,X)}{\kappa_L\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)} \Bigg\{ - \frac{1}{\kappa_L} \qquad (101)\\
&- \Bigg[\frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right) d_1 \frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau)+\sigma_Y(\tau))}} \frac{\partial \Phi\left(\Sigma_Y(\tau)^{-1}\rho_\kappa\right)}{\partial \rho_\kappa}\\
&+ \frac{(1-\kappa_T\xi)}{\kappa_L} K^j_M(t)\mathcal{K}(\tau,X)\frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau)+\sigma_K(\tau))}} \frac{\partial \Phi\left(\Sigma_K(\tau)^{-1}\rho_\kappa\right)}{\partial \rho_\kappa}\Bigg] \Phi\left(\Sigma_M^{-1}\rho_\kappa\right) \frac{\partial \rho_\kappa}{\partial K^j_L(t+\tau)}\Bigg\}\\
&+ \frac{2\chi\mathcal{M}(2\tau,X)}{\kappa_L\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)}\Bigg[-\frac{CCB(X)}{\mathcal{M}(2\tau,X)} + \frac{1}{2}\left(\gamma^j_{0,\tau}+\sigma_0\varepsilon^j_{\tau,t}\right) - \frac{1}{\kappa_L}K^j_L(t+\tau) + \left(L^j_{2\tau,t+\tau} + L^j_{2\tau,t}\right)\\
&- \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)\left(c^j + d_0 + d_1\mu(\tau,X)\right)\\
&- \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right) d_1 \Phi\left(\Sigma_Y(\tau)^{-1}\rho_\kappa\right)\frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau)+\sigma_Y(\tau))}}\\
&- \frac{(1-\kappa_T\xi)}{\kappa_L}K^j_M(t)\mathcal{K}(\tau,X)\Phi\left(\Sigma_K(\tau)^{-1}\rho_\kappa\right)\frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau)+\sigma_K(\tau))}}\Bigg]\frac{\partial\Phi\left(\Sigma_M^{-1}\rho_\kappa\right)}{\partial\rho_\kappa}\frac{\partial\rho_\kappa}{\partial K^j_L(t+\tau)}
\end{aligned}
$$

Note

$$
\frac{\partial \Phi\left(\Sigma_K(\tau)^{-1}\rho_\kappa\right)}{\partial \rho_\kappa} = -\frac{\exp\left\{-\frac{1}{2}\rho'_\kappa\left(\sigma_K(\tau)^{-1}\right)\rho_\kappa\right\}}{\sqrt{(2\pi)^N det(\sigma_K(\tau))}}\left(\sigma_K(\tau)^{-1}\right)\rho_\kappa < 0. \qquad (102)
$$

There are four competing effects of an increase in the capital for the loan desk's:

1. a direct negative effect on the capital constraint, since more capital lessens the capital constraint.

2. a negative effect through the cumulative probability that the capital constraint binds using the probability distribution for the stochastic discount factor $\Phi\left(\Sigma_M^{-1}\rho_\kappa\right)$, by Proposition 5.3.

3. a positive indirect effect through $\Phi\left(\Sigma_Y(\tau)^{-1}\rho_\kappa\right)$, the cumulative probability that the capital constraint binds using the probability distributions for the future factors, Y in (5) from the paper.

4. a positive indirect effect through $\Phi\left(\Sigma_K(\tau)^{-1}\rho_\kappa\right)$, the cumulative probabilities that the capital constraint binds using the probability distribution for the capital for the trading desk (35) in the paper.

To satisfy the second order condition for the loan desk's to issue equity or pay dividends, we must have $\Delta_\kappa < 0$. In addition, to have an interior solution for the issuing of equity or payment of dividends (43) or (44), we make the following assumption:

Assumption 1: The absolute value of the sum of effects 1. and 2. is greater than the sum of effects 3. and 4.

Next we want to know how changes in the current capital of the trading desk influences the expected marginal value of capital for the loan desk (65) in the paper responds to a change in the trading desk's capital, $\Delta_{KM} = \frac{\partial EMV(X, K_M^j(t), K_L^j(t+\tau))}{\partial K_M^j}$.

$$\Delta_{KM} = \frac{\partial \mathcal{M}(2\tau, X) E_t\left[p_M(2\tau, Y)\left[r^{j\kappa}_{\tau,t+\tau} - r^{j*}_{\tau,t+\tau}\right]^+\right]}{\partial K_M^j(t)} = \frac{2\chi \mathcal{M}(2\tau, X)}{\kappa_L\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)}\left\{ -\frac{(1-\kappa_T\xi)}{\kappa_L}\right. \tag{103}$$

$$\times \mathcal{K}(\tau, X)\Phi\left(\Sigma_K(\tau)^{-1}\rho_\kappa\right)\frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau)+\sigma_K(\tau))}}\Phi\left(\Sigma_M^{-1}\rho_\kappa\right)$$

$$+\left[\frac{2\chi}{\kappa_L\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)}\left\{\frac{1}{2}\left(\gamma^j_{0,\tau} + \sigma_0\varepsilon^j_{\tau,t}\right) - \frac{1}{\kappa_L}K_L^j(t+\tau) + \left(L^j_{2\tau,t+\tau} + L^j_{2\tau,t}\right)\right.\right.$$

$$\left. - \frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)\left(c^j + d_0 + d_1\mu(\tau, X)\right)\right\}$$

$$-\frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)d_1\Phi\left(\Sigma_Y(\tau)^{-1}\rho_\kappa\right)\frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau)+\sigma_Y(\tau))}}$$

$$\left.-\frac{(1-\kappa_T\xi)}{\kappa_L}K_M^j(t)\mathcal{K}(\tau, X)\Phi\left(\Sigma_K(\tau)^{-1}\rho_\kappa\right)\frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau)+\sigma_K(\tau))}}\right]\frac{\partial\Phi\left(\Sigma_M^{-1}\rho_\kappa\right)}{\partial\rho_\kappa}\frac{\partial\rho_\kappa(K_L^j(t+\tau), \Omega_{t,\tau})}{\partial K_M^j(t)}$$

$$-\left[\frac{1}{2}\left(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t}\right)d_1\frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau)+\sigma_Y(\tau))}}\right.$$

$$\left.\left.+\frac{(1-\kappa_T\xi)}{\kappa_L}K_M^j(t)\mathcal{K}(\tau, X)\frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau)+\sigma_K(\tau))}}\right]\frac{\partial\Phi\left(\Sigma_K^{-1}\rho_\kappa\right)}{\partial\rho_\kappa}\frac{\partial\rho_\kappa(K_L^j(t+\tau), \Omega_{t,\tau})}{\partial K_M^j(t)}\Phi\left(\Sigma_M^{-1}\rho_\kappa\right)\right\}.$$

The four effects of this change are the same as for a change in the capital for the loan desk's. The key change is that the changes in the three cumulative probabilities for the stochastic discount factor, yield curve factor and capital for the trading desk now have different qualitative effects, based on the size of $\frac{\partial \rho_\kappa}{\partial K_L^j(t+\tau)} > 0$ relative to $\frac{\partial \rho_\kappa}{\partial K_M^j(t)} > 0$ from Proposition (5.3) in the paper.

The impact of a change in the interest rate factors $\Delta_X = \frac{\partial EMV(X, K_M^j(t), K_L^j(t+\tau))}{\partial X}$ is given by

$$\begin{aligned}
\Delta_X &= \frac{2\chi}{\kappa_L\left(\gamma_{1,\tau}^j - \sigma_1 \varepsilon_{\tau,t}^j\right)} \frac{\partial CCB(X)}{\partial X} \\
&+ \frac{2\chi \mathcal{M}(2\tau, X)}{\kappa_L\left(\gamma_{1,\tau}^j - \sigma_1 \varepsilon_{\tau,t}^j\right)} \Bigg\{ \Bigg[\frac{1}{2}\left(\gamma_{0,\tau}^j + \sigma_0 \varepsilon_{\tau,t}^j\right) - \frac{1}{\kappa_L} K_L^j(t+\tau) + \left(L_{2\tau,t+\tau}^j + L_{2\tau,t}^j\right) \\
&- \frac{1}{2}\left(\gamma_{1,\tau}^j - \sigma_1 \varepsilon_{\tau,t}^j\right)\left(c^j + d_0 + d_1 \mu(\tau, X)\right) - \frac{1}{2}\left(\gamma_{1,\tau}^j - \sigma_1 \varepsilon_{\tau,t}^j\right) d_1 \Phi\left(\Sigma_Y(\tau)^{-1} \rho_\kappa\right) \frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau) + \sigma_Y(\tau))}} \\
&- \frac{(1-\kappa_T \xi)}{\kappa_L} K_M^j(t) \mathcal{K}(\tau, X) \Phi\left(\Sigma_K(\tau)^{-1} \rho_\kappa\right) \frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau) + \sigma_K(\tau))}} \Bigg] \\
&\times \left[\frac{\partial \Phi\left(\Sigma_M^{-1} \rho_\kappa\right)}{\partial \rho_\kappa} \frac{\partial \rho_\kappa(K_L^j(t+\tau), \Omega_{t,\tau})}{\partial X} + (X - \mu_{\mathcal{M}}) \Phi\left(\Sigma_M^{-1} \rho_\kappa\right) \right] \\
&+ \Bigg[-\frac{1}{2}\left(\gamma_{1,\tau}^j - \sigma_1 \varepsilon_{\tau,t}^j\right) d_1 \frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau) + \sigma_Y(\tau))}} \frac{\partial \Phi\left(\Sigma_Y^{-1} \rho_\kappa\right)}{\partial \rho_\kappa} \frac{\partial \rho_\kappa(K_L^j(t+\tau), \Omega_{t,\tau})}{\partial X} \\
&- \frac{(1-\kappa_T \xi)}{\kappa_L} K_M^j(t) \mathcal{K}(\tau, X) \frac{\sqrt{det(\sigma_M(2\tau))}}{\sqrt{det(\sigma_M(2\tau) + \sigma_K(\tau))}} \frac{\partial \Phi\left(\Sigma_K^{-1} \rho_\kappa\right)}{\partial \rho_\kappa} \frac{\partial \rho_\kappa(K_L^j(t+\tau), \Omega_{t,\tau})}{\partial X} \\
&- \frac{(1-\kappa_T \xi)}{\kappa_L} K_M^j(t) \mathcal{K}(\tau, X) (X - \mu_{\mathcal{K}}) \Phi\left(\Sigma_K(\tau)^{-1} \rho_\kappa\right) \\
&- \frac{1}{2}\left(\gamma_{1,\tau}^j - \sigma_1 \varepsilon_{\tau,t}^j\right) d_1 e^{-A^{\mathcal{P}}(\tau - t)} \Bigg] \times \Phi\left(\Sigma_M^{-1} \rho_\kappa\right) \Bigg\}
\end{aligned} \tag{104}$$

There are four effects of changes in the yield curve factors.

1. a direct negative effect through the decrease in the cost of the counter cyclical buffer by (80).

2. a negative effect through a change in the conditional expectation of the gross rate of capital growth for the trading desk, $\mathcal{K}(\tau, X)$, for $X > \mu_{\mathcal{K}}$, and the change in the future deposit rate from (51) in the paper resulting from the change in the expected value of the yield curve factors by (4) in the paper.

3. a positive effect through a change in the cumulative probability distribution, $\Phi\left(\Sigma_Y^{-1} \rho_\kappa\right)$

for the yield curve factors, and the gross growth rate of the gross growth rate of trading desk's capital, $\Phi\left(\Sigma_K^{-1}\rho_\kappa\right)$, by Proposition 5.3 in the paper.

4. a positive effect through the cumulative probability distribution, $\Phi\left(\Sigma_M^{-1}\rho_\kappa\right)$, and the conditional expectated value, $\mathcal{M}(2\tau, X)$, of the stochastic discount factor, when $X > \mu_{\mathcal{M}} > \mu_{\mathcal{K}}$ by Proposition 5.3 in the paper.

4.2 The Capital Option Value under Liquidity Constraint

We also need to find the option value of the liquidity constraint to complete the calculations in (65) in the paper. Recall from (22) from the paper.

$$L^j_{l,t+\tau} = \frac{1}{\alpha_\tau}\left[K^j_L(t+\tau) + (1-\alpha_T\xi)K^j_M(t+\tau) + \alpha_R R^j_{t+\tau} - \alpha_{2\tau}\left(L^j_{2\tau,t+\tau} + L^j_{2\tau,t}\right)\right]$$

The following calculations are similar to those for the capital constraint. We use the rules for future capital of the trading desk (36) in the paper and the yield curve factors (17).

The payoff on the option is

$$\begin{aligned}
&p_M(2\tau,Y)\left[L^j_{l,t+\tau} - \frac{1}{2}\left(\gamma^j_{0,\tau} + \sigma_0\varepsilon^j_{\tau,t}\right) - \frac{1}{2}\left(-\gamma^j_{1,\tau} + \sigma_1\varepsilon^j_{\tau,t}\right)\left(c^j + r^D_{\tau,t+\tau}\right)\right] = p_M(2\tau,Y)\left[\frac{1}{\alpha_\tau}K^j_L(t+\tau)\right.\\
&+ \frac{1}{\alpha_\tau}(1-\alpha_T\xi)K^j_M(t+\tau) + \frac{1}{\alpha_\tau}\alpha_R\left(r_0 + r_1X(t+\tau)\right) - \frac{1}{\alpha_\tau}\alpha_{2\tau}\left(L^j_{2\tau,t+\tau} + L^j_{2\tau,t}\right)\\
&\left.- \frac{1}{2}\left(\gamma^j_{0,\tau} + \sigma_0\varepsilon^j_{\tau,t}\right) - \frac{1}{2}\left(-\gamma^j_{1,\tau} + \sigma_1\varepsilon^j_{\tau,t}\right)c^j - \frac{1}{2}\left(-\gamma^j_{1,\tau} + \sigma_1\varepsilon^j_{\tau,t}\right)\left(d_0 + d_1X(t+\tau)\right)\right]\\
&= \left\{\frac{1}{\alpha_\tau}K^j_L(t+\tau) - \frac{1}{\alpha_\tau}\alpha_{2\tau}\left(L^j_{2\tau,t+\tau} + L^j_{2\tau,t}\right) + \frac{1}{\alpha_\tau}\alpha_R r_0 - \frac{1}{2}\left(\gamma^j_{0,\tau} + \sigma_0\varepsilon^j_{\tau,t}\right)\right.\\
&\left.- \frac{1}{2}\left(-\gamma^j_{1,\tau} + \sigma_1\varepsilon^j_{\tau,t}\right)\left(c^j + d_0\right) + \left[\frac{1}{\alpha_\tau}\alpha_R r_1 - \frac{1}{2}\left(-\gamma^j_{1,\tau} + \sigma_1\varepsilon^j_{\tau,t}\right)d_1\right]\mu(X)\right\}p_M(2\tau,Y)\\
&+ \left[\frac{1}{\alpha_\tau}\alpha_R r_1 - \frac{1}{2}\left(-\gamma^j_{1,\tau} + \sigma_1\varepsilon^j_{\tau,t}\right)d_1\right]p_M(2\tau,Y)Y_\tau\\
&+ \frac{1}{\alpha_\tau}(1-\alpha_T\xi)K^j_M(t)p_M(2\tau,Y)\exp\left\{\int_t^{t+\tau}\left[\mathcal{C}_1(\tau) + \mathcal{C}_2(\tau)X(s) + \frac{1}{2}X(s)'\mathcal{C}_3(\tau)X(s)\right]ds\right.\\
&\left.+ \int_t^{t+\tau_i}\left[\mathcal{C}_4(\tau) + \mathcal{C}_5(\tau)X(s)\right]d\epsilon_s\right\}
\end{aligned}$$

$$
= \left\{ \frac{1}{\alpha_\tau} K_L^j(t+\tau) - \frac{\alpha_{2\tau}}{\alpha_\tau}\left(L_{2\tau,t+\tau}^j + L_{2\tau,t}^j\right) - \frac{1}{2}\left(\gamma_{0,\tau}^j + \sigma_0 \varepsilon_{\tau,t}^j\right) - \frac{1}{2}\left(-\gamma_{1,\tau}^j + \sigma_1 \varepsilon_{\tau,t}^j\right)\left(c^j + d_0 + d_1 \mu(X)\right)
$$

$$
+ \frac{\alpha_R}{\alpha_\tau}\left(r_0 + r_1 \mu(X)\right) \right\} \frac{\exp\left\{ -\frac{1}{2} Y' \sigma_M(2\tau)^{-1} Y \right\}}{\sqrt{(2\pi)^N \, det(\sigma_M(2\tau))}}
$$

$$
+ \left[\frac{\alpha_R}{\alpha_\tau} r_1 - \frac{1}{2}\left(-\gamma_{1,\tau}^j + \sigma_1 \varepsilon_{\tau,t}^j\right) d_1 \right] \frac{\exp\left\{ -\frac{1}{2} Y' \left(\sigma_M(2\tau)^{-1} + \sigma_Y(\tau)^{-1}\right) Y \right\}}{\sqrt{(2\pi)^N \, det(\sigma_M(2\tau) + \sigma_Y(\tau))}}
$$

$$
+ \frac{(1-\alpha_T \xi)}{\alpha_\tau} K_M^j(t) \mathcal{K}(\tau, X) \frac{\exp\left\{ -\frac{1}{2} Y' \left(\sigma_M(2\tau)^{-1} + \sigma_K(\tau)^{-1}\right) Y \right\}}{\sqrt{(2\pi)^N \, det(\sigma_M(2\tau) + \sigma_K(\tau))}}.
$$

The option value of the liquidity constraint becomes

$$
E_t\left[p_M(2\tau, Y)\left[r_{\tau,t+\tau}^{jl} - r_{\tau,t+\tau}^{j*} \right]^+ \right] = \frac{2\chi}{\alpha_\tau\left(-\gamma_{1,\tau}^j + \sigma_1 \varepsilon_{\tau,t}^j\right)} \int_{-\infty}^{\infty} \left\{ \left\{ \frac{1}{\alpha_\tau} K_L^j(t+\tau) - \frac{\alpha_{2\tau}}{\alpha_\tau}\left(L_{2\tau,t+\tau}^j + L_{2\tau,t}^j\right) \right.\right.
$$

$$
\left. - \frac{1}{2}\left(\gamma_{0,\tau}^j + \sigma_0 \varepsilon_{\tau,t}^j\right) - \frac{1}{2}\left(-\gamma_{1,\tau}^j + \sigma_1 \varepsilon_{\tau,t}^j\right)\left(c^j + d_0 + d_1 \mu(X)\right) + \frac{\alpha_R}{\alpha_\tau}\left(r_0 + r_1 \mu(X)\right) \right\}
$$

$$
\times \frac{\exp\left\{ -\frac{1}{2} Y' \sigma_M(2\tau)^{-1} Y \right\}}{\sqrt{(2\pi)^N \, det(\sigma_M(2\tau))}}
$$

$$
+ \left[\frac{\alpha_R}{\alpha_\tau} r_1 - \frac{1}{2}\left(-\gamma_{1,\tau}^j + \sigma_1 \varepsilon_{\tau,t}^j\right) d_1 \right] \frac{\exp\left\{ -\frac{1}{2} Y' \left(\sigma_M(2\tau)^{-1} + \sigma_Y(\tau)^{-1}\right) Y \right\}}{\sqrt{(2\pi)^N \, det(\sigma_M(2\tau) + \sigma_Y(\tau))}}
$$

$$
\left. + (1-\alpha_T \xi) K_M^j(t) \mathcal{K}(\tau, X) \frac{\exp\left\{ -\frac{1}{2} Y' \left(\sigma_M(2\tau)^{-1} + \sigma_K(\tau)^{-1}\right) Y \right\}}{\sqrt{(2\pi)^N \, det(\sigma_M(2\tau) + \sigma_K(\tau))}} \right\}^+ dY.
$$

Find ρ_l for each $\left\{K_L^j(t+\tau), K_M^j(t), X, \varepsilon_i^j\right\}$ such that

$$G\left(\rho_l, K_L^j(t+\tau), K_L^j(t+\tau), K_M^j(t), X, \varepsilon_i^j\right) = \Bigg\{\frac{1}{\alpha_\tau}K_L^j(t+\tau) - \frac{\alpha_{2\tau}}{\alpha_\tau}\left(L_{2\tau,t+\tau}^j + L_{2\tau,t}^j\right)$$
$$-\frac{1}{2}\left(\gamma_{0,\tau}^j + \sigma_0\varepsilon_{\tau,t}^j\right) - \frac{1}{2}\left(-\gamma_{1,\tau}^j + \sigma_1\varepsilon_{\tau,t}^j\right)\left(c^j + d_0 + d_1\mu(X)\right) + \frac{\alpha_R}{\alpha_\tau}\left(r_0 + r_1\mu(X)\right)\Bigg\} \tag{105}$$
$$\frac{1}{\sqrt{(2\pi)^N\, det(\sigma_M(2\tau))}}\exp\left\{-\frac{1}{2}\rho_l'\sigma_M(2\tau)^{-1}\rho_l\right\}$$
$$+\frac{\left[\frac{\alpha_R}{\alpha_\tau}r_1 - \frac{1}{2}\left(-\gamma_{1,\tau}^j + \sigma_1\varepsilon_{\tau,t}^j\right)d_1\right]}{\sqrt{(2\pi)^N\, det(\sigma_M(2\tau)+\sigma_Y(\tau))}}\exp\left\{-\frac{1}{2}\rho_l'\left[\sigma_M(2\tau)^{-1} + \sigma_Y(\tau)^{-1}\right]\rho_l\right\}$$
$$+\frac{(1-\alpha_T\xi)}{\alpha_\tau}\frac{K_M^j(t)\mathcal{K}(\tau, X)}{\sqrt{(2\pi)^N\, det(\sigma_M(2\tau)+\sigma_K(\tau))}}\exp\left\{-\frac{1}{2}\rho_l'\left[\sigma_M(2\tau)^{-1} + \sigma_K(\tau)^{-1}\right]\rho_l\right\}. \tag{106}$$

The Greeks Δ_l and Γ_l are the same as in the case of the capital constraint with κ replaced by l.

As a result the option value of capital is given by

$$\mathcal{M}(2\tau, X)E_t\left[p_M(2\tau, Y)\left[r_{\tau,t+\tau}^{jl} - r_{\tau,t+\tau}^{j*}\right]^+\right] = \frac{2\chi\mathcal{M}(2\tau, X)}{\alpha_\tau\left(\gamma_{1,\tau}^j - \sigma_1\varepsilon_{\tau,t}^j\right)}\Bigg\{\Bigg\{\frac{1}{2}\left(\gamma_{0,\tau}^j + \sigma_0\varepsilon_{\tau,t}^j\right)$$
$$-\frac{1}{\alpha_\tau}K_L^j(t+\tau) + \frac{\alpha_{2\tau}}{\alpha_\tau}\left(L_{2\tau,t+\tau}^j + L_{2\tau,t}^j\right) - \frac{1}{2}\left(\gamma_{1,\tau}^j - \sigma_1\varepsilon_{\tau,t}^j\right)\left(c^j + d_0 + d_1\mu(X)\right)$$
$$-\frac{\alpha_R}{\alpha_\tau}\left(r_0 + r_1\mu(X)\right)\Bigg\} \times \Phi\left(\Sigma_M^{-1}\rho_l\right) + \left[\frac{\alpha_R}{\alpha_\tau}r_1 - \frac{1}{2}\left(-\gamma_{1,\tau}^j + \sigma_1\varepsilon_{\tau,t}^j\right)d_1\right]\Phi\left(\left(\Sigma_M(\tau)^{-1} + \Sigma_Y(\tau)^{-1}\right)\rho_l\right)$$
$$+\frac{(1-\alpha_T\xi)}{\alpha_\tau}K_M^j(t)\mathcal{K}(\tau, X)\Phi\left(\left(\Sigma_M(\tau)^{-1} + \Sigma_K(\tau)^{-1}\right)\rho_l\right)\Bigg\}. \tag{107}$$

The option value of the loan desk's capital under both the capital (100) and liquidity constraints (107) have the same functional form when one uses the parameters of each constraint. Thus, the comparative statics for each option value are the same as in the previous subsection.

The last term we have to calculate for the expected marginal value of the loan desk's capital is the marginal effect of the deposit rate.

$$E_t\left[p_M(2\tau,Y)\left[r^D(t+\tau)\tau\right]\right]=E_t\left[p_M(2\tau,Y)\left(d_0+d_1X(t+\tau)\right)\tau\right]$$
$$=E_t\left[p_M(2\tau,Y)\left(d_0+d_1\left[e^{-A^{\mathcal{P}}\tau}X(t)-\left(I-e^{-A^{\mathcal{P}}\tau}\right)\left(A^{\mathcal{P}}\right)^{-1}\gamma^{\mathcal{P}}+\int_t^{\tau}e^{-A^{\mathcal{P}}(\tau-v)}\Sigma_X d\epsilon_v\right]\right)\tau\right]$$
$$=\left(d_0+d_1\mu(X)+1\right)\tau.$$

In the first step we use the expression for the deposit rate (51) in the paper. In the second step we use the solution for the factors (17). The third step separates the two random expressions. The last step applies the intertemporal rate of substitution (10) from the paper and the price of an τ period zero coupon bond (11) from the paper to evaluate the first term in terms of current factors.

We can now put the three terms together. For any change over one period we have

$$\mathcal{M}(2\tau,X)E_t\left\{p_M(2\tau,Y)\left[r^D(t+\tau)\tau+2\chi Max\left[\frac{1}{\alpha_\tau}\left(r^{jl}_{\tau,t+\tau}-r^{j*}_{\tau,t+\tau}\right)^+;\frac{1}{\kappa_L}\left(r^{j\kappa}_{\tau,t+\tau}-r^{j*}_{\tau,t+\tau}\right)^+\right]\right.\right.$$
$$\left.\left.+(\chi-1)\tau\right]\right\}=\left(d_0+d_1\mu(X)+1\right)\mathcal{M}(2\tau,X)\tau$$
$$+\sum_{i=1}^{S}Pr\left(\varepsilon^j_{\tau,t}=\varepsilon^j_i\right)\frac{2\chi\mathcal{M}(2\tau,X)}{\left(\gamma^j_{1,\tau}-\sigma_1\varepsilon^j_i\right)}Max\left\{\frac{1}{\alpha_\tau}\left\{\left\{\frac{1}{2}\left(\gamma^j_{0,\tau}+\sigma_0\varepsilon^j_{\tau,t}\right)-\frac{1}{\alpha_\tau}K^j_L(t+\tau)\right.\right.\right.$$
$$\left.+\frac{\alpha_{2\tau}}{\alpha_\tau}\left(L^j_{2\tau,t+\tau}+L^j_{2\tau,t}\right)-\frac{1}{2}\left(\gamma^j_{1,\tau}-\sigma_1\varepsilon^j_{\tau,t}\right)\left(c^j+d_0+d_1\mu(X)\right)-\frac{\alpha_R}{\alpha_\tau}\left(r_0+r_1\mu(X)\right)\right\}$$
$$\times\Phi\left(\Sigma_M^{-1}\rho_l\right)+\left[\frac{\alpha_R}{\alpha_\tau}r_1-\frac{1}{2}\left(-\gamma^j_{1,\tau}+\sigma_1\varepsilon^j_{\tau,t}\right)d_1\right]\Phi\left(\Sigma_{MY}^{-1}\rho_l\right)$$
$$\left.+\frac{(1-\alpha_T\xi)}{\alpha_\tau}K^j_M(t)\mathcal{K}(\tau,X)\Phi\left(\Sigma_{MK}^{-1}\rho_l\right)\right\};$$
$$\frac{1}{\kappa_L}\frac{CCB(X)}{\mathcal{M}(2\tau,X)}+\frac{1}{\kappa_L}\left\{\frac{1}{2}\left(\gamma^j_{0,\tau}+\sigma_0\varepsilon^j_{\tau,t}\right)-\frac{1}{\kappa_L}K^j_L(t+\tau)+\left(L^j_{2\tau,t+\tau}+L^j_{2\tau,t}\right)\right.$$
$$\left.-\frac{1}{2}\left(\gamma^j_{1,\tau}-\sigma_1\varepsilon^j_{\tau,t}\right)\left(c^j+d_0+d_1\mu(\tau,X)\right)\right\}\Phi\left(\Sigma_M^{-1}\rho_\kappa\right)$$
$$-\frac{1}{2}\left(\gamma^j_{1,\tau}-\sigma_1\varepsilon^j_{\tau,t}\right)d_1\mathcal{M}(2\tau,X)\Phi\left(\left(\Sigma_M(\tau)^{-1}+\Sigma_Y(\tau)^{-1}\right)\rho_\kappa\right)$$
$$\left.-\frac{(1-\kappa_T\xi)}{\kappa_L}K^j_M(t)\mathcal{K}(\tau,X)\Phi\left(\left(\Sigma_M(2\tau)^{-1}+\Sigma_K(\tau)^{-1}\right)\rho_\kappa\right)\right\}.$$

Here t can be any time period. Also, the shock to loan demand $\varepsilon^j_{\tau,t}$ has a discrete distribution with S values ε^j_i. This corresponds to (75) in the text when the two period loans are ignored. The Δ in the text corresponds to either Δ_κ or Δ_l depending on which constraint (22) or (23) is binding. The same is true for (104) and (101) for the change in the interest rate factors or the trading desk's capital stock.

5 Choice of Capital for the Trading and Loan Desks

We can now discuss how the COO chooses the amount of future capital for the loan desk. We start in a terminal period $t+(n-1)\tau+\tau$ and choose capital for the last period based on the first order condition for choosing new issues of capital or payment of dividends.[8] We use $(\chi-1)$ since most of the time a bank pays dividends but seldom issues equity.

As a result, the capital allocated to the loan desk for time $n\tau$ under (44) in the paper is given by[9]

$$\begin{aligned}
&\mathcal{M}(\tau,X)(\chi-1)\tau-(d_0+d_1\mu(X(t+(n-1)\tau))+1)\,\mathcal{M}(2\tau,X(t+(n-1)\tau))\tau\\
&-\sum_{i=1}^{S}Pr\left(\varepsilon^j_{n\tau,t}=\varepsilon^j_i\right)\frac{2\chi\mathcal{M}(2\tau,X(t+(n-1)\tau))}{\left(\gamma^j_{1,\tau}-\sigma_1\varepsilon^j_i\right)}\Bigg\{\frac{1}{\kappa_L}\frac{CCB(X(t+(n-1)\tau))}{\mathcal{M}(2\tau,X(t+(n-1)\tau))}+\frac{1}{\kappa_L}\Bigg\{\frac{1}{2}\left(\gamma^j_{0,\tau}+\sigma_0\varepsilon^j_{\tau,t}\right)\\
&-\frac{1}{\kappa_L}K^j_L(t+\tau)+\left(L^j_{2\tau,t+\tau}+L^j_{2\tau,t}\right)-\frac{1}{2}\left(\gamma^j_{1,\tau}-\sigma_1\varepsilon^j_{\tau,t}\right)\left(c^j+d_0+d_1\mu(\tau,X(t+(n-1)\tau))\right)\Bigg\}\\
&\times\Phi\left(\Sigma_M^{-1}\rho_\kappa((n-1)\tau)\right)-\frac{1}{2}\left(\gamma^j_{1,\tau}-\sigma_1\varepsilon^j_{\tau,t}\right)d_1\Phi\left(\left(\Sigma_M(\tau)^{-1}+\Sigma_Y(\tau)^{-1}\right)\rho_\kappa((n-1)\tau)\right)\\
&-\frac{(1-\kappa_T\xi)}{\kappa_L}K^j_M(t)\mathcal{K}(\tau,X(t+(n-1)\tau))\Phi\left(\left(\Sigma_M(\tau)^{-1}+\Sigma_K(\tau)^{-1}\right)\rho_\kappa((n-1)\tau)\right)\Bigg\}=0,
\end{aligned} \tag{108}$$

which yields a maximum when

$$\Delta_\kappa<0.$$

Here, Δ_κ is given by (101). This is a fixed point problem that yields the optimal choice of capital in the next to last period, $K^{j*}_L(t+n\tau)$.

[8]We write the period as $t+(n-1)\tau+\tau$ rather than $t+n\tau$, since in general the loan desk's capital is chosen in the previous period $t+(n-j)\tau$ and is available in the next period $t+(n+1-j)\tau$.

[9]To save space we only include the option value of the capital constraint. If The liquidity constraints binds then replace l with κ.

Theorem 5.1. *The bank's choice of capital for the loan desk is optimal when* (108) *holds for the time period* $[(n-1)\tau, n\tau]$ *and* $\Delta_\kappa < 0$.

By (81) the marginal value of capital for the lending desk in period $t+(n-1)\tau$ is

$$\frac{\partial V}{\partial K_L^j(t+(n-1)\tau)} = \mathcal{M}(\tau, X)\left[r^D(t+(n-1)\tau)\tau + \lambda_1(t+(n-1)\tau) + \lambda_2(t+(n-1)\tau)\right]$$
$$+ \mathcal{M}(2\tau, X)E_{t+(n-1)\tau}\left[p_M(2\tau, Y)\left[r^D(t+n\tau)\tau + \lambda_1^*(t+n\tau) + \lambda_2^*(t+n\tau)\right]\right]$$
$$= \mathcal{M}(\tau, X)\left[r^D(t+(n-1)\tau)\tau + \lambda_1(t+(n-1)\tau) + \lambda_2(t+(n-1)\tau) + (\chi-1)\tau\right]. \tag{109}$$

As a result, the choice of capital (44) of the paper in period $t+(n-2)\tau$ is $K_L^j(t+(n-1)\tau)$ satisfies

$$\mathcal{M}(\tau, X)(\chi-1)\tau = \mathcal{M}(2\tau, X)E_{(t+(n-2)\tau)}\Bigg[p_M(2\tau, Y)\bigg(r^D(t+(n-1)\tau)\tau$$
$$+ \lambda_1(t+(n-1)\tau) + \lambda_2(t+(n-1)\tau) + (\chi-1)\tau\bigg)\Bigg]. \tag{110}$$

By(81), (109), and (110) the marginal value of capital in period $t+(n-2)\tau$ is

$$\frac{\partial V}{\partial K_L^j(t+(n-2)\tau)} = \mathcal{M}(\tau, X)\left[r^D(t+(n-2)\tau)\tau + \lambda_1(t+(n-2)\tau) + \lambda_2(t+(n-2)\tau)\right]$$
$$+ \mathcal{M}(2\tau, X)E_{t+(n-2)\tau}\Bigg[p_M(2\tau, Y)\bigg[r^D(t+(n-1)\tau)\tau + \lambda_1^*(t+(n-1)\tau)$$
$$+ \lambda_2^*(t+(n-1)\tau) + (\chi-1)\tau\bigg]\Bigg]$$
$$= \mathcal{M}(\tau, X)\left[r^D(t+(n-2)\tau)\tau + \lambda_1(t+(n-2)\tau) + \lambda_2(t+(n-2)\tau) + (\chi-1)\tau\right] \tag{111}$$

This result has the same form as (109).

As a result, the choice of capital in period $t+(n-3)\tau$ is $K_L^j(t+(n-2)\tau)$

$$(\chi-1)\tau = \mathcal{M}(2\tau, X)E_t\Bigg[p_M(2\tau, Y)\bigg(r^D(t+(n-2)\tau)\tau + \lambda_1(t+(n-2)\tau)$$
$$+ \lambda_2(t+(n-2)\tau) + (\chi-1)\tau\bigg)\Bigg]. \tag{112}$$

Thus, in general the optimal condition for the loan desk's capital is

$$(\chi-1)\tau = \mathcal{M}(2\tau, X)E_t\Bigg[p_M(2\tau, Y)\bigg(r^D(t+\tau)\tau + \lambda_1(t+\tau) + \lambda_2(t+\tau) + (\chi-1)\bigg)\Bigg]. \tag{113}$$

Theorem 5.2. *The bank's choice of capital for the loan desk's is optimal when* (113) *holds for all* n *and* $\Delta_\kappa < 0$.

This corresponds to Proposition (5.4) in the paper.

We can use the optimal condition (113) to see how the changes in the level, slope and curvature of the yield curve impacts the optimal capital of the loan desk.

$$\frac{\partial K_L^j(t+\tau)}{\partial K_M^j(t)} = -\frac{1}{\Delta_\kappa}\frac{\partial \mathcal{M}(2\tau, X)E_t\left[p_M(2\tau, Y)\left[r_{\tau,t+\tau}^{j\kappa} - r_{\tau,t+\tau}^{j*}\right]^+\right]}{\partial K_M^j(t)} = -\frac{\Delta_{KM}}{\Delta_\kappa}, \tag{114}$$

where the partial derivatives are given by (101) and (103).

$$\frac{\partial K_L^j(t+\tau)}{\partial X} = \frac{(\chi-1)\tau}{\Delta_\kappa}\mathcal{M}(2\tau, X)\left(\sigma_{\mathcal{M}}(\tau)\right)^{-1}\Big(X - \mu_{\mathcal{M}}(\tau)\Big) - \frac{\Delta_X}{\Delta_\kappa}, \tag{115}$$

where the partial derivatives are given by (101) and (104). We can also determine how the capital of the loan desk changes over several periods.

Theorem 5.3. *The impulse response of the optimal loan desk's capital to the level, slope and curvature of the term structure is determined by* (114) *and* (115).

Proof. If the level, slope or curvature increases at time t, then the expected percentage change in the trading desk's capital at time $t+\tau$ is $-\left(\sigma_{\mathcal{K}}(\tau)\right)^{-1}\Big(X - \mu_{\mathcal{K}}(\tau)\Big)$ following (34) in the paper. The change in the loan desk's capital at time $t+\tau$ is given by (115). At time $t+2\tau$ the trading desk's expected percentage change in the capital is $-\left(\sigma_{\mathcal{K}}(2\tau)\right)^{-1}\Big(X - \mu_{\mathcal{K}}(2\tau)\Big)$. In addition the change in the loan desk's capital is now

$$\begin{aligned}\frac{\partial E_t\left(K_L^j(t+2\tau)\right)}{\partial X} =& - E_t\left(\frac{\Delta_{KM}(t+2\tau)}{\Delta_\kappa(t+2\tau)}\right)\mathcal{K}(2\tau, X)\left(\sigma_{\mathcal{K}}(2\tau)\right)^{-1}\Big(X - \mu_{\mathcal{K}}(2\tau)\Big)\\ &+ E_t\left(\frac{\partial K_L^j(t+2\tau)}{\partial X_\tau}\right)e^{-A^{\mathcal{P}}(\tau-t)}.\end{aligned}$$

Here, the partial derivatives are given by (114) and (115) at time 2τ.

At time $t+k\tau$ for $k \geq 3$ the trading desk's expected percentage change in the capital is

$-\left(\sigma_{\mathcal{K}}(k\tau)\right)^{-1}\left(X-\mu_{\mathcal{K}}(k\tau)\right)$, while the change in the loan desk's capital is

$$E_t\left(\frac{\partial K_L^j(t+k\tau)}{\partial X}\right) = -E_t\left(\frac{\Delta_{KM}(t+k\tau)}{\Delta_\kappa(t+k\tau)}\right)\mathcal{K}(k\tau, X)\left(\sigma_{\mathcal{K}}(k\tau)\right)^{-1}\left(X-\mu_{\mathcal{K}}(k\tau)\right) + E_t\left(\frac{\partial K_L^j(t+k\tau)}{\partial X_{(k-1)\tau}}\right)e^{-A^{\mathcal{P}}((k-1)\tau-t)}.$$

■

5.1 Optimal K_M^j

To solve the COO's problem we use the first order conditions (45) to (47) in the paper. Here, the marginal value of the trading desk's capital is given by

$$\frac{\partial V}{\partial K_M^j(t)} = \mathcal{M}(\tau, X)\left[\mathcal{K}(\tau, X) + \xi\frac{r^p}{2\bar{D}}\left[\bar{D} - \xi K_M^j(t)\right] + (1-\xi)r^D(t)\tau + (1-\xi\kappa_T)\lambda_1(t) + (1-\xi\alpha_T)\lambda_2(t)\right] + \mathcal{M}(2\tau, X)E_t\left[p_M(2\tau, Y)\frac{\partial V}{\partial K_M^j(t+\tau)}\right].$$

In addition, the marginal value of the loan desk is given by (81) given the optimal choice of capital for the trading desk.

$$\frac{\partial V}{\partial K_L^j(t)} - EMV(X, K_M^j, K_L^j(\tau, K_M^j, X)) = \mathcal{M}(\tau, X)\left[r^D(t)\tau + \lambda_1(t) + \lambda_2(t)\right]. \tag{116}$$

Now substitute the marginal value of capital for the loan desk into the marginal value for the trading desk to yield

$$\frac{\partial V}{\partial K_M^j(t)} = \frac{\partial V}{\partial K_L^j(t)} - \left[EMV(X, K_M^j, K_L^j(\tau, K_M^j, X)) - \mathcal{M}(2\tau, X)E_t\left[p_M(2\tau, Y)\frac{\partial V}{\partial K_M^j(t+\tau)}\right]\right] + \mathcal{M}(\tau, X)\left[\mathcal{K}(\tau, X) + \xi\frac{r^p}{2\bar{D}}\left[\bar{D} - \xi K_M^j(t)\right] - \xi r^D(t)\tau - \xi\kappa_T\lambda_1(t) - \xi\alpha_T\lambda_2(t)\right]. \tag{117}$$

If the bank has both a trade and loan desk in the future, then $\frac{\partial V}{\partial K_M^j(t+\tau)} - \frac{\partial V}{\partial K_L^j(t+\tau)} = 0$, then

$$\frac{\partial V}{\partial K_M^j(t)} - \frac{\partial V}{\partial K_L^j(t)} = \mathcal{M}(\tau, X)\left[\mathcal{K}(\tau, X) + \xi\frac{r^p}{2\bar{D}}\left[\bar{D} - \xi K_M^j(t)\right] - \xi r^D(t)\tau - \xi\kappa_T\lambda_1(t) - \xi\alpha_T\lambda_2(t)\right]. \tag{118}$$

This leads to (72) in the paper under (47) in the paper. (73) in the paper follows from setting this equation to zero using (47) in the paper.

Consider this condition at K_M^{jN} given by (71) in the paper.

$$\begin{aligned}\frac{\partial V}{\partial K_M^j(t)}-\frac{\partial V}{\partial K_L^j(t)} &= \mathcal{M}(\tau,X)\left[\mathcal{K}(\tau,X)+\xi\frac{r^p}{2\bar{D}}\left[\bar{D}-\xi K_M^{jN}(t)\right]-\xi r^D(t)\tau-\xi\kappa_T\lambda_1(t)-\xi\alpha_T\lambda_2(t)\right]\gtreqless 0\\ &=\left[\mathcal{K}(\tau,X)+\xi\frac{r^p}{2}-\xi^2\frac{r^p}{2\bar{D}}\bar{D}\left[\frac{2}{\xi^2 r^p}\mathcal{K}(\tau,X)+1\right]-\xi r^D(t)\tau-\xi\kappa_T\lambda_1(t)-\xi\alpha_T\lambda_2(t)\right]\gtreqless 0\\ &=\left[\mathcal{K}(\tau,X)-\mathcal{K}(\tau,X)+\xi\frac{r^p}{2}-\xi^2\frac{r^p}{2}-\xi r^D(t)\tau-\xi\kappa_T\lambda_1(t)-\xi\alpha_T\lambda_2(t)\right]\gtreqless 0\\ &\Rightarrow r^p(1-\xi)\gtreqless 2\left[r^D(t)\tau+\kappa_T\lambda_1(t)+\alpha_T\lambda_2(t)\right].\end{aligned} \tag{119}$$

This corresponds to (75) in the paper.

Suppose the capital constraint is binding, then

$$\begin{aligned}\frac{\partial V}{\partial K_M^j(t)}-\frac{\partial V}{\partial K_L^j(t)} &= \mathcal{K}(\tau,X)+\xi\frac{r^p}{2\bar{D}}\left[\bar{D}-\xi K_M^j(t)\right]-\xi(d_0+d_1X)-\frac{2\xi\chi\tau}{\kappa_L\left(\gamma_{1,\tau}^j-\sigma_1\varepsilon_{t,t}^j\right)}\left\{\frac{\gamma_{0,\tau}^j+\sigma_0\varepsilon_{t,t}^j}{2}\right.\\ &\quad\left.-\frac{1}{\kappa_L}\left[K^j(t)-\kappa_T\xi K_M^j(t)-c_b\left(\frac{\bar{P}_{3\tau,t}}{P_{3\tau,t}}-1\right)^+\right]-\frac{1}{2}\left(\gamma_{1,\tau}^j-\sigma_1\varepsilon_{t,t}^j\right)\left(c^j+d_0+d_1X\right)\right\}=0\\ \Rightarrow K_M^j(t) &= \frac{2\kappa_L^2\left(\gamma_{1,\tau}^j-\sigma_1\varepsilon_{t,t}^j\right)\bar{D}}{\xi^2\left[2\kappa_T\chi\bar{D}\tau+r^p\kappa_L^2\left(\gamma_{1,\tau}^j-\sigma_1\varepsilon_{t,t}^j\right)\right]}\left(\mathcal{K}(\tau,X)+\frac{r^p}{2}-\xi(d_0+d_1X)\right)\\ &\quad-\frac{4\xi\chi\kappa_L\bar{D}\tau}{\xi^2\left[2\kappa_T\chi\bar{D}\tau+r^p\kappa_L^2\left(\gamma_{1,\tau}^j-\sigma_1\varepsilon_{t,t}^j\right)\right]}\left\{\frac{\gamma_{0,\tau}^j+\sigma_0\varepsilon_{t,t}^j}{2}-\frac{1}{\kappa_L}\left[K^j(t)-c_b\left(\mathcal{P}(t,X)-1\right)^+\right]\right.\\ &\quad\left.-\frac{1}{2}\left(\gamma_{1,\tau}^j-\sigma_1\varepsilon_{t,t}^j\right)\left(c^j+d_0+d_1X\right)\right\}.\end{aligned}$$

This corresponds to (78) in the paper.

We can also find how changes in factors influence the capital for the trading desk.

$$\begin{aligned}\frac{\partial K_M^j(t)}{\partial X} &= -\frac{2\kappa_L^2\left(\gamma_{1,\tau}^j-\sigma_1\varepsilon_{t,t}^j\right)\bar{D}}{\xi^2\left[2\kappa_T\chi\bar{D}\tau+r^p\kappa_L^2\left(\gamma_{1,\tau}^j-\sigma_1\varepsilon_{t,t}^j\right)\right]}\left(\mathcal{K}(\tau,X)\left(\sigma_{\mathcal{K}}(\tau)\right)^{-1}\left(X-\mu_{\mathcal{K}}(\tau)\right)+\xi d_1\right)\\ &\quad-\frac{4\xi\chi\kappa_L\bar{D}\tau}{\xi^2\left[2\kappa_T\chi\bar{D}\tau+r^p\kappa_L^2\left(\gamma_{1,\tau}^j-\sigma_1\varepsilon_{t,t}^j\right)\right]}\left\{\frac{1}{\kappa_L}c_b\left(\mathcal{P}(t,X)b_{s\tau}\mid X<\bar{X}\right)-\frac{1}{2}\left(\gamma_{1,\tau}^j-\sigma_1\varepsilon_{t,t}^j\right)d_1\right\}.\end{aligned} \tag{120}$$

If the capital constraint is not binding, then

$$\begin{aligned}\frac{\partial V}{\partial K_M^j(t)} - \frac{\partial V}{\partial K_L^j(t)} &= \mathcal{K}(\tau, X) + \xi \frac{r^p}{2\bar{D}} \left[\bar{D} - \xi K_M^j(t)\right] - \xi(d_0 + d_1 X) = 0 \\ \Rightarrow K_M^j(t) &= \frac{2\bar{D}}{\xi^2 r^p} \left(\mathcal{K}(\tau, X) + \xi \frac{r^p}{2} - \xi(d_0 + d_1 X) \right).\end{aligned} \tag{121}$$

The impact of X is

$$\frac{\partial K_M^j(t)}{\partial X} = - \frac{2\bar{D}}{\xi^2 r^p} \left(\mathcal{K}(\tau, X) \left(\sigma_{\mathcal{K}}(\tau)\right)^{-1} \left(X - \mu_{\mathcal{K}}(\tau) \right) + \xi d_1 \right). \tag{122}$$

6 Calibrating The Bank Parameters

This section explains how the parameters of the simulation in the paper are determined. First we set the parameters for the liquidity (19) and capital (21) constraints. We then determine the parameters for the bank using the 500 largest commercial banks in the United States from Quarter I 2001 to Quarterly IV 2007. Finally, we provide evidence on the relation between monetary policy and the yield curve factors.

6.1 Regulatory Constraints

To determine the parameters for the regulatory constraint we use Michael King (2010, pp. 10-11) who provides a simplified model of the NSFR.

$$NSFR = \frac{Equity + Debt_{>1yr} + Liabs_{>1yr} + 0.85 StableDeposits_{<1yr} + 0.70 OtherDeposits}{0.05 GovtDebt + 0.50 CorpLoans_{<1yr} + 0.85 RetLoans_{<1yr} + OtherAssets} > 1. \tag{123}$$

which implies

$$\begin{aligned}&Equity + Debt_{>1yr} + Liabs_{>1yr} + 0.85 StableDeposits_{<1yr} + 0.70 OtherDeposits \\ &> 0.05 GovtDebt + 0.50 CorpLoans_{<1yr} + 0.85 RetLoans_{<1yr} + OtherAssets.\end{aligned}$$

In our model we have

$$
\begin{aligned}
K^j &\equiv Equity \\
L^j_\tau &\equiv 0.50 CorpLoans_{<1yr} + 0.85 RetLoans_{<1yr} \\
L^j_{2\tau,t} + L^j_{2\tau,t-\tau} &\equiv OtherAssets \\
D^j &\equiv StableDeposits_{<1yr} \\
OL^j &\equiv Debt_{>1yr} + Liabs_{>1yr} + 0.70 OtherDeposits \\
\omega K^j_M &\equiv GovtDebt
\end{aligned}
$$

These variables are stated relative to total assets in Table 12 for the average of the 574 bank holding companies with more than \$1 billion in assets as of March 31, 2015. Bank Capital is Total Equity capital. Short term loans, L^j_τ is the Commercial and Industrial Loans plus Loans to Individuals minus Automobile loans. We subtract off Auto Loans since they tend to be longer than one year. For GovtDebt, ωK^j_M we use U. S. Government Securities plus Securities Issued by states and Political Subdivisions. For bank reserves, R^j_t we take Currency and Coin in Domestic Offices plus Balances due from Federal Reserve Banks. For $StableDeposits_{<1yr}$ we use Transaction Accounts + NonTransaction Accounts - Total Time Deposits.

In terms of our model of the NSFR we take[10]

$$
K^j_t \geq 0.05\omega K^j_M + 0.675 L^j_\tau + L^j_{2\tau,t} + L^j_{2\tau,t-\tau} - 0.85 D^j - \text{ Other Liabilities } . \tag{124}
$$

The 0.68 is based on equal weight for corporate and retail loans.

From the balance sheet constraint we have

$$
D^j = R^j + L^j_\tau + 0.675 L^j_{2\tau,t} + L^j_{2\tau,t-\tau} - \text{ Other Liabilities } - K^j_t .
$$

We place the same weight on short term loans. Otherwise these loans would reduce capital requirements.

$$
\begin{aligned}
K^j_t \geq & 0.027\left(T^j_{\tau,t}P_{\tau,t} + T^j_{2\tau,t}P_{2\tau,t} + T^j_{3\tau,t}P_{3\tau,t} + T^j_{4\tau,t}P_{4\tau,t}\right) + 0.055 L^j_\tau + 0.08\left(L^j_{2\tau,t} + L^j_{2\tau,t-\tau}\right) \\
& - 0.459 R^j - 0.069 \text{ Other Liabilities } .
\end{aligned} \tag{125}
$$

[10] Since commercial and retail loans are consolidated in the model we take the weight to be the average of 0.5 and 0.85.

$$K_t^j \geq \alpha_\tau L_{\tau,t}^j + \alpha_{2\tau}\left(L_{2\tau,t}^j + L_{2\tau,t-\tau}^j\right) - \alpha_R R_t^j + \alpha_T\left(T_{\tau,t}^j P_{\tau,t} + T_{2\tau,t}^j P_{2\tau,t} + T_{3\tau,t}^j P_{3\tau,t} + T_{4\tau,t}^j P_{4\tau,t}\right). \quad (126)$$

The parameters for the regulatory constraints are given in Table 13. This Table corresponds to Table 10 in the paper.

Table 12: . Average Accounting Ratios for Commercial Banks with more than a Billion $ (March 31, 2015).

Variable	%
$\frac{K_t^j}{A}$	11.22
$\frac{L_{\tau,t}^j}{A}$	15.59
$\frac{L_{2\tau,t}^j + L_{2\tau,t-\tau}^j}{A}$	58.82
$\frac{R_t^j}{A}$	10.4
$\frac{\left(T_{2\tau,t}^j P_{2\tau,t} + T_{3\tau,t}^j P_{3\tau,t}\right)}{A}$	15.19
$\frac{D^j}{A}$	56.24
$\frac{OL^j}{A}$	32.54

Table 13: Parameters for Regulatory Constraints (125) and (126).

α_τ	$\alpha_{2\tau}$	α_R	α_T	κ_T	κ_L	b
0.055	0.08	0.459	0.027	0.0	0.08	0.02

6.2 Benchmark Parameters for Banking Model

Next we identify the parameters for the bank. To set the deposit rate parameters we use the data from 500 largest U. S. Commercial Banks from 2001 Quarter I to 2007 Quarter IV. To obtain the deposit rate parameters in

$$r_{\tau,t}^{Dj} = d_0 + d_1 X_1(t) + \epsilon_{rD,j,t}.$$

A panel regression of the interest expense on deposits relative to the first interest rate factor $X_1(t)$ is provided in Table 14. Bank fixed effects are included in the regression. We use interest expense on total deposits relative to total deposits, which has a mean value of 0.0078. We use the estimates in Table 14 for the deposit rate parameters in Table 16.

Table 14: . Panel Regression of deposit rate on first state variable.

Variable/Statistic	$Constant$	$X(1)$
β	0.0111	0.0282
T-Stat	(7.4196)	(4.6694)
$adj\bar{R}^2$	0.2487	

Table 15: . Panel Regression of bank reserves on first state variable.

Variable/Statistic	$Constant$	$X_1(t)$
β	0.1340	0.3936
T-Stat	17.4619	12.6935
$\bar{R}^2$	0.5715	

Table 16: Parameters for Deposits (51) and Reserves (52) in the paper.

d_0	d_1	r_0	r_1	c^j
0.0111	0.0282	0.1340	0.3936	0.0378

To find parameters for the bank reserves we run the panel regression with bank fixed effects, and using the same set of banks and time period.

$$R_t^j = r_0 + r_1 X_1(t) + \epsilon_{R,j,t}.$$

We estimate this relation using a panel regression with bank fixed effects in Table 15 for the 500 largest U. S. Commercial Banks from 2001 Quarter I to 2007 Quarter IV. The dependent variable is cash balances plus deposits due from other depository institutions including the Federal Reserve. These estimates are included in the Table 16 for the parameters used in the simulations. In the paper this Table corresponds to Table 7.

For reserves we use cash balances plus deposits due from other depository institutions divided by total assets for the 500 largest commercial banks using data from 2001 to 2007. This number includes balances due from depository institutions which is not part of reserves. Yet, cash items in process of collection plus balances due from Federal Reserve divided by total assets, 10.83%, is 80% of cash balances plus deposits due from other depository institutions divided by total assets for the commercial banks with more than one Billion $ as of March $31, 2015$.[11]

We estimate the demand for loans (48) in the paper.

$$L^{d,j}_{\tau,t} = \gamma^j_{0,\tau} - \gamma^j_{1,\tau} r^j_{\tau,t} + \sigma(r^j_{\tau,t})\varepsilon^j_{\tau,t} \text{ with } \sigma(r^j_{\tau,t}) = \sigma_0 + \sigma_1 r^j_{\tau,t}.$$

To estimate the loan demand we use Commercial and Industrial Loans divided by bank assets for the 500 largest Commercial Banks from 2001 Quarter I to 2007 Quarter IV. Table 17 contains the results for a panel regression with bank fixed effects. We also control for bank size by including the logarithm of bank assets.

Table 17: . Panel Regression of Commercial and Industrial Loans/Assets on Interest Income.

Variable/Statistic	$Constant$	$r^j_{\tau,t}$	$\ln(Assets)$
β	0.373368	−0.029707	−0.017113
T-Stat	22.98413	(4.2635)	15.4891
$adj\bar{R}^2$	0.8739	S.E.	0.0331

The slope of the demand curve is $-\gamma^j_{1,\tau} + \sigma_1\varepsilon^j_{\tau,t}$. As a result, the inverse of the elasticity of demand is

$$\begin{aligned} -\frac{1}{\epsilon} = \frac{\partial r^j_{\tau,t}}{\partial L^{d,j}_{\tau,t}} \frac{L^{d,j}_{\tau,t}}{r^j_{\tau,t}} =& \frac{1}{-\gamma^j_{1,\tau} + \sigma_1\varepsilon^j_{\tau,t}} \frac{\gamma^j_{0,\tau} + \sigma_0\varepsilon^j_{\tau,t} + \left(-\gamma^j_{1,\tau} + \sigma_1\varepsilon^j_{\tau,t}\right) r^j_{\tau,t}}{r^j_{\tau,t}} \\ =& 1 + \frac{\gamma^j_{0,\tau} + \sigma_0\varepsilon^j_{\tau,t}}{r^j_{\tau,t}(-\gamma^j_{1,\tau} + \sigma_1\varepsilon^j_{\tau,t})} < 0 \\ \Rightarrow& MR = \left(1 + \frac{\partial r^j_{\tau,t}}{\partial L^{d,j}_{\tau,t}} \frac{L^{d,j}_{\tau,t}}{r^j_{\tau,t}}\right) r^j_{\tau,t} = r^j_{\tau,t}\left(1 - \frac{1}{\epsilon}\right) = 2r^j_{\tau,t} - \frac{\gamma^j_{0,\tau} + \sigma_0\varepsilon^j_{\tau,t}}{(\gamma^j_{1,\tau} - \sigma_1\varepsilon^j_{\tau,t})}. \end{aligned} \tag{127}$$

[11] We divided the parameters from the level of reserves regression by the total average across all banks and time periods.

Here, $\frac{1}{\epsilon} = -\frac{\partial r^j_{\tau,t}}{\partial L^{d,j}_{\tau,t}} \frac{L^{d,j}_{\tau,t}}{r^j_{\tau,t}}$ is the inverse of elasticity of demand. When loans are not constrained the first order condition is

$$\frac{\partial \pi^j_L}{\partial m^j_{\tau,t}} = \left[2\left(-\gamma^j_{1,\tau} + \sigma_1 \varepsilon^j_{\tau,t}\right) r^j_{\tau,t} - \left(c^j + r^D_{\tau,t}\right)\left(-\gamma^j_{1,\tau} + \sigma_1 \varepsilon^j_{\tau,t}\right) + \gamma^j_{0,\tau} + \sigma_0 \varepsilon^j_{\tau,t}\right] \tau$$

$$= \left(-\gamma^j_{1,\tau} + \sigma_1 \varepsilon^j_{\tau,t}\right)\left[-\left(c^j + r^D_{\tau,t}\right) + 2r^j_{\tau,t} + \frac{\gamma^j_{0,\tau} + \sigma_0 \varepsilon^j_{\tau,t}}{\left(-\gamma^j_{1,\tau} + \sigma_1 \varepsilon^j_{\tau,t}\right)}\right] = 0$$

$$\Rightarrow r^j_{\tau,t} = \frac{1}{2}\left[\left(c^j + r^D_{\tau,t}\right) + \frac{\gamma^j_{0,\tau} + \sigma_0 \varepsilon^j_{\tau,t}}{\left(\gamma^j_{1,\tau} - \sigma_1 \varepsilon^j_{\tau,t}\right)}\right]. \quad (128)$$

We pick the values of the parameters $\gamma^j_{0,\tau}$ and $\gamma^j_{0,\tau}$ using the average data across the 500 U.S. Commercial Banks from Quarter I of 2001 to Quarter IV of 2007. We want to match the average value of the commercial and industrial loans relative to assets $L^j/A^j = 0.1212$, the loan rate of $r^j_{\tau,t} = 0.0643$ using the ratio of interest income and fees to loans for commercial and industrial loans. $c^j = 0.0376$, for the average non-interest expenses divided by total assets. The average interest expense on deposits to total deposit ratio is used to set $r^D_{\tau,t} = 0.0165$. Finally, we want the coefficients to yield the optimal unconstrained loan rate of $r^j_{\tau,t} = 0.0643$ when the uncertainty is zero. This leads to the relations:

1. $\frac{\gamma^j_{0,\tau}}{2\left(\gamma^j_{1,\tau}\right)} = r^j_{\tau,t} - \frac{1}{2}\left(c^j + r^D_{\tau,t}\right) = 0.03719.$

2. $\gamma^j_{0,\tau} = \gamma^j_{1,\tau} r^j_{\tau,t} + L^j = \gamma^j_{1,\tau} r^j_{\tau,t} + L^j = \frac{\gamma^j_{0,\tau}}{2(0.03719)} r^j_{\tau,t} + L^j.$

3. Yields $\gamma^j_{0,\tau} = 0.8972$ and $\gamma^j_{1,\tau} = 12.0621$.

4. The Marginal Revenue is $2r^j_{\tau,t} - \frac{\gamma^j_{0,\tau}}{\gamma^j_{1,\tau}} = 2 \times 0.0643 - \frac{0.1526}{2.0516} = 0.0543.$

Table 18: Parameters for Loan Demand (48) and (49) in the paper.

$\gamma_{0,\tau}$	$\gamma_{1,\tau}$	σ_0	σ_1	z_0	z_1
0.8972	12.0621	0.0331	0.2067	-0.6150	0.00035

If we use the parameters from the regression we have $\gamma_0 = 0.1121$ and $\gamma_1 = 0.030$. However, Marginal revenue turns out to be too big, since $\gamma_0/\gamma_1 = 3.7367$. Consequently, we choose the parameters in Table 18 which are based on the unconstrained profit maximization condition and the loan demand curve.

From the 500 Largest U.S. Commercial banks we have total charge offs to total assets is 0.00566 with a standard deviation of 0.0133. The standard error of the regressions in Table 17 is 0.0331. We want the standard error $\sigma_0 + \sigma_1 r^j_{\tau,t} = 0.0331 + 0.0133 = 0.0464$ so that $\sigma_0 = 0.0331$ and $\sigma_1 = 0.0133/r^j_{\tau,t} = 0.2067$ Consider a two point distribution with payoff z_0 with probability $p = 0.00566$ and z_1 with probability $1 - p$. Suppose we know mean x and variance y, then basic algebra shows

$$z_0 = x - \sqrt{\frac{1-p}{p}y} \text{ and } z_1 = x + \sqrt{\frac{p}{1-p}y}.$$

These leads to the solutions $z_0 = -0.6150$ and $z_1 = 0.0035$. $r^j_{\tau,t}(z_0) = 0.0631$ and $r^j_{\tau,t}(z_1) = 0.0643$ when the interest rate is kept at its optimum. This completes Table 18 which is reproduced in Table 8 of the paper.

6.3 Yields Factors and Macroeconomic Variables

Recent work by Joslin, Priebsch and Singleton (2014) has examined the relation between the principle components of yields data and macroeconomic variables. They find that the level is positively affected by economic growth and inflation.[12] This corresponds to the usual result that interest rates increase during booms to the business cycle and when inflation increases. At the same time a higher level of economic growth leads to a flatter slope for a positive sloped yield curve, since a central bank would want to raise short term interest rates when the economy is expanding too fast. In addition, they find that higher inflation increases the slope of the yield curve by a smaller magnitude relative to economic growth. While the third principle component is not affected by economic growth and inflation, it does have a negative impact on economic growth and inflation. Consequently, higher curvature signals lower economic growth and inflation, which in turn leads to lower level and a larger slope for the yield curve. Thus, there is a significant connection among the latent factors and economic growth and inflation, which corresponds to how the yield curve behaves over the business cycle.

We examine such connections between yields factors and economic variables using our data in the sample period 1990M01-2013M12. Specifically, we base our analysis within the

[12] They use the Chicago Federal Reserve index of economic activity for economic growth.

well established monetary policy framework of the Taylor rule (1993). Since we illustrate the implications of the model using a one factor model, we investigate how a one factor model captures monetary policy. To this end, we first regress the short term 3 Month Treasury yield on the deviation of inflation ΔP_{t-1} from its target ΔP^*, as well as the GDP gap $Y^{gap} = \frac{Y^P - Y}{Y^P}$.

$$r_{3mo,t} = \Delta P_{t-1} + r_t^* + a_{\Delta P}(\Delta P_{t-1} - \Delta P^*) + a_Y Y^{gap}.$$

To measure the Taylor variables we use Real Potential Gross Domestic Product, Billions of Chained 2009 Dollars, Quarterly, Not Seasonally Adjusted from FRED, Y^P and Monthly real GDP from Macroeconomic Analysis http://www.macroadvisers.com/monthly-gdp/, Y. We interpolate the quarterly potential GDP to obtain the monthly observations. For inflation, ΔP we use Consumer Price Index for All Urban Consumers: All Items Less Food and Energy, Change from Year Ago, Index 1982-84=100, Monthly, Seasonally Adjusted from FRED. Finally, we use the 3-month yield to maturity on Treasury security as a proxy for the short term rate.[13] As a result, Y^{gap} is negative during a recession. r_t^* is the natural real rate of interest. In addition, the Taylor principle assumes that $a_{\Delta P} = 0.5$ and $a_Y = 0.5$. As a result a 1% increase in inflation leads to a $1\frac{1}{2}$% increase in the short term interest rate, while a 1% increase in the output gap leads to a $\frac{1}{2}$% increase in short term interest rates.

The results of the Taylor rule regression are in Table 19. Note that for this exercise and all of the following exercises using factors, the t-test statistics are calculated using Newey-West HAC (Heteroskedasticity and Autocorrelation Consistent) estimator because the yields data are typically highly persistent. The regression is run for data from April 1992 to December 2013 because monthly real GDP starts from April 1992. We obtain the expected signs for the impact of inflation and the GDP gap. Yet, the Taylor principle does not hold since a change in the inflation rate leads to a less than 1% increase in short term interest rates, so that the Federal Reserve is less aggressive in combating inflation relative to the Taylor principle. Furthermore, the response of the short rate to inflation rate is not significant.[14]

Because the short rate variation reflects both the level and slope movements of the yield curve, we study how each of the factors responds to the macroeconomic variables using both the empirically constructed factors and the term structure latent factors obtained through

[13] We could also use the Federal Funds rate and obtain similar results.

[14] Note, however, that we use lagged variables in the Taylor-rule regression here as a simple specification to illustrate how yields factors are generally related to the economic variables.

Table 19: . OLS Regression of 3 Month yield on Taylor Rule Variables

Variable/Statistic	Y_{t-1}^{gap}	ΔP_{t-1}
β	0.6005	0.4299
T-Stat	(7.1273)	(1.1759)
$\bar{R}^2$	0.6641	
D-W	0.0993	

Kalman filter within the affine term structure model. Both constructions are presented in the previous subsection.

First we focus on the empirical factors as constructed in the previous section by following Diebold and Li (2006). Recall the level explains 95.5% of the variation of the yield curve, while slope and curvature explain 4.2% and 0.2% of this variation, respectively. We estimate the Taylor rule regressions using the level, slope and curvature factors of the yield curve as the dependent variable and the results are given in Tables 20, 21 and 22. First, the level factor regression yields quantitatively similar results to that using the 3-month short rate in Table 19. A positive and significant response coefficient estimate associated with the output gap suggests that the level factor tends to rise (decline) in response to economic booms (recessions). Although the response coefficient estimate associated with the inflation rate is also consistent with economic intuitions it is again not significant and the magnitude does not satisfy the Taylor principle. The adjusted R-square of the level factor regression turns out to be higher than those of both the slope and curvature factors regressions. Despite a lower adjusted R-square, the slope factor regression also yields results consistent with economic intuitions. The negative and significant response coefficient estimate of the output gap suggests that the Fed tends to cut (raise) the short rate leading to larger (smaller) slope factor during recessions (booms). Again, although the response coefficient estimate of the inflation rate is consistent with Taylor rule it is not significant. Turning to the curvature factor, the regression results reveal that it appears to be significantly positively correlated with the output gap.

For comparison purposes we also run the same Taylor rule regressions using the latent factors obtained in the affine term structure model via the Kalman Filter, and the results are given in Tables 23, 24, and 25. The Taylor rule regressions results are broadly similar to those using the empirical factors. However, the adjusted R-square for the first latent factor that is closely related to the level factor becomes lower than those for the second and third

Table 20: . OLS Regression of Level on Taylor Rule Variables

Variable/Statistic	Y_{t-1}^{gap}	ΔP_{t-1}
β	0.4648	0.2615
T-Stat	(4.1191)	(0.9649)
$\bar{R}^2$	0.4991	
D-W	0.0814	

Table 21: . OLS Regression of Slope on Taylor Rule Variables

Variable/Statistic	Y_{t-1}^{gap}	ΔP_{t-1}
β	-0.1357	-0.1684
T-Stat	(-2.8873)	(-0.6095)
$\bar{R}^2$	0.2134	
D-W	0.1381	

Table 22: . OLS Regression of Curvature on Taylor Rule Variables

Variable/Statistic	Y_{t-1}^{gap}	ΔP_{t-1}
β	0.1438	0.1307
T-Stat	(4.2511)	(0.7840)
$\bar{R}^2$	0.3337	
D-W	0.1417	

factors.

Overall, the above regression results suggest yield curve factors contain useful information about how monetary policy reacts to economic conditions. For example, during economic booms, consistently the level factor rises and the slope factor declines. This suggests that during economic booms when the Fed raises the policy rate, it shifts up the whole yield curve leading to a rise of the level factor, and at the same time, such a policy raises the short end of the yield curve more than the long end so that the yield curve also becomes flatter. These observations suggest that it is entirely possible to use one factor such as the level factor to capture the monetary policy.

Table 23: . OLS Regression of First Latent Variable on Taylor Rule Variables

Variable/Statistic	Y_{t-1}^{gap}	ΔP_{t-1}
β	0.1657	0.0899
T-Stat	(1.2073)	(0.2284)
$\bar{R}^2$	0.0967	
D-W	0.0504	

Table 24: . OLS Regression of Second Latent Variable on Taylor Rule Variables

Variable/Statistic	Y_{t-1}^{gap}	ΔP_{t-1}
β	-0.5046	-0.1869
T-Stat	(-5.2797)	(-1.0252)
$\bar{R}^2$	0.5000	
D-W	0.2640	

Table 25: . OLS Regression of Third Latent Variable on Taylor Rule Variables

Variable/Statistic	Y_{t-1}^{gap}	ΔP_{t-1}
β	0.9083	0.5300
T-Stat	(8.2229)	(1.0401)
$\bar{R}^2$	0.6308	
D-W	0.1591	