Portfolio Decisions in Continuous Time

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Abstract

In this paper we solve an adjusted version of a continuous time asset allocation model that was written and approximated by Campbell, Chacko, Rodriguez, and Viciera (2004) and solved by Chen, Cosimano, and Himonas (2007). We first examine why equations in the original model must be altered to account for the impossibility of a negative expected equity premium, or the excess return over a riskless asset an investor receives for holding a risky asset. Using the new equations, we express the model as an ordinary differential equation which can be solved using the methods developed by CCH.

Contents

1 Introduction

How to optimally invest assets given their riskiness and expected return is one of the fundamental concerns of any investor. Empirical data has shown that short term returns are unpredictable. However, in his book *Asset Pricing: Revised Edition* John Cochrane shows that long term expected returns of the bond and stock markets can be forecasted. John Campbell, George Chacko, Jorge Rodriguez, and Luis Viceira (2004) further this argument and develop a model for determining optimal overall allocation in *Strategic Asset Allocation in a Continuous-Time VAR Model.* However, CCRV were only able to approximate the ODE in their model. Yu Chen, Thomas Cosimano, and Alex Himonas (2007) solve for the ordinary differential equation,

$$H''+\frac{2}{\sigma_\mu^2}\left[\kappa\left(\theta-\mu(t)\right)+\frac{1-\gamma}{\gamma}\rho\sigma_\mu\frac{(\mu(t)-r)}{\sigma_S}\right]H'-\frac{2}{\sigma_\mu^2}\left[\beta\psi+(1-\psi)r+\frac{(1-\psi)}{2\gamma}\left(\frac{\mu(t)-r}{\sigma_S}\right)^2\right]H+\frac{2}{\sigma_\mu^2}\beta^\psi$$
$$=\left[\left(1+\frac{1-\gamma}{(1-\psi)}\right)+\rho^2\frac{(1-\gamma)^2}{\gamma(1-\psi)}\right]\frac{(H')^2}{H(x)}, \tag{1.1}$$

which they can solve by developing a computer program.[1]

The following are three of the processes of the CCRV model that were used to solve for the ODE, (1.1). First, the return on a riskless asset such as a bond is

$$\frac{dB(t)}{B(t)}=rdt, \tag{1.2}$$

the return on a risky asset like a stock is

$$\frac{dS(t)}{S(t)}=\mu(t)dt+\sigma_S d\omega_{S,t}, \tag{1.3}$$

such that the derivative of the instantaneous mean $\mu(t)$ is

$$d\mu(t)=\kappa(\theta-\mu(t))dt+\sigma_\mu d\omega_{\mu,t}. \tag{1.4}$$

Note $\omega_{S,t}$ and $\omega_{\mu,t}$ are Brownian motions.

These equations involve stochastic processes because the returns are being measured in continuous time. Thus the $\sigma_S d\omega_{S,t}$ and $\sigma_\mu d\omega_{\mu,t}$ terms account for the randomness of stocks. A stock is

[1] Throughout this paper, CCRV is used to refer to John Campbell, George Chacko, Jorge Rodriguez, and Luis Viceira (2004) and CCH is used to refer to Yu Chen, Thomas Cosimano, and Alex Himonas (2007).

equally likely to go up or down in the next time period, so stochastic terms are used to reflect this uncertainty. However, when we recall that variables in stochastic calculus are normally distributed, we realize this implies $\frac{dB(t)}{B(t)}$, $\frac{dS(t)}{S(t)}$, and $d\mu(t)$ follow a normal distribution curve. Thus the instantaneous mean, $\mu(t)$, can be negative. Figure 1 is a graph of $\mu(t)$ based on variables determined by Chen, Cosimano, and Himonas (2008) and found in the appendix.

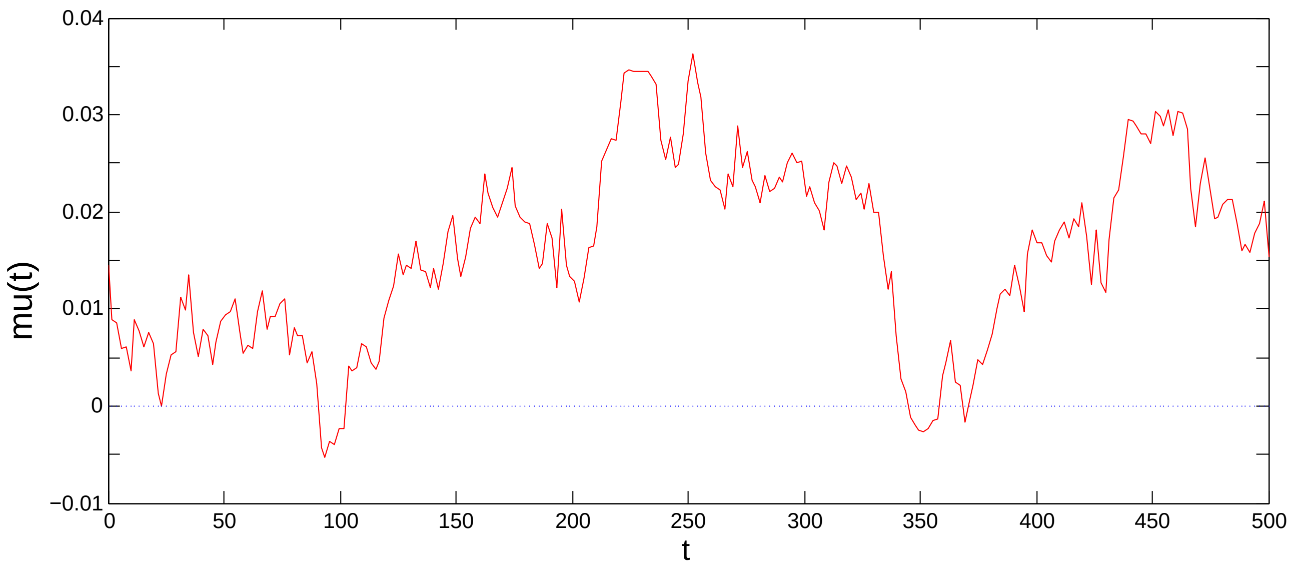

Figure 1

As this graph shows, $\mu(t)$ fluctuates and is usually positive, but it can be negative. When the mean expected return is negative, the mean expected return on risky assets has fallen below the expected return on riskless assets and an arbitrage opportunity has been created. In an arbitrage opportunity, an investor can make money without investing any money or risking losing any money. In the case of negative mean expected returns on stocks, investors can buy bonds with an expectation of a higher return on them than on stocks and face no risk whatsoever in attaining this higher return. If this circumstance were to occur in real life, no investor would ever buy stocks. After all, why would anyone buy a risky asset they will lose money on, when they could buy a riskless asset with a higher return? Obviously people still buy stocks in the real world which implies there is not an expectation of negative mean returns. Thus in order to use CCRV's model and to eliminate the possibility of negative mean expected returns, we will now adjust the equations and solve for the new differential equation by the same methods used to find equation (1.1).

2 Derivations of Equations for the Model

Before we alter $\frac{dS(t)}{S(t)}$ and $d\mu(t)$, note that the instantaneous return on a riskless asset is still

$$\frac{dB(t)}{B(t)} = rdt. \tag{2.5}$$

Notice if this equation is solved for the expected return on a riskless asset, we get

$$\int_0^t \frac{dB(t)}{B(t)} = \int_0^t rdt$$

which equals

$$\ln(B(t)) - \ln(B(0)) = rt.$$

When we simplify this equation using the rules of exponentials and logs that $e^{\ln(x)} = x$ and $\ln(a) - \ln(b) = \ln(a/b)$, we get

$$e^{\ln(B(t)/B(0))} = e^{rt}$$

which simplifies to

$$B(t)/B(0) = e^{rt}. \tag{2.6}$$

Therefore we have

$$B(t) = B(0)e^{rt}. \tag{2.7}$$

Financially, this equation makes sense as the future value of a riskless asset, $B(t)$, is simply the initial value of B at time $t = 0$ times e^{rt}, or the future value factor.[2] Furthermore, the value of B at time 0 is known and it must be positive or else there is an arbitrage opportunity. As the value of the exponential function only returns values greater than zero, the expected return on a riskless asset, $B(t)$, is always positive.

[2]Himonas and Howard (2003) p.131 shows how the continuously compounded interest factor is derived from the discrete case. Specifically, in the discrete case, $(1 + \frac{r}{n})^{nt}$ is the interest factor where r is the interest rate, n is the number of periods per year, and t is the number of years. If we make a change of variables and take the limit of this equation as n goes to infinity, we get $\lim_{n\to\infty}(1 + \frac{r}{n})^{nt} = e^{rt}$, or the continuously compounded interest factor.

In the case of the risky asset, we will now subtract the instantaneous return on the riskless asset, $\frac{dB(t)}{B(t)}$, from the instantaneous return on the risky asset to get

$$\frac{dS(t)}{S(t)} - \frac{dB(t)}{B(t)} = \mu(t)dt + \sigma_S d\omega_{S,t} \tag{2.8}$$

which simplifies to

$$\frac{dS(t)}{S(t)} = (\mu(t) + r)dt + \sigma_S d\omega_{S,t}. \tag{2.9}$$

In finance the value, $\frac{dS(t)}{S(t)} - \frac{dB(t)}{B(t)}$, is called the equity premium. It is the excess return on stocks above the return on a riskless asset that investors receive for investing money in stocks instead of bonds. As mentioned before, this value must always be positive or there is an arbitrage opportunity. Chen, Cosimano, and Himonas (2008) show that from the first quarter of 1947 to the fourth quarter of 2006 the average yearly return on riskless assets was 1.000% and the average yearly return on risky assets was 6.808%. This gives us an average yearly equity premium during this period of 5.808%. In other words, on average during this period the return on risky assets was 5.808% higher than the return on riskless assets.

Based on equation (2.8), another expression of the equity premium is $\mu(t)dt + \sigma_S d\omega_{S,t}$. As discussed previously, the stochastic term $\sigma_S d\omega_{S,t}$ accounts for the uncertainty of prices at the future time t. Thus because prices are known at time 0, there is no uncertainty so $\sigma_S d\omega_{S,t}$ is 0, and $\mu(0)$ is the expected equity premium which must be positive. Before we examine this, let us solve equation (2.9) for $S(t)$ by first taking the integral of both sides,

$$\int_0^t \frac{dS(t)}{S(t)} = \int_0^t (\mu(t) + r)dt + \sigma_S d\omega_{S,t} \tag{2.10}$$

which equals

$$\ln(S(t)) - \ln(S(0)) = \int_0^t (\mu(t) + r)dt + \sigma_S d\omega_{S,t}. \tag{2.11}$$

By using the rules of logs and exponentials again, we get

$$e^{\ln(S(t)/S(0))} = e^{\int_0^t (\mu(t)+r)dt + \sigma_S d\omega_{S,t}} \tag{2.12}$$

which simplifies to

$$S(t) = S(0)e^{\int_0^t (\mu(t)+r)dt + \sigma_S d\omega_{S,t}}. \tag{2.13}$$

The value of a risky asset such as a stock cannot be negative. A stock can have negative returns, but once a company goes bankrupt the stock is worth \$0, not a negative number, because creditors cannot come after the shareholders. Thus equation (2.13) shows us $S(t)$ must be positive because the exponential function only returns values greater than zero and the value of the risky asset S is known at time 0, so it must be positive too.

Finally to ensure the equity premium is always positive, $\frac{dS(t)}{S(t)} = (\mu(t) + r)dt + \sigma_S d\omega_{S,t}$ is now subject to

$$d(\ln \mu(t)) = \kappa(\ln(\theta) - \ln \mu(t))dt + \sigma_\mu d\omega_{\mu,t} \tag{2.14}$$

where $\sigma_S d\omega_{S,t}$ and $\sigma_\mu d\omega_{\mu,t}$ are negatively correlated.[3]

We can solve this equation to show that $\mu(t)$ is always positive by recognizing that because $\ln(\theta)$ is a constant with a derivative of zero, equation (2.14) is equivalent to

$$d(\ln \mu(t) - \ln(\theta)) = -\kappa(\ln \mu(t) - \ln(\theta))dt + \sigma_\mu d\omega_{\mu,t}. \tag{2.15}$$

Using equation (2.15) we first add $\kappa(\ln \mu(t) - \ln(\theta))dt$ to both sides

$$d(\ln \mu(t) - \ln(\theta)) + \kappa(\ln \mu(t) - \ln(\theta))dt = \sigma_\mu d\omega_{\mu,t}. \tag{2.16}$$

Next we multiply by $e^{\kappa t}$, the integrating factor,[4] to get

$$e^{\kappa t}d(\ln \mu(t) - \ln(\theta)) + e^{\kappa t}\kappa(\ln \mu(t) - \ln(\theta))dt = e^{\kappa t}\sigma_\mu d\omega_{\mu,t}. \tag{2.17}$$

Let us note that

$$d[e^{\kappa t}(\ln \mu(t) - \ln(\theta))] = e^{\kappa t}d(\ln \mu(t) - \ln(\theta)) + (\kappa e^{\kappa t}dt)(\ln \mu(t) - \ln(\theta)), \tag{2.18}$$

which rearranges to

$$d[e^{\kappa t}(\ln \mu(t) - \ln(\theta))] = e^{\kappa t}d(\ln \mu(t) - \ln(\theta)) + e^{\kappa t}\kappa(\ln \mu(t) - \ln(\theta))dt. \tag{2.19}$$

[3]The relationship between $\sigma_S d\omega_{S,t}$ and $\sigma_\mu d\omega_{\mu,t}$ is discussed in Cochrane (2005). In short, when stock prices are high, there is an expectation that the stock prices will fall back to the mean so there are lower expected returns. The opposite occurs when stock prices are low, expected returns are high because there is an expectation that the stock prices will tend up toward the mean. Thus the value of stocks and expected returns are negatively correlated.

[4]William E. Boyce and Richard C. DiPrima, *Elementary Differential Equations*, 7th ed., (New York: John Wiley & Sons, Inc., 2001), 32.

Because the right side of (2.19) is equal to the left side of (2.17), we can equate the left side of (2.19) to the right side of (2.17),

$$d[e^{\kappa t}(\ln\mu(t) - \ln(\theta))] = e^{\kappa t}\sigma_\mu d\omega_{\mu,t}. \tag{2.20}$$

Taking the integral of both sides

$$\int_0^t d[e^{\kappa s}(\ln\mu(s) - \ln(\theta))] = \int_0^t e^{\kappa s}\sigma_\mu d\omega_{\mu,s}, \tag{2.21}$$

but by the Fundamental Theorem of Calculus we can simplify the left side of equation (2.21) to get

$$e^{\kappa s}[\ln\mu(s) - \ln(\theta)]\big|_0^t = \int_0^t e^{\kappa s}\sigma_\mu d\omega_{\mu,s}. \tag{2.22}$$

Expanding the left side we get

$$e^{\kappa t}[\ln\mu(t) - \ln(\theta)] - e^0[\ln\mu(0) - \ln(\theta)] = \int_0^t e^{\kappa s}\sigma_\mu d\omega_{\mu,s}, \tag{2.23}$$

which simplifies to

$$e^{\kappa t}[\ln\mu(t) - \ln(\theta)] = [\ln\mu(0) - \ln(\theta)] + \int_0^t e^{\kappa s}\sigma_\mu d\omega_{\mu,s}. \tag{2.24}$$

Finally we use the rule of logarithms that $\ln(a) - \ln(b) = \ln(a/b)$ to simplify the log terms and divide both sides by $e^{\kappa t}$,

$$\ln(\mu(t)/\theta) = e^{-\kappa t}[\ln(\mu(0)/\theta)] + \int_0^t e^{\kappa(s-t)}\sigma_\mu d\omega_{\mu,s}. \tag{2.25}$$

Recalling that $e^{\ln(x)} = x$, we take the exponential of both sides

$$e^{\ln(\mu(t)/\theta)} = e^{e^{-\kappa t}[\ln(\mu(0)/\theta)] + \int_0^t e^{\kappa(s-t)}\sigma_\mu d\omega_{\mu,s}}, \tag{2.26}$$

and simplify to

$$\mu(t)/\theta = e^{e^{-\kappa t}[\ln(\mu(0)/\theta)] + \int_0^t e^{\kappa(s-t)}\sigma_\mu d\omega_{\mu,s}}. \tag{2.27}$$

Finally we multiply by θ

$$\mu(t) = \theta e^{e^{-\kappa t}[\ln(\mu(0)/\theta)] + \int_0^t e^{\kappa(s-t)}\sigma_\mu d\omega_{\mu,s}}. \tag{2.28}$$

Because the exponential function only returns positive values and θ is a positive constant, $\mu(t)$ is always positive as shown by Figure 2. On the graph, note that t is shown in terms of quarters, not years, and the quarterly value for θ is the dotted line on the graph.

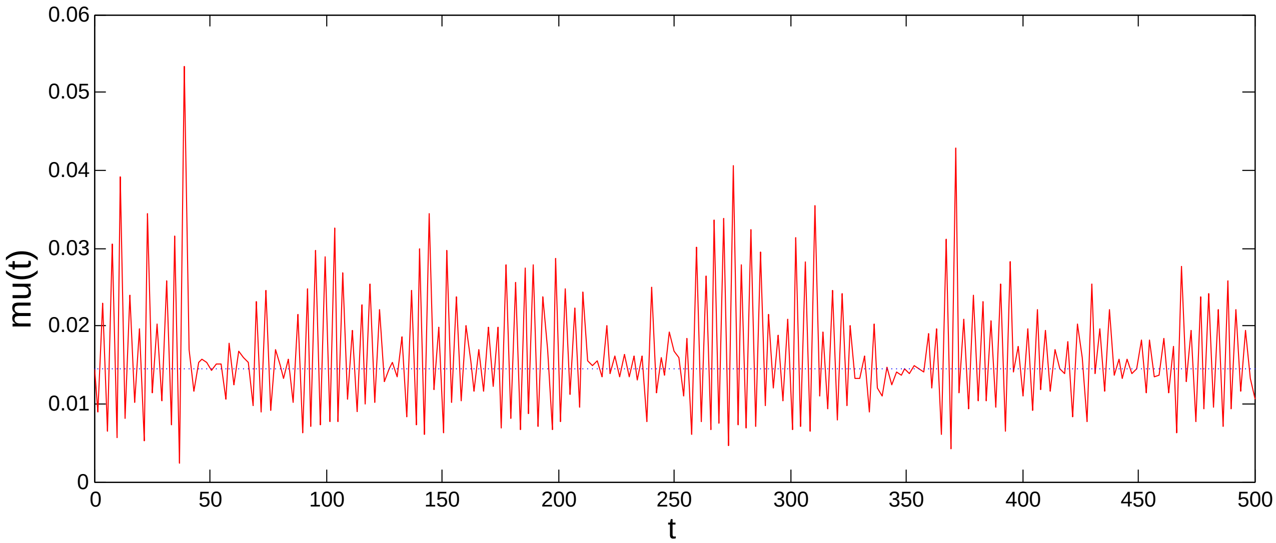

Figure 2

As Figure 2 shows, $\mu(t)$ is now always positive. Notice by Shreve (2004) we have

$$\begin{aligned}\lim_{t\to\infty}\mu(t) &= \lim_{t\to\infty} E_t\left[\theta e^{e^{-\kappa t}[\ln(\mu(0)/\theta)]+\int_0^t e^{\kappa(s-t)}\sigma_\mu d\omega_{\mu,s}}\right] \\ &= \theta \lim_{t\to\infty} E_t\left[e^{\int_0^t e^{-\kappa(t-s)}\sigma_\mu d\omega_{\mu,s}}\right] \\ &= \theta \lim_{t\to\infty}\left[e^{\frac{1}{4\kappa}\sigma_\mu^2\left(1-e^{-2\kappa t}\right)}\right] \\ &= \theta\big(e^{\frac{1}{4\kappa}\sigma_\mu^2}\big).\end{aligned} \tag{2.29}$$

Thus $\theta\big(e^{\frac{1}{4\kappa}\sigma_\mu^2}\big)$ is the long run expected equity premium. Based on the parameter values determined by Chen, Cosimano, and Himonas (2008) and located in the appendix, $\frac{1}{4\kappa}\sigma_\mu^2$ is close to 0, so $e^{\frac{1}{4\kappa}\sigma_\mu^2}$ is close to 1, which means the equity premium is close to θ. Therefore, for the sake of simplicity, we refer to θ as the equity premium throughout this paper. Chen, Cosimano, and Himonas (2008) estimate the value of θ to be 1.452% per quarter. Notice in Figure 2 that $\mu(t)$ is highly concentrated around $\theta = 1.452\%$ as expected. In addition, the random movements of $\mu(t)$ reflect the fact that μ is dependent on a stochastic process that is equally likely to go up or down at each time period.

Later in our calculations we will need an explicit equation for $d\mu(t)$, so we will determine its value now. Because $d\ln\mu(t)$ includes a stochastic term $d\omega_{\mu,t}$, the one dimensional case of Ito's Lemma as it is shown in the appendix must be used. For our equation,

$$df = d\ln\mu(t) = \kappa(\ln(\theta) - \ln\mu(t))dt + \sigma_\mu d\omega_{\mu,t} \tag{2.30}$$

and

$$f = \ln\mu(t). \tag{2.31}$$

To use Ito's Lemma we have that $f_t = 0$, $f_\mu = \frac{1}{\mu(t)}$, and $f_{\mu\mu} = -\frac{1}{\mu(t)^2}$. Thus when we equate $d\ln\mu(t)$ with df of Ito's Lemma we get

$$\kappa(\ln(\theta) - \ln\mu(t))dt + \sigma_\mu d\omega_{\mu,t} = \frac{1}{\mu(t)}d\mu + \frac{1}{2}(-\frac{1}{\mu(t)^2})d\mu d\mu. \tag{2.32}$$

Now we need $d\mu(t)$, so let us suppose

$$d\mu(t) = \mu(t)\left[\frac{1}{2}\sigma_\mu^2 + \kappa(\ln(\theta) - \ln\mu(t))\right]dt + \mu(t)\sigma_\mu d\omega_{\mu,t}. \tag{2.33}$$

Then $(d\mu)^2 = \mu(t)^2\sigma_\mu^2 dt$ by the rules of stochastic calculus in the appendix. Entering these values for $d\mu$ and $(d\mu)^2$ into $\frac{1}{\mu(t)}d\mu + \frac{1}{2}(-\frac{1}{\mu(t)^2})d\mu d\mu$ gives us

$$\frac{1}{\mu(t)}\left\{\mu(t)\left[\frac{1}{2}\sigma_\mu^2 + \kappa(\ln(\theta) - \ln\mu(t))\right]dt + \mu(t)\sigma_\mu d\omega_{\mu,t}\right\} + \frac{1}{2}\left(-\frac{1}{\mu(t)^2}\right)\mu(t)^2\sigma_\mu^2 dt. \tag{2.34}$$

This simplifies to

$$\left[\frac{1}{2}\sigma_\mu^2 + \kappa(\ln(\theta) - \ln\mu(t))\right]dt + \sigma_\mu d\omega_{\mu,t} - \frac{1}{2}\sigma_\mu^2 dt \tag{2.35}$$

which equals

$$\kappa(\ln(\theta) - \ln\mu(t))dt + \sigma_\mu d\omega_{\mu,t}, \tag{2.36}$$

or the left side of equation (2.32). Therefore because Ito's Lemma is satisfied,

$$d\mu(t) = \mu(t)\left[\frac{1}{2}\sigma_\mu^2 + \kappa(\ln(\theta) - \ln\mu(t))\right]dt + \mu(t)\sigma_\mu d\omega_{\mu,t}. \tag{2.37}$$

Now that we have $\frac{dB(t)}{B(t)}$, $\frac{dS(t)}{S(t)}$, and $d\mu(t)$, we must consider some of the other equations used by CCRV in their calculations. One such equation is the change in wealth at each time t. In order to determine the change in an investor's wealth, we must consider the return on bonds, the return on stocks, and the amount the investor consumes. Thus an investor's wealth at time t is

$$dW(t) = W(t)\left[\frac{dB(t)}{B(t)}\right] + \alpha(t)W(t)\left[\frac{dS(t)}{S(t)} - \frac{dB(t)}{B(t)}\right] - C(t)dt. \tag{2.38}$$

At time t the investor's wealth is $W(t)$, the percentage of the investor's wealth invested in stocks is $\alpha(t)$, and the investor's consumption is $C(t)$. In words this equation means that at time t the change in an investor's wealth is his total wealth times the return on bonds plus the percentage of his wealth invested in stocks times his total wealth times the equity premium minus the amount the investor chooses to consume.

If we substitute our equations for $\frac{dB(t)}{B(t)}$ and $\frac{dS(t)}{S(t)}$ we get

$$dW(t) = rW(t)dt + \alpha(t)W(t)\left[(\mu(t) + r)dt - rdt + \sigma_S d\omega_{S,t}\right] - C(t)dt, \tag{2.39}$$

which simplifies to

$$dW(t) = rW(t)dt + \alpha(t)W(t)\left[\mu(t)dt + \sigma_S d\omega_{S,t}\right] - C(t)dt. \tag{2.40}$$

Now that we have an equation for $dW(t)$, we can find the values of $(dW)^2$, $(d\mu)^2$, and $dWd\mu$ which we will need later. Using the properties of stochastic calculus from the appendix, we get

$$(dW)^2 = \alpha(t)^2 W(t)^2 \sigma_S^2 dt, \tag{2.41}$$

$$(d\mu)^2 = (\mu(t))^2 \sigma_\mu^2 dt, \tag{2.42}$$

$$dWd\mu = \alpha(t)W(t)\sigma_S d\omega_{S,t}\mu(t)\sigma_\mu d\omega_{\mu,t}. \tag{2.43}$$

Because $d\omega_{\mu,t}$ and $d\omega_{S,t}$ are negatively correlated, $d\omega_{\mu,t}d\omega_{S,t} = \rho dt$ as shown in the appendix and discussed earlier. Therefore, equation (2.43) simplifies to

$$dWd\mu = \alpha(t)W(t)\mu(t)\sigma_S\sigma_\mu\rho dt. \tag{2.44}$$

Using these equations for $d\mu(t)$, $dW(t)$, $(dW)^2$, $(d\mu)^2$, and $dWd\mu$, we can now begin to solve the CCRV model for an ordinary differential equation.

3 Solving the Model for an Ordinary Differential Equation

We now return to the ultimate goal of the investor: to optimally consume and invest wealth over a lifetime given the riskiness and expected return of assets. We will first consider the one-dimensional case, $V(x(t))$, which is solved in CCH and Chow (1997). In order to do this we need a reward function, $r(x(t), u(t))$, that takes into account variables such as a person's income, how they invest or spend this income, and the expected returns on bonds and stocks.

As shown by CCH, in discrete time the investor's goal of maximum lifetime utility, V, can be expressed by the equation

$$V(x(t)) = \max_{u(t)} E_t \left[\sum_{t}^{\infty} \lambda^{-t} r(x(t), u(t)) \right] \tag{3.45}$$

such that

$$x(t+1) = f(x(t), u(t)) + \epsilon(t+1). \tag{3.46}$$

In this equation we are finding the maximum expected value of the sum of the present values of the investor's future utility at each time t. $r(x(t), u(t))$ is the optimal utility of the investor at each time period t. r depends on $x(t)$, which includes all the variables that will be influenced by random shocks, and $u(t)$. $u(t)$ includes variables that the investor chooses so they are not influenced by random shocks since the investor does not know what these shocks will be. Because of the random shocks, we need the error term, $\epsilon(t+1)$, to account for random movements in the equation of $x(t+1)$. Finally, $\lambda^{-t} = \frac{1}{(1+R)^t}$ so it is the present value factor when R is a constant interest rate. λ^{-t} means that for every time period t, the reward function r will be discounted by $(1 + R)$.[5]

When we expand equation (3.45) to continuous time, we get

$$V(x(t)) = \max_{u(t)} E_t \left[\int_{t}^{\infty} e^{-\beta(\tau - t)} r(x(\tau), u(\tau)) d\tau \right] \tag{3.47}$$

subject to

$$dx = f(x(t), u(t))dt + S(x(t), u(t))d\omega. \tag{3.48}$$

[5]Zvi Bodie, Alex Kane, and Alan J. Marcus, *Essentials of Investments*, 6th ed., (New York: McGraw-Hill Irwin, 2007), 284.

Following the calculations of CCH, we now split the integral into two with one integral being on the small interval from t to $t + dt$ so we get

$$V(x(t)) = \max_{u(t)} E_t \left[\int_t^{t+dt} e^{-\beta(\tau - t)} r(x(\tau), u(\tau)) d\tau \right] + E_{t+dt} \left[\int_{t+dt}^{\infty} e^{-\beta(\tau - t)} r(x(\tau), u(\tau)) d\tau \right]. \quad (3.49)$$

If we make the interval small enough from t to $t+dt$, we can say this integral is equal to $r(x(t), u(t))dt$. In addition, $\max_{u(t+dt)} E_{t+dt} \left[\int_{t+dt}^{\infty} e^{-\beta(\tau-t)} r(x(\tau), u(\tau)) d\tau \right]$ is $V(x(t + dt))$, but we are considering the expectation at t so we must multiply $V(x(t + dt))$ by the discount factor $e^{-\beta dt}$. Our simplified equation is now

$$V(x(t)) = \max_{u(t)} E_t \left[r(x(t), u(t))dt + e^{-\beta dt} V(x(t + dt)) \right]. \quad (3.50)$$

If we recognize that $dV(x(t)) = V(x(t + dt)) - V(x(t))$, then our equation becomes

$$V(x(t)) = \max_{u(t)} E_t \left[r(x(t), u(t))dt + e^{-\beta dt}(dV(x(t)) + V(x(t))) \right]. \quad (3.51)$$

Because this equation is once again in discrete time, the discount factor must be appropriately adjusted. The continuous factor $e^{-\beta dt} \simeq 1 + \ln(e^{-\beta dt}) = 1 - \beta dt$ in discrete time. Inserting this into equation (3.51) gives us

$$V(x(t)) = \max_{u(t)} E_t \left[r(x(t), u(t))dt + (1 - \beta dt)(dV(x(t)) + V(x(t))) \right]. \quad (3.52)$$

When we distribute the $(1 - \beta dt)$ term we get

$$V(x(t)) = \max_{u(t)} E_t \left[r(x(t), u(t))dt + dV(x(t)) - 0 + V(x(t)) - \beta dt V(x(t)) \right] \quad (3.53)$$

because by Ito's Rule $[-\beta dt dV(x(t))] = 0$. Next we add $\beta dt V(x(t)) - V(x(t))$ to both sides to get

$$V(x(t)) + \beta dt V(x(t) - V(x(t)) = \max_{u(t)} E_t \left[r(x(t), u(t))dt + dV(x(t)) \right]. \quad (3.54)$$

Finally we simplify and divide by dt,

$$\beta V(x(t)) = \max_{u(t)} E_t \left[r(x(t), u(t)) + \frac{dV(x(t))}{dt} \right]. \quad (3.55)$$

Now we must find $dV(x(t))$. Because $V(x(t))$ is subject to a stochastic process $dx = f(x(t), u(t))dt + S(x(t), u(t))d\omega$, we must once again use Ito's Lemma which means that

$$dV(x(t)) = V_t dt + V_x dx + \frac{1}{2} V_{xx} dx dx. \quad (3.56)$$

Recalling the rules of stochastic calculus in the appendix,

$$dxdx = (f(x(t), u(t))dt)^2 + 2(f(x(t), u(t))dt)(S(x(t), u(t))d\omega) + (S(x(t), u(t))d\omega)^2, \tag{3.57}$$

which simplifies to

$$dxdx = (S(x(t), u(t)))^2 dt. \tag{3.58}$$

Inputting $dx = f(x(t), u(t))dt + S(x(t), u(t))d\omega$ and $dxdx = (S(x(t), u(t)))^2 dt$, gives us

$$dV(x(t)) = V_t dt + V_x f dt + V_x S d\omega + \frac{1}{2} V_{xx} (S)^2 dt. \tag{3.59}$$

Finally we group like terms to get

$$dV(x(t)) = [V_t + V_x f + \frac{1}{2} V_{xx} (S)^2] dt + V_x S d\omega. \tag{3.60}$$

Inputting $dV(x(t))$ into equation (3.55) and distributing the $\frac{1}{dt}$ results in

$$\beta V(x(t)) = \max_{u(t)} \left[r(x(t), u(t)) + V_x f(x(t), u(t)) + \frac{1}{2} V_{xx} (S(x(t), u(t)))^2 \right]. \tag{3.61}$$

This equation (3.61) is known as the Bellman Equation. It optimizes both the investor's current period reward function and the investor's future utility.

In the two-dimensional version of the Bellman Equation, the Hamilton-Jacobi-Bellman Equation, $V(x(t))$ is $J(W(t), \mu(t))$. However, unlike V, the function J is unknown. Regardless of this, CCH and Malliaris and Brock (1982) show J can still be solved with methods similar to those that we used to solve V. Because of all the information that is relevant to the optimization problem, CCRV use the two-dimensional function J in order to maximize the investor's lifetime utility. The optimization problem is

$$J(W(t), \mu(t)) = \max_{\alpha(t), C(t)} E_t \left[\int_t^{\infty} f(C(\tau), J(W(\tau), \mu(\tau))) d\tau \right] \tag{3.62}$$

subject to

$$d\mu(t) = \mu(t) \left[\frac{1}{2} \sigma_\mu^2 + \kappa(\ln(\theta) - \ln \mu(t)) \right] dt + \mu(t) \sigma_\mu d\omega_{\mu,t} \tag{3.63}$$

and

$$dW(t) = rW(t)dt + \alpha(t)W(t) \left[\mu(t)dt + \sigma_S d\omega_{S,t} \right] - C(t)dt. \tag{3.64}$$

In these equations, $C(t)$ is consumption, $W(t)$ is wealth, $\alpha(t)$ is the percentage of W invested in stocks, and $1-\alpha(t)$ is the percentage of W invested in bonds. Note there is no explicit discount factor in the equation for J as there is in the equation for V. Later, an estimate of J will be introduced, but for now it can be assumed that the discount factor of J is already embedded in J.

CCH solve J similar to how we solved $V(x(t))$, by first splitting the integral in two with one integral over the small interval from t to $t+dt$ to get

$$J(W(t),\mu(t)) = \max_{\alpha(t),C(t)} E_t\left[\int_t^{t+dt} f(C(\tau), J(W(\tau),\mu(\tau)))d\tau\right] + E_{t+dt}\left[\int_{t+dt}^{\infty} f(C(\tau), J(W(\tau),\mu(\tau)))d\tau\right]. \tag{3.65}$$

The second half of the equation simplifies because it is simply J at $t+dt$. Notice in this case that we do not need to multiply by a discount factor to adjust from the expectation $t+dt$ to t,

$$J(W(t),\mu(t)) = \max_{\alpha(t),C(t)} E_t\left[\int_t^{t+dt} f(C(\tau), J(W(\tau),\mu(\tau)))d\tau + J(W(t+dt),\mu(t+dt))\right]. \tag{3.66}$$

We are once again able to simplify the first half of the equation as follows

$$J(W(t),\mu(t)) = \max_{\alpha(t),C(t)} E_t[f(C(t), J(W(t),\mu(t)))dt + J(W(t+dt),\mu(t+dt))]. \tag{3.67}$$

Recognize that $dJ(W(t),\mu(t)) = J(W(t+dt),\mu(t+dt)) - J(W(t),\mu(t))$ so

$$J(W(t),\mu(t)) = \max_{\alpha(t),C(t)} E_t[f(C(t), J(W(t),\mu(t)))dt + dJ(W(t),\mu(t)) + J(W(t),\mu(t))], \tag{3.68}$$

Subtracting $J(W(t),\mu(t))$ from both sides gives us

$$0 = \max_{\alpha(t),C(t)} E_t[f(C(t), J(W(t),\mu(t)))dt + dJ(W(t),\mu(t))]. \tag{3.69}$$

Finally we divide by dt to get

$$0 = \max_{\alpha(t),C(t)} E_t\left[f(C(t), J(W(t),\mu(t))) + \frac{dJ(W(t),\mu(t))}{dt}\right]. \tag{3.70}$$

Similar to how we solved for $dV(x(t))$ we must now solve for $dJ(W(t),\mu(t))$, but this time we must use the two dimensional version of Ito's Lemma from the appendix because both $W(t)$ and $\mu(t)$ contain stochastic processes. Therefore

$$dJ(W,\mu) = J_W dW + J_\mu d\mu + \frac{1}{2}(J_{WW} dWdW + 2J_{W\mu} dWd\mu + J_{\mu\mu} d\mu d\mu) \tag{3.71}$$

which can be rewritten as

$$dJ(W,\mu) = dW\frac{\partial J}{\partial W} + d\mu\frac{\partial J}{\partial \mu} + \frac{1}{2}\left((dW)^2\frac{\partial^2 J}{\partial W \partial W} + 2dWd\mu\frac{\partial^2 J}{\partial W \partial \mu} + (d\mu)^2\frac{\partial^2 J}{\partial \mu \partial \mu}\right). \tag{3.72}$$

We now plug equation (3.72) into equation (3.70)

$$0 = \max_{\alpha(t),C(t)} E_t\left[f(C(t),J(W(t),\mu(t))) + \frac{1}{dt}\left[dW\frac{\partial J}{\partial W} + d\mu\frac{\partial J}{\partial \mu} + \frac{1}{2}\left((dW)^2\frac{\partial^2 J}{\partial W \partial W} + 2dWd\mu\frac{\partial^2 J}{\partial W \partial \mu} + (d\mu)^2\frac{\partial^2 J}{\partial \mu \partial \mu}\right)\right]\right]. \tag{3.73}$$

Putting all this together gives us

$$0 = \max_{\alpha(t),C(t)} E_t\left\{f(C(t),J(t)) + \frac{1}{dt}\left[\left[(rdt + \alpha(t)\mu(t)dt + \alpha(t)\sigma_S d\omega_{S,t})W(t) - C(t)dt\right]\frac{\partial J}{\partial W} + \left[\mu(t)\left[\frac{1}{2}\sigma_\mu^2 + \kappa(\ln(\theta) - \ln\mu(t))\right]dt + \mu(t)\sigma_\mu d\omega_{\mu,t}\right]\frac{\partial J}{\partial \mu} + \frac{1}{2}\left[\alpha(t)^2W(t)^2\sigma_S^2 dt\frac{\partial^2 J}{\partial W \partial W} + 2\alpha(t)W(t)\rho\mu(t)\sigma_S\sigma_\mu dt\frac{\partial^2 J}{\partial W \partial \mu} + \mu(t)^2\sigma_\mu^2 dt\frac{\partial^2 J}{\partial \mu \partial \mu}\right]\right]\right\}. \tag{3.74}$$

When we distribute the $\frac{1}{dt}$, the Hamilton-Jacobi-Bellman Equation simplifies to

$$0 = \max_{\alpha(t),C(t)}\left\{f(C(t),J(t)) + \left[(r + \alpha(t)\mu(t))W(t) - C(t)\right]\frac{\partial J}{\partial W} + \mu(t)\left[\frac{1}{2}\sigma_\mu^2 + \kappa(\ln(\theta) - \ln\mu(t))\right]\frac{\partial J}{\partial \mu} + \frac{1}{2}\left[\alpha(t)^2W(t)^2\sigma_S^2\frac{\partial^2 J}{\partial W \partial W} + 2\alpha(t)W(t)\rho\mu(t)\sigma_S\sigma_\mu\frac{\partial^2 J}{\partial W \partial \mu} + \mu(t)^2\sigma_\mu^2\frac{\partial^2 J}{\partial \mu \partial \mu}\right]\right\}. \tag{3.75}$$

In words this equation means that the investor wants to optimize his reward function, f, in the current period and optimize all his future utility by choosing the optimal amount of consumption, $C(t)$, and the optimal investment in stocks and bonds, $\alpha(t)$ and $1 - \alpha(t)$ respectively.

The reward function, f, used by CCH and CCRV is cited as that of Duffie and Epstein (1992) and Kreps and Porteus (1978):

$$f(C,J) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J\left[\left(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}} - 1\right]. \tag{3.76}$$

When we solve for $C(t)$, we will need $\frac{\partial f(C,J)}{\partial C}$, so

$$\frac{\partial f(C,J)}{\partial C} = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J\left[\left(1-\frac{1}{\psi}\right)\left(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}}\right)^{-\frac{1}{\psi}}\frac{1}{((1-\gamma)J)^{\frac{1}{1-\gamma}}}\right].$$

This simplifies to

$$\frac{\partial f(C,J)}{\partial C} = \beta(C)^{-\frac{1}{\psi}}((1-\gamma)J)\left[((1-\gamma)J)^{\frac{1}{\psi(1-\gamma)}}\right]\left[((1-\gamma)J)^{-\frac{1}{1-\gamma}}\right],$$

and finally gives us

$$\frac{\partial f(C,J)}{\partial C} = \beta\,(C)^{-\frac{1}{\psi}}\,((1-\gamma)J)^{\frac{1-\gamma\psi}{\psi(1-\gamma)}}. \tag{3.77}$$

Because we want to maximize (3.75) with respect to $C(t)$, we take the partial derivative with respect to $C(t)$ and include the value for $\frac{\partial f(C,J)}{\partial C}$ from equation (3.77). Thus

$$\beta\,(C)^{-\frac{1}{\psi}}\,((1-\gamma)J)^{\frac{1-\gamma\psi}{\psi(1-\gamma)}} - \frac{\partial J}{\partial W} = 0, \tag{3.78}$$

and solving this for $C(t)$ gives us

$$C(t) = \beta^{\psi}\left(\frac{\partial J}{\partial W}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}. \tag{3.79}$$

We also want to maximize $\alpha(t)$, so we now take the partial derivative of (3.75) with respect to $\alpha(t)$,

$$\mu(t)W(t)\frac{\partial J}{\partial W} + W(t)^2\alpha(t)\sigma_S^2\frac{\partial^2 J}{\partial W\partial W} + W(t)\rho\mu(t)\sigma_S\sigma_\mu\frac{\partial^2 J}{\partial W\partial\mu} = 0. \tag{3.80}$$

Finally we solve for $\alpha(t)$

$$\alpha(t) = \frac{-1}{W(t)\frac{\partial^2 J}{\partial W\partial W}}\left\{\frac{\mu(t)}{\sigma_S^2}\frac{\partial J}{\partial W} + \frac{\rho\mu(t)\sigma_\mu}{\sigma_S}\frac{\partial^2 J}{\partial W\partial\mu}\right\}. \tag{3.81}$$

Now we plug our values for $C(t)$ and $\alpha(t)$ into the Hamilton-Jacobi-Bellman equation (3.75)

$$\begin{aligned} 0 = {} & f\left(\beta^{\psi}\left(\frac{\partial J}{\partial W}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}, J(t)\right) + \mu(t)\left[\frac{1}{2}\sigma_\mu^2 + \kappa(\ln(\theta) - \ln\mu(t))\right]\frac{\partial J}{\partial\mu} + \\ & \left[\left(r - \frac{1}{W(t)\frac{\partial^2 J}{\partial W\partial W}}\left\{\frac{\mu(t)}{\sigma_S^2}\frac{\partial J}{\partial W} + \frac{\rho\mu(t)\sigma_\mu}{\sigma_S}\frac{\partial^2 J}{\partial W\partial\mu}\right\}(\mu(t))\right)W(t) - \left(\beta^{\psi}\left(\frac{\partial J}{\partial W}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}\right)\right]\frac{\partial J}{\partial W} \\ & + \frac{1}{2}\left[\left[\frac{-1}{W(t)\frac{\partial^2 J}{\partial W\partial W}}\left\{\frac{\mu(t)}{\sigma_S^2}\frac{\partial J}{\partial W} + \frac{\rho\mu(t)\sigma_\mu}{\sigma_S}\frac{\partial^2 J}{\partial W\partial\mu}\right\}\right]^2 W(t)^2\sigma_S^2\frac{\partial^2 J}{\partial W\partial W} + \right. \\ & \left. 2\left[\frac{-1}{W(t)\frac{\partial^2 J}{\partial W\partial W}}\left\{\frac{\mu(t)}{\sigma_S^2}\frac{\partial J}{\partial W} + \frac{\rho\mu(t)\sigma_\mu}{\sigma_S}\frac{\partial^2 J}{\partial W\partial\mu}\right\}\right]W(t)\rho\mu(t)\sigma_S\sigma_\mu\frac{\partial^2 J}{\partial W\partial\mu} + \mu(t)^2\sigma_\mu^2\frac{\partial^2 J}{\partial\mu\partial\mu}\right]. \end{aligned} \tag{3.82}$$

Expand the squared term

$$0 = f\left(\beta^{\psi}\left(\frac{\partial J}{\partial W}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}, J(t)\right) + \mu(t)\left[\frac{1}{2}\sigma_{\mu}^{2} + \kappa(\ln(\theta) - \ln\mu(t))\right]\frac{\partial J}{\partial \mu} + \tag{3.83}$$

$$\left[\left(r - \frac{1}{W(t)\frac{\partial^2 J}{\partial W \partial W}}\left\{\frac{\mu(t)}{\sigma_S^2}\frac{\partial J}{\partial W} + \frac{\rho\mu(t)\sigma_{\mu}}{\sigma_S}\frac{\partial^2 J}{\partial W \partial \mu}\right\}(\mu(t))\right)W(t) - \left(\beta^{\psi}\left(\frac{\partial J}{\partial W}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}\right)\right]\frac{\partial J}{\partial W}$$

$$+\frac{1}{2}\left[\left(\frac{-1}{W(t)\frac{\partial^2 J}{\partial W \partial W}}\right)^2\left[\left(\frac{\mu(t)}{\sigma_S^2}\frac{\partial J}{\partial W}\right)^2 + 2\left(\frac{\mu(t)}{\sigma_S^2}\frac{\partial J}{\partial W}\right)\left(\frac{\rho\mu(t)\sigma_{\mu}}{\sigma_S}\frac{\partial^2 J}{\partial W \partial \mu}\right)\right.\right.$$

$$\left.+\left(\frac{\rho\mu(t)\sigma_{\mu}}{\sigma_S}\frac{\partial^2 J}{\partial W \partial \mu}\right)^2\right]W(t)^2\sigma_S^2\frac{\partial^2 J}{\partial W \partial W} +$$

$$\left.2\left[\frac{-1}{W(t)\frac{\partial^2 J}{\partial W \partial W}}\left\{\frac{\mu(t)}{\sigma_S^2}\frac{\partial J}{\partial W} + \frac{\rho\mu(t)\sigma_{\mu}}{\sigma_S}\frac{\partial^2 J}{\partial W \partial \mu}\right\}\right]W(t)\rho\mu(t)\sigma_S\sigma_{\mu}\frac{\partial^2 J}{\partial W \partial \mu} + \mu(t)^2\sigma_{\mu}^2\frac{\partial^2 J}{\partial \mu \partial \mu}\right].$$

Cancel and combine like terms

$$0 = f\left(\beta^{\psi}\left(\frac{\partial J}{\partial W}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}, J(t)\right) + \mu(t)\left[\frac{1}{2}\sigma_{\mu}^{2} + \kappa(\ln(\theta) - \ln\mu(t))\right]\frac{\partial J}{\partial \mu} + \tag{3.84}$$

$$\left[\left(r - \frac{1}{W(t)\frac{\partial^2 J}{\partial W \partial W}}\left\{\frac{\mu(t)}{\sigma_S^2}\frac{\partial J}{\partial W} + \frac{\rho\mu(t)\sigma_{\mu}}{\sigma_S}\frac{\partial^2 J}{\partial W \partial \mu}\right\}(\mu(t))\right)W(t) - \left(\beta^{\psi}\left(\frac{\partial J}{\partial W}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}\right)\right]\frac{\partial J}{\partial W}$$

$$+\frac{1}{2}\left[\frac{1}{\frac{\partial^2 J}{\partial W \partial W}}\left\{\frac{\mu(t)}{\sigma_S}\frac{\partial J}{\partial W} + \rho\mu(t)\sigma_{\mu}\frac{\partial^2 J}{\partial W \partial \mu}\right\}^2\right.$$

$$\left.-2\left[\frac{1}{\frac{\partial^2 J}{\partial W \partial W}}\left\{\frac{\mu(t)}{\sigma_S}\frac{\partial J}{\partial W} + \rho\mu(t)\sigma_{\mu}\frac{\partial^2 J}{\partial W \partial \mu}\right\}\right]\rho\mu(t)\sigma_{\mu}\frac{\partial^2 J}{\partial W \partial \mu} + \mu(t)^2\sigma_{\mu}^2\frac{\partial^2 J}{\partial \mu \partial \mu}\right].$$

Expand all parentheses

$$0 = f\left(\beta^{\psi}\left(\frac{\partial J}{\partial W}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}, J(t)\right) + \mu(t)\left[\frac{1}{2}\sigma_{\mu}^{2} + \kappa(\ln(\theta) - \ln\mu(t))\right]\frac{\partial J}{\partial \mu} + rW(t)\frac{\partial J}{\partial W} \tag{3.85}$$

$$-\frac{1}{\frac{\partial^2 J}{\partial W \partial W}}\left(\frac{\mu(t)^2}{\sigma_S^2}\left(\frac{\partial J}{\partial W}\right)^2\right) - \frac{1}{\frac{\partial^2 J}{\partial W \partial W}}\left(\frac{\rho\mu(t)^2\sigma_{\mu}}{\sigma_S}\frac{\partial^2 J}{\partial W \partial \mu}\right)\frac{\partial J}{\partial W} - \left(\beta^{\psi}\left(\frac{\partial J}{\partial W}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}\right)\frac{\partial J}{\partial W}$$

$$+\frac{1}{2}\frac{1}{\frac{\partial^2 J}{\partial W \partial W}}\left(\frac{\mu(t)}{\sigma_S}\frac{\partial J}{\partial W}\right)^2 + \frac{1}{\frac{\partial^2 J}{\partial W \partial W}}\left(\frac{\mu(t)}{\sigma_S}\frac{\partial J}{\partial W}\right)\left(\rho\mu(t)\sigma_{\mu}\frac{\partial^2 J}{\partial W \partial \mu}\right) + \frac{1}{2}\frac{1}{\frac{\partial^2 J}{\partial W \partial W}}\left(\rho\mu(t)\sigma_{\mu}\frac{\partial^2 J}{\partial W \partial \mu}\right)^2$$

$$-\frac{1}{\frac{\partial^2 J}{\partial W \partial W}}\left(\frac{\mu(t)}{\sigma_S}\frac{\partial J}{\partial W}\right)\left(\rho\mu(t)\sigma_{\mu}\frac{\partial^2 J}{\partial W \partial \mu}\right) - \frac{1}{\frac{\partial^2 J}{\partial W \partial W}}\left(\rho\mu(t)\sigma_{\mu}\frac{\partial^2 J}{\partial W \partial \mu}\right)^2 + \frac{1}{2}\mu(t)^2\sigma_{\mu}^2\frac{\partial^2 J}{\partial \mu \partial \mu}.$$

Combine and rearrange terms

$$0 = f\left(\beta^{\psi}\left(\frac{\partial J}{\partial W}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}, J(t)\right) - \left(\beta^{\psi}\left(\frac{\partial J}{\partial W}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}\right)\frac{\partial J}{\partial W} \quad (3.86)$$
$$+ rW(t)\frac{\partial J}{\partial W} + \mu(t)\left[\frac{1}{2}\sigma_\mu^2 + \kappa(\ln(\theta) - \ln\mu(t))\right]\frac{\partial J}{\partial \mu} + \frac{1}{2}\mu(t)^2\sigma_\mu^2\frac{\partial^2 J}{\partial\mu\partial\mu}$$
$$- \frac{1}{2}\frac{1}{\frac{\partial^2 J}{\partial W\partial W}}\left[\left(\frac{\mu(t)}{\sigma_S}\frac{\partial J}{\partial W}\right)^2 + 2\left(\frac{\mu(t)}{\sigma_S}\frac{\partial J}{\partial W}\right)\left(\rho\mu(t)\sigma_\mu\frac{\partial^2 J}{\partial W\partial\mu}\right) + \left(\rho\mu(t)\sigma_\mu\frac{\partial^2 J}{\partial W\partial\mu}\right)^2\right].$$

Plugging $C(t)$ into the reward function $f(C(t), J(t))$ gives us

$$f\left(\beta^{\psi}\left(\frac{\partial J}{\partial W}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}, J(t)\right) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J \quad (3.87)$$
$$\times\left[\left(\frac{\beta^{\psi}\left(\frac{\partial J}{\partial W}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}}{((1-\gamma)J)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}} - 1\right]$$
$$= \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J$$
$$\times\left[\left(\beta^{\psi-1}\left(\frac{\partial J}{\partial W}\right)^{1-\psi}\right)[(1-\gamma)J]^{-\frac{\gamma(\psi-1)}{1-\gamma}} - 1\right].$$

In order to solve (3.86), we need a better idea of what $J(W(t), \mu(t))$, or the investor's lifetime utility, is. Based on the work by Merton (1990), CCRV guess

$$J(W(t), \mu(t)) = H(\mu(t))^{-\frac{1-\gamma}{1-\psi}}\frac{W(t)^{1-\gamma}}{1-\gamma} \quad (3.88)$$

where $H(\mu(t))$ includes the discount factor as discussed previously.

From equation (3.88), we can now solve for the partial derivatives in (3.86).

$$\frac{\partial J}{\partial W} = \frac{(1-\gamma)}{1-\gamma}H(\mu(t))^{-\frac{1-\gamma}{1-\psi}}W(t)^{-\gamma} = \frac{(1-\gamma)}{1-\gamma}H(\mu(t))^{-\frac{1-\gamma}{1-\psi}}\frac{W(t)^{1-\gamma}}{W(t)} = \frac{(1-\gamma)J}{W(t)}, \quad (3.89)$$

$$\frac{\partial^2 J}{\partial W\partial W} = -\gamma\frac{(1-\gamma)}{1-\gamma}H(\mu(t))^{-\frac{1-\gamma}{1-\psi}}W(t)^{-\gamma-1} = -\gamma\frac{(1-\gamma)}{1-\gamma}H(\mu(t))^{-\frac{1-\gamma}{1-\psi}}\frac{W(t)^{1-\gamma}}{W(t)^2} = \frac{-\gamma(1-\gamma)J}{W(t)^2}, \quad (3.90)$$

$$\frac{\partial J}{\partial \mu} = -\frac{1-\gamma}{1-\psi}\frac{W(t)^{1-\gamma}}{1-\gamma}H(\mu(t))^{-\frac{1-\gamma}{1-\psi}-1}H' = -\frac{1-\gamma}{1-\psi}\frac{W(t)^{1-\gamma}}{1-\gamma}H(\mu(t))^{-\frac{1-\gamma}{1-\psi}}\frac{H'}{H} = -\frac{1-\gamma}{1-\psi}J\frac{H'}{H}, \quad (3.91)$$

$$\frac{\partial^2 J}{\partial W\partial\mu} = \frac{(1-\gamma)}{W(t)}\frac{\partial J}{\partial\mu} = -\frac{1-\gamma}{1-\psi}\frac{(1-\gamma)}{W(t)}J\frac{H'}{H}, \quad (3.92)$$

and

$$\frac{\partial^2 J}{\partial\mu\partial\mu} = -\frac{1-\gamma}{1-\psi}\frac{\partial J}{\partial\mu}\frac{H'}{H} - \frac{1-\gamma}{1-\psi}J\frac{H''H-(H')^2}{H^2} = \left(\frac{1-\gamma}{1-\psi}\right)^2 J\frac{(H')^2}{H^2} - \frac{1-\gamma}{1-\psi}J\frac{H''H-(H')^2}{H^2}. \tag{3.93}$$

Plugging (3.89) into (3.87) we get

$$\begin{aligned} f\left(\beta^\psi\left(\frac{(1-\gamma)J}{W(t)}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}, J(t)\right) &= \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J \\ &\times\left[\left(\beta^{\psi-1}\left(\frac{(1-\gamma)J}{W(t)}\right)^{1-\psi}\right)[(1-\gamma)J]^{-\frac{\gamma(\psi-1)}{1-\gamma}} - 1\right] \\ &= \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J \\ &\times\left[(\beta W(t))^{\psi-1}[(1-\gamma)J]^{\frac{(1-\psi)}{1-\gamma}} - 1\right]. \end{aligned} \tag{3.94}$$

Now we substitute the partial derivatives for J and (3.94) into (3.86)

$$\begin{aligned} 0 &= \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J\left[(\beta W(t))^{\psi-1}[(1-\gamma)J]^{\frac{(1-\psi)}{1-\gamma}} - 1\right] \\ &- \left(\beta^\psi\left(\frac{(1-\gamma)J}{W(t)}\right)^{-\psi}((1-\gamma)J)^{\frac{1-\gamma\psi}{(1-\gamma)}}\right)\frac{(1-\gamma)J}{W(t)} + rW(t)\frac{(1-\gamma)J}{W(t)} \\ &+ \mu(t)\left[\frac{1}{2}\sigma_\mu^2 + \kappa(\ln(\theta) - \ln\mu(t))\right]\left(-\frac{1-\gamma}{1-\psi}J\frac{H'}{H}\right) + \frac{1}{2}\mu(t)^2\sigma_\mu^2\left(\left(\frac{1-\gamma}{1-\psi}\right)^2 J\frac{(H')^2}{H^2} - \frac{1-\gamma}{1-\psi}J\frac{H''H-(H')^2}{H^2}\right) \\ &- \frac{1}{2}\frac{1}{\frac{-\gamma(1-\gamma)J}{W(t)^2}}\left[\left(\frac{\mu(t)}{\sigma_S}\frac{(1-\gamma)J}{W(t)}\right)^2 + 2\left(\frac{\mu(t)}{\sigma_S}\frac{(1-\gamma)J}{W(t)}\right)\left(\rho\mu(t)\sigma_\mu\left(-\frac{1-\gamma}{1-\psi}\frac{(1-\gamma)}{W(t)}J\frac{H'}{H}\right)\right)\right. \\ &+ \left.\left(\rho\mu(t)\sigma_\mu\left(-\frac{1-\gamma}{1-\psi}\frac{(1-\gamma)}{W(t)}J\frac{H'}{H}\right)\right)^2\right]. \end{aligned} \tag{3.95}$$

Simplifying this equation

$$
\begin{aligned}
0 = & \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J\left[(\beta W(t))^{\psi-1}\left[(1-\gamma)J\right]^{\frac{(1-\psi)}{1-\gamma}}-1\right] \\
& -\left(\beta^{\psi}\left(\frac{(1-\gamma)J}{W(t)}\right)^{1-\psi}\left((1-\gamma)J\right)^{\frac{1-\gamma\psi}{(1-\gamma)}}\right)+r(1-\gamma)J \\
& -\mu(t)\left[\frac{1}{2}\sigma_\mu^2+\kappa(\ln(\theta)-\ln\mu(t))\right]\left(\frac{1-\gamma}{1-\psi}J\frac{H'}{H}\right)+\frac{1}{2}\mu(t)^2\sigma_\mu^2\left(\left(\frac{1-\gamma}{1-\psi}\right)^2 J\frac{(H')^2}{H^2}-\frac{1-\gamma}{1-\psi}J\frac{H''H-(H')^2}{H^2}\right) \\
& +\frac{1}{2}\frac{1}{\frac{\gamma(1-\gamma)J}{W(t)^2}}\left[\left(\frac{\mu(t)}{\sigma_S}\frac{(1-\gamma)J}{W(t)}\right)^2-2\rho\mu(t)\sigma_\mu\left(\frac{\mu(t)}{\sigma_S}\frac{(1-\gamma)J}{W(t)}\right)\left(\frac{1-\gamma}{1-\psi}\frac{(1-\gamma)}{W(t)}J\frac{H'}{H}\right)\right. \\
& \left.+\left(\rho\mu(t)\sigma_\mu\frac{1-\gamma}{1-\psi}\frac{(1-\gamma)}{W(t)}J\frac{H'}{H}\right)^2\right].
\end{aligned}
\tag{3.96}
$$

Plugging (3.88) into (3.96) we get

$$
\begin{aligned}
0 = & \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J\left[(\beta W(t))^{\psi-1}\left[H^{-\frac{1-\gamma}{1-\psi}}W(t)^{1-\gamma}\right]^{\frac{(1-\psi)}{1-\gamma}}-1\right] \\
& -\left(\beta^{\psi}\left(\frac{(1-\gamma)J}{W(t)^{1-\psi}}\right)\left(H^{-\frac{1-\gamma}{1-\psi}}W(t)^{1-\gamma}\right)^{\frac{1-\psi}{(1-\gamma)}}\right)+r(1-\gamma)J \\
& -\mu(t)\left[\frac{1}{2}\sigma_\mu^2+\kappa(\ln(\theta)-\ln\mu(t))\right]\left(\frac{1-\gamma}{1-\psi}J\frac{H'}{H}\right)+\frac{1}{2}\mu(t)^2\sigma_\mu^2\left(\left(\frac{1-\gamma}{1-\psi}\right)^2 J\frac{(H')^2}{H^2}-\frac{1-\gamma}{1-\psi}J\frac{H''H-(H')^2}{H^2}\right) \\
& +\frac{1}{2}\frac{1}{\frac{\gamma(1-\gamma)J}{W(t)^2}}\left[\left(\frac{\mu(t)}{\sigma_S}\frac{(1-\gamma)J}{W(t)}\right)^2-2\rho\mu(t)\sigma_\mu\left(\frac{\mu(t)}{\sigma_S}\frac{(1-\gamma)J}{W(t)}\right)\left(\frac{1-\gamma}{1-\psi}\frac{(1-\gamma)}{W(t)}J\frac{H'}{H}\right)\right. \\
& \left.+\left(\rho\mu(t)\sigma_\mu\frac{1-\gamma}{1-\psi}\frac{(1-\gamma)}{W(t)}J\frac{H'}{H}\right)^2\right].
\end{aligned}
\tag{3.97}
$$

Cancel exponents

$$
\begin{aligned}
0 = & \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J\left[(\beta W(t))^{\psi-1}\left[H^{-1}W(t)^{1-\psi}\right]-1\right] \\
& -\left(\beta^{\psi}\left(\frac{(1-\gamma)J}{W(t)^{1-\psi}}\right)\left(H^{-1}W(t)^{1-\psi}\right)\right)+r(1-\gamma)J \\
& -\mu(t)\left[\frac{1}{2}\sigma_\mu^2+\kappa(\ln(\theta)-\ln\mu(t))\right]\left(\frac{1-\gamma}{1-\psi}J\frac{H'}{H}\right)+\frac{1}{2}\mu(t)^2\sigma_\mu^2\left(\left(\frac{1-\gamma}{1-\psi}\right)^2 J\frac{(H')^2}{H^2}-\frac{1-\gamma}{1-\psi}J\frac{H''H-(H')^2}{H^2}\right) \\
& +\frac{1}{2}\frac{1}{\frac{\gamma(1-\gamma)J}{W(t)^2}}\left[\left(\frac{\mu(t)}{\sigma_S}\frac{(1-\gamma)J}{W(t)}\right)^2-2\rho\mu(t)\sigma_\mu\left(\frac{\mu(t)}{\sigma_S}\frac{(1-\gamma)J}{W(t)}\right)\left(\frac{1-\gamma}{1-\psi}\frac{(1-\gamma)}{W(t)}J\frac{H'}{H}\right)\right. \\
& \left.+\left(\rho\mu(t)\sigma_\mu\frac{1-\gamma}{1-\psi}\frac{(1-\gamma)}{W(t)}J\frac{H'}{H}\right)^2\right].
\end{aligned}
\tag{3.98}
$$

Divide by $(1-\gamma)J$ and simplify

$$
\begin{aligned}
0 = {} & \frac{\beta\psi}{\psi-1}\left[\beta^{\psi-1}H^{-1}-1\right]-\left(\beta^{\psi}H^{-1}\right)+r \\
& -\mu(t)\left[\frac{1}{2}\sigma_\mu^2+\kappa(\ln(\theta)-\ln\mu(t))\right]\left(\frac{1}{1-\psi}\frac{H'}{H}\right)+\frac{1}{2}\mu(t)^2\sigma_\mu^2\left(\frac{1-\gamma}{(1-\psi)^2}\frac{(H')^2}{H^2}-\frac{1}{1-\psi}\frac{H''H-(H')^2}{H^2}\right) \\
& +\frac{1}{2}\frac{1}{\gamma}\left[\left(\frac{\mu(t)}{\sigma_S}\right)^2-2\rho\mu(t)\sigma_\mu\left(\frac{\mu(t)}{\sigma_S}\right)\left(\frac{1-\gamma}{1-\psi}\frac{H'}{H}\right)+\left(\rho\mu(t)\sigma_\mu\frac{1-\gamma}{1-\psi}\frac{H'}{H}\right)^2\right].
\end{aligned}
\tag{3.99}
$$

Multiply by $1-\psi$ to get

$$
\begin{aligned}
0 = {} & -\beta\psi\left[\beta^{\psi-1}H^{-1}-1\right]-\left(\beta^{\psi}(1-\psi)H^{-1}\right)+r(1-\psi) \\
& -\mu(t)\left[\frac{1}{2}\sigma_\mu^2+\kappa(\ln(\theta)-\ln\mu(t))\right]\frac{H'}{H}+\frac{1}{2}\mu(t)^2\sigma_\mu^2\left(\frac{1-\gamma}{(1-\psi)}\frac{(H')^2}{H^2}-\frac{H''H-(H')^2}{H^2}\right) \\
& +\frac{1-\psi}{2\gamma}\left(\frac{\mu(t)}{\sigma_S}\right)^2-\frac{(1-\gamma)}{\gamma}\rho\mu(t)\sigma_\mu\left(\frac{\mu(t)}{\sigma_S}\right)\frac{H'}{H}+\frac{(1-\gamma)^2}{2\gamma(1-\psi)}\left(\rho\mu(t)\sigma_\mu\frac{H'}{H}\right)^2.
\end{aligned}
\tag{3.100}
$$

Finally we simplify to get

$$
\begin{aligned}
0 = {} & \beta\psi-\beta^{\psi}H^{-1}+r(1-\psi)-\mu(t)\left[\frac{1}{2}\sigma_\mu^2+\kappa(\ln(\theta)-\ln\mu(t))\right]\frac{H'}{H}+ \\
& \frac{\mu(t)^2\sigma_\mu^2}{2}\left[\left(1+\frac{1-\gamma}{1-\psi}\right)\left(\frac{H'}{H}\right)^2-\frac{H''}{H}\right]+\frac{1-\psi}{2\gamma}\left(\frac{\mu(t)}{\sigma_S}\right)^2- \\
& \frac{(1-\gamma)}{\gamma}\rho\mu(t)\sigma_\mu\left(\frac{\mu(t)}{\sigma_S}\right)\frac{H'}{H}+\frac{(1-\gamma)^2}{2\gamma(1-\psi)}\left(\rho\mu(t)\sigma_\mu\frac{H'}{H}\right)^2.
\end{aligned}
\tag{3.101}
$$

After combining H functions we are left with

$$
\begin{aligned}
0 = {} & \beta\psi-\frac{\beta^{\psi}}{H}+r(1-\psi)+\left[-\mu(t)\left[\frac{1}{2}\sigma_\mu^2+\kappa(\ln(\theta)-\ln\mu(t))\right]-\frac{(1-\gamma)}{\gamma}\rho\mu(t)\sigma_\mu\left(\frac{\mu(t)}{\sigma_S}\right)\right]\frac{H'}{H}+ \\
& \left[\frac{\mu(t)^2\sigma_\mu^2}{2}\left(1+\frac{1-\gamma}{1-\psi}\right)+\rho^2\mu(t)^2\sigma_\mu^2\frac{(1-\gamma)^2}{2\gamma(1-\psi)}\right]\left(\frac{H'}{H}\right)^2-\frac{\mu(t)^2\sigma_\mu^2}{2}\frac{H''}{H}+\frac{1-\psi}{2\gamma}\left(\frac{\mu(t)}{\sigma_S}\right)^2.
\end{aligned}
\tag{3.102}
$$

After multiplying by $H\frac{2}{\mu(t)^2\sigma_\mu^2}$ and grouping like terms, we have

$$
\begin{aligned}
0 = {} & \left[\beta\psi\frac{2}{\mu(t)^2\sigma_\mu^2}+r(1-\psi)\frac{2}{\mu(t)^2\sigma_\mu^2}+\frac{1-\psi}{\sigma_S^2\sigma_\mu^2\gamma}\right]H-\beta^{\psi}\frac{2}{\mu(t)^2\sigma_\mu^2}- \\
& 2\left[\frac{\sigma_\mu^2+2\kappa(\ln(\theta)-\ln\mu(t))}{2\mu(t)\sigma_\mu^2}+\frac{(1-\gamma)}{\gamma\sigma_S\sigma_\mu}\rho\right]H'+\left[\left(1+\frac{1-\gamma}{1-\psi}\right)+\rho^2\frac{(1-\gamma)^2}{\gamma(1-\psi)}\right]\left(\frac{(H')^2}{H}\right)-H''.
\end{aligned}
\tag{3.103}
$$

Finally, by factoring out $\frac{2}{\mu(t)^2\sigma_\mu^2}$ and rearranging the terms, we get the new ordinary differential equation we want to solve:

$$H'' + \frac{2}{\mu(t)^2\sigma_\mu^2}\left[\frac{\mu(t)\sigma_\mu^2}{2} + \mu(t)\kappa\left(\ln(\theta) - \ln\mu(t)\right) + \frac{(1-\gamma)}{\gamma\sigma_S}\rho\sigma_\mu\mu(t)^2\right]H' - \tag{3.104}$$

$$\frac{2}{\mu(t)^2\sigma_\mu^2}\left[\beta\psi + r\left(1-\psi\right) + \frac{1-\psi}{2\gamma\sigma_S^2}\mu(t)^2\right]H + \frac{2}{\mu(t)^2\sigma_\mu^2}\beta^\psi = \left[\left(1 + \frac{1-\gamma}{1-\psi}\right) + \rho^2\frac{(1-\gamma)^2}{\gamma(1-\psi)}\right]\left(\frac{(H')^2}{H}\right).$$

By comparing the new differential equation (3.104) to the original differential equation (1.1) of CCRV and CCH, we see that there are only a few differences. An extra $\frac{1}{\mu(t)^2}$ term is now present in the coefficients of H', H, and the constant. The $\kappa(\theta - \mu(t))$ term in the original coefficient of H' is now $\frac{\mu(t)\sigma_\mu^2}{2} + \mu(t)\kappa\left(\ln(\theta) - \ln\mu(t)\right)$ because of the revised value of $d\mu(t)$. Also $\frac{1-\gamma}{\gamma}\rho\sigma_\mu\frac{(\mu(t)-r)}{\sigma_S}$ in the original coefficient of H' is now $\frac{(1-\gamma)}{\gamma}\rho\sigma_\mu\frac{\mu(t)^2}{\sigma_S}$, and in the original coefficient of H, $\frac{1-\psi}{2\gamma}\left(\frac{\mu(t)-r}{\sigma_S}\right)^2$ is now $\frac{1-\psi}{2\gamma}\left(\frac{\mu(t)}{\sigma_S}\right)^2$. In both of these cases, $\mu(t) - r$ becomes $\mu(t)$ due to the revised value of $\frac{dS(t)}{S(t)}$.

4 Overview of the Mathematical Problem

4.1 Initial Conditions

For the initial conditions of the ODE, we need $H(\mu(0)) = H_0$ and $H'(\mu(0)) = H_1$. To find these, we use the same method as CCH, specifically, we first focus on the revised motion of wealth equation,

$$dW(t) = rW(t)dt + \alpha(t)^*W(t)\left[\mu(t)dt + \sigma_S d\omega_{S,t}\right] - C(t)^*dt. \tag{4.105}$$

Using CCRV's equation's for consumption, J, and $\frac{\partial J}{\partial W}$, or equations (3.79), (3.88), and (3.89) respectively, we can solve for $C(t)^*$, the optimal level of consumption,

$$\begin{aligned} C(t)^* &= \beta^\psi\left(\frac{\partial J}{\partial W}\right)^{-\psi}\left((1-\gamma)J\right)^{\frac{1-\gamma\psi}{(1-\gamma)}} \\ &= \beta^\psi\left(\frac{(1-\gamma)J}{W(t)}\right)^{-\psi}\left((1-\gamma)J\right)^{\frac{1-\gamma\psi}{(1-\gamma)}} \\ &= \left(\beta W(t)\right)^\psi\left((1-\gamma)J\right)^{\frac{1-\psi}{1-\gamma}} \\ &= \left(\beta W(t)\right)^\psi\left((1-\gamma)H(\mu(t))^{-\frac{1-\gamma}{1-\psi}}\frac{W(t)^{1-\gamma}}{1-\gamma}\right)^{\frac{1-\psi}{1-\gamma}} \\ &= W(t)\frac{\beta^\psi}{H(\mu(t))}. \end{aligned} \tag{4.106}$$

Now we solve for $\alpha(t)^*$, the optimal investment in stocks where $1-\alpha(t)^*$ is the optimal investment in bonds, by using our original equation for $\alpha(t)$, (3.81). We input the values of $\frac{\partial J}{\partial W}$, $\frac{\partial^2 J}{\partial W \partial W}$, and $\frac{\partial^2 J}{\partial W \partial \mu}$, or equations (3.89), (3.90), and (3.92) respectively, to get

$$\begin{aligned}
\alpha(t)^* &= \frac{-1}{W(t)\frac{\partial^2 J}{\partial W \partial W}} \left\{ \frac{\mu(t)}{\sigma_S^2} \frac{\partial J}{\partial W} + \frac{\rho \mu(t) \sigma_\mu}{\sigma_S} \frac{\partial^2 J}{\partial W \partial \mu} \right\} \\
&= \frac{-1}{W(t)\frac{-\gamma(1-\gamma)J}{W(t)^2}} \left\{ \frac{\mu(t)}{\sigma_S^2} \frac{(1-\gamma)J}{W(t)} + \frac{\rho \mu(t) \sigma_\mu}{\sigma_S} \left(-\frac{1-\gamma}{1-\psi} \frac{(1-\gamma)}{W(t)} J \frac{H'}{H} \right) \right\} \\
&= \frac{1}{\frac{\gamma(1-\gamma)}{W(t)}} \left\{ \frac{\mu(t)}{\sigma_S^2} \frac{(1-\gamma)}{W(t)} - \frac{\rho \mu(t) \sigma_\mu}{\sigma_S} \frac{1-\gamma}{1-\psi} \frac{(1-\gamma)}{W(t)} \frac{H'}{H} \right\} \\
&= \frac{\mu(t)}{\gamma \sigma_S^2} - \frac{\rho \mu(t) \sigma_\mu}{\gamma \sigma_S} \frac{1-\gamma}{1-\psi} \frac{H'}{H} \\
&= \frac{\mu(t)}{\gamma} \left\{ \frac{1}{\sigma_S^2} - \frac{\rho \sigma_\mu}{\sigma_S} \frac{1-\gamma}{1-\psi} \frac{H'}{H} \right\}.
\end{aligned} \tag{4.107}$$

We now plug $C(t)^*$, equation (4.106), and $\alpha(t)^*$, equation (4.107) into equation (4.105) to find the motion of wealth for the optimal values of consumption and investment in stocks and bonds,

$$\begin{aligned}
dW(t) &= rW(t)dt + \frac{\mu(t)}{\gamma} \left\{ \frac{1}{\sigma_S^2} - \frac{\rho \sigma_\mu}{\sigma_S} \frac{1-\gamma}{1-\psi} \frac{H'}{H} \right\} W(t) \left[\mu(t)dt + \sigma_S d\omega_{S,t} \right] - W(t) \frac{\beta^\psi}{H(\mu(t))} dt \\
&= W(t) \left[r - \frac{\beta^\psi}{H(\mu(t))} \right] dt + \frac{\mu(t) W(t)}{\gamma} \left\{ \frac{1}{\sigma_S^2} - \frac{\rho \sigma_\mu}{\sigma_S} \frac{1-\gamma}{1-\psi} \frac{H'}{H} \right\} \left[\mu(t)dt + \sigma_S d\omega_{S,t} \right] \\
&= W(t) \left[r - \frac{\beta^\psi}{H(\mu(t))} + \frac{\mu(t)^2}{\gamma} \left\{ \frac{1}{\sigma_S^2} - \frac{\rho \sigma_\mu}{\sigma_S} \frac{1-\gamma}{1-\psi} \frac{H'}{H} \right\} \right] dt + \frac{\mu(t) W(t)}{\gamma} \left\{ \frac{1}{\sigma_S^2} - \frac{\rho \sigma_\mu}{\sigma_S} \frac{1-\gamma}{1-\psi} \frac{H'}{H} \right\} \sigma_S d\omega_{S,t}.
\end{aligned} \tag{4.108}$$

Finally, dividing $dW(t)$ by $W(t)$ gives us

$$\frac{dW(t)}{W(t)} = \left[r - \frac{\beta^\psi}{H(\mu(t))} + \frac{\mu(t)^2}{\gamma} \left\{ \frac{1}{\sigma_S^2} - \frac{\rho \sigma_\mu}{\sigma_S} \frac{1-\gamma}{1-\psi} \frac{H'}{H} \right\} \right] dt + \frac{\mu(t)}{\gamma} \left\{ \frac{1}{\sigma_S^2} - \frac{\rho \sigma_\mu}{\sigma_S} \frac{1-\gamma}{1-\psi} \frac{H'}{H} \right\} \sigma_S d\omega_{S,t}. \tag{4.109}$$

Following CCH's example, we can now solve for H_0 and H_1 by using the equations for consumption and alpha. We can solve for H from equation (4.106),

$$C(t) = W(t) \frac{\beta^\psi}{H(\mu(t))}. \tag{4.110}$$

Solving this for H gives us

$$H(\mu(t)) = \beta^\psi \frac{W(t)}{C(t)}, \tag{4.111}$$

$$H_0 = H(\mu(0)) = \beta^{\psi} \frac{W(0)}{C(0)}. \tag{4.112}$$

Next we solve for H' from equation (4.107),

$$\alpha(t) = \frac{\mu(t)}{\gamma} \left\{ \frac{1}{\sigma_S^2} - \frac{\rho\sigma_\mu}{\sigma_S} \frac{1-\gamma}{1-\psi} \frac{H'}{H} \right\}, \tag{4.113}$$

$$H'(\mu(t)) = \frac{(1-\psi)}{\rho\sigma_\mu(1-\gamma)} \left(\frac{1}{\sigma_S} - \frac{\alpha(t)\gamma\sigma_S}{\mu(t)} \right) H(\mu(t)), \tag{4.114}$$

$$H'(\mu(0)) = \frac{(1-\psi)}{\rho\sigma_\mu(1-\gamma)} \left(\frac{1}{\sigma_S} - \frac{\alpha(0)\gamma\sigma_S}{\mu(0)} \right) H(\mu(0)). \tag{4.115}$$

The final condition of the ODE is called the transversality condition and is described by CCH. As $T \to \infty$, the condition must converge to zero for the optimal conditions of the ODE. This means the investor wants to optimally consume and invest his wealth until infinity when he expects to die and leave nothing behind. The transversality condition is

$$0 = \lim_{T\to\infty} E_t \left[e^{-\beta T} J(W(T), \mu(T)) \right], \tag{4.116}$$

which when we input the equation for J is

$$0 = \lim_{T\to\infty} E_t \left[e^{-\beta T} H(\mu(T))^{-\frac{1-\gamma}{1-\psi}} \frac{W(T)^{1-\gamma}}{1-\gamma} \right], \tag{4.117}$$

subject to $dW(t)$ and $d\mu(t)$.

4.2 Summary of the Mathematical Problem

We want to solve the following ODE

$$y''(x) + a(x)y'(x) + b(x)y(x) + g(x) = k\frac{(y'(x))^2}{y(x)}, \tag{4.118}$$

subject to two initial conditions

$$y(x(0)) = y_0 = \beta^{\psi} \frac{W(0)}{C(0)}, \tag{4.119}$$

and

$$y'(x(0)) = y_1 = \frac{(1-\psi)}{\rho\sigma_\mu(1-\gamma)} \left(\frac{1}{\sigma_S} - \frac{\alpha(0)\gamma\sigma_S}{x(0)} \right) y_0. \tag{4.120}$$

We also must ensure the transversality condition described in CCH holds for the optimal choices, so

$$\lim_{T\to\infty} E_t\left[e^{-\beta T} y(x(T))^{-\frac{1-\gamma}{1-\psi}} \frac{W(T)^{1-\gamma}}{1-\gamma}\right] = 0, \tag{4.121}$$

where the stochastic process for $x(t)$ is

$$dx(t) = x(t)\left[\frac{1}{2}\sigma_\mu^2 + \kappa(\ln(\theta) - \ln(x(t)))\right]dt + x(t)\sigma_\mu d\omega_{\mu,t}, \tag{4.122}$$

and the stochastic process for wealth is

$$\frac{dW(t)}{W(t)} = \left[r - \frac{\beta^\psi}{y(x)} + \frac{x(t)^2}{\gamma}\left\{\frac{1}{\sigma_S^2} - \frac{\rho\sigma_\mu}{\sigma_S}\frac{1-\gamma}{1-\psi}\frac{y'(x)}{y(x)}\right\}\right]dt + \frac{x(t)}{\gamma}\left\{\frac{1}{\sigma_S^2} - \frac{\rho\sigma_\mu}{\sigma_S}\frac{1-\gamma}{1-\psi}\frac{y'(x)}{y(x)}\right\}\sigma_S d\omega_{S,t}. \tag{4.123}$$

The coefficients for the ODE (4.118) are given by

$$a(x) \equiv \frac{2}{x(t)^2\sigma_\mu^2}\left[\frac{x(t)\sigma_\mu^2}{2} + x(t)\kappa\left(\ln(\theta) - \ln x(t)\right) + \frac{(1-\gamma)}{\gamma\sigma_S}\rho\sigma_\mu x(t)^2\right], \tag{4.124}$$

$$b(x) \equiv -\frac{2}{x(t)^2\sigma_\mu^2}\left[\beta\psi + r\left(1-\psi\right) + \frac{1-\psi}{2\sigma_S^2\gamma}x(t)^2\right], \tag{4.125}$$

$$g(x) \equiv \frac{2}{x(t)^2\sigma_\mu^2}\beta^\psi, \tag{4.126}$$

and

$$k \equiv \left(1 + \frac{1-\gamma}{1-\psi}\right) + \rho^2\frac{(1-\gamma)^2}{\gamma(1-\psi)}, \tag{4.127}$$

where k is the only constant.

5 Conclusion

In this paper we derived an ordinary differential equation that can be solved to determine an investor's optimal consumption and investment in stocks and bonds per period over his lifetime. The original ordinary differential equation found by CCRV relied on equations that occasionally permitted the expected return on stocks to be less than the expected return on bonds. If this were to happen, an arbitrage opportunity would result and no one would ever buy risky assets. Thus in this paper we altered the equations for $\frac{dS(t)}{S(t)}$ and $d\mu(t)$ before completing the calculations to ensure the equity premium is always positive which would prevent such an event from ever happening.

The new ODE can now be solved by the method developed by CCH which breaks the problem down into power series and solves it with a computer program. Although very similar to the original ODE found by CCRV and CCH, the new ODE contains $\ln(\theta)$ and $\ln \mu(t)$ terms not found in the original ODE in addition to more $\mu(t)^2$ terms. Thus, the computer program to solve the new ODE will have to be changed slightly from the old computer program.

Once this differential equation is solved using CCH's computer program, extensions of this problem can be explored and hopefully determined. For example, the model currently assumes the investor lives until infinity and has exactly \$0 when he dies. We can now begin to consider possible implications and solutions to the problem if the investor chooses to leave behind a specific sum of money, or the investor dies before infinity. We can try to solve the model for more realistic investment conditions using the derivation and solution of this ordinary differential equation as a starting point.

6 Appendix

6.1 Quarterly Parameter Values

The following quarterly values were estimated by Chen, Cosimano, and Himonas (2008) using data from the first quarter of 1947 to the fourth quarter of 2006.

$r = 0.00250$	$\kappa = 0.015346$
$\theta = 0.01452$	$\sigma_S = 0.080204$
$\sigma_\mu = 0.002161$	$\rho = -0.9583$

6.2 Ito's Lemma

The one-dimensional Ito's Lemma says that if a function $f(t, \mu(t))$ has a continuous derivative at t, two continuous derivatives at μ, and is subject to a stochastic process $d\mu$, then

$$df = f_t dt + f_\mu d\mu + \frac{1}{2} f_{\mu\mu} d\mu d\mu.^{6} \qquad (6.128)$$

[6] Jaksa Cvitanic and Fernando Zapatero, *Introduction to the Economics and Mathematics of Financial Markets* (Cambridge: The MIT Press, 2004), 70.

The two-dimensional Ito's Lemma says that if a function $J(W(t), \mu(t))$ has two continuous derivatives on each of W and μ and has continuous mixed partial derivatives between W and μ, then

$$dJ(W, \mu) = J_W dW + J_\mu d\mu + \frac{1}{2}(J_{WW} dW dW + 2J_{W\mu} dW d\mu + J_{\mu\mu} d\mu d\mu).^7 \tag{6.129}$$

6.3 Properties of Stochastic Calculus

From the lecture notes of Cosimano and Himonas for their Mathematical Methods in Financial Economics course at Notre Dame, we know that in stochastic calculus there are rules that can help us simplify our equations:

$(dt)^2 = 0$	$dt d\omega_{S,t} = 0$
$dt d\omega_{\mu,t} = 0$	$(d\omega_{\mu,t})^2 = dt$
$(d\omega_{S,t})^2 = dt$	$d\omega_{S,t} d\omega_{\mu,t} = \rho dt$

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[7] Cvitanic and Zapatero, 72.

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