# Solving Ramsey Problems with Nonlinear Projection Methods*

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**Abstract**

This paper applies nonlinear projection methods to solve Ramsey problems in a stochastic monetary economy. The presence of nonlinear distortions in the Ramsey problem requires the use of a solution procedure which captures these effects. The nonlinear projection method, even with low-order Chebyshev polynomials as employed in this paper, is able to capture a significant portion of the Jensen's inequality effects. As an example of the usefulness of nonlinear projection methods, we examine Barro's (1987, 1979) conjecture that welfare gains are available from policy smoothing with debt. Increases in the volatility of distortionary monetary policy are more than offset by declines in the volatility of distortionary labor taxes so that introduction of debt is welfare enhancing.

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## 1. Introduction

This paper applies nonlinear projection methods to solve Ramsey problems in a stochastic monetary economy. As an example of the usefulness of nonlinear projection methods, this paper examines whether the existence of debt is welfare enhancing through smoothing the response of distortionary fiscal and monetary policy to economic shocks, as suggested by Barro (1979, 1987). To answer this question, the traditional general equilibrium framework of macroeconomics is combined with the public finance approach from Ramsey (1927) to calibrate and simulate a stochastic monetary model under economies with and without debt. The model employed is a combination of a cash-in-advance model and a stochastic growth model where the household derives utility from leisure and consumption while the government raises revenue to finance its exogenous stochastic spending through distortionary means: a tax on labor income and the ability to print money.[1] The government also has the ability to issue debt.

The model captures the loss from distortionary government policy within the nonlinearity of the labor supply equation since the contemporaneous tax on labor income and money growth are determinants of optimal household labor supply in equilibrium. Therefore, the labor supply equation formalizes the assumption of a quadratic loss function over distortionary taxes and inflation as discussed in Bohn (1988) and Barro and Gordon (1981). Shocks that cause variations in fiscal and monetary policy are transmitted through optimal labor supply to output, remaining household allocations, and the equilibrium price system while feeding back into the government budget constraint through tax revenue. Equilibrium decisions by households, firms, and the government are then passed into future periods through the price level and interest rate equations. When choosing a combination of fiscal and monetary policy, the government must take into account the relationship between this policy mix and household labor supply to minimize distortions. Optimal policies, or Ramsey policies, maximize consumer welfare while minimizing distortions within the system.

The presence of nonlinear distortions to labor requires the use of a simulation procedure which captures these effects. As discussed in Kim and Kim (2003), traditional log-linearization techniques may not produce accurate welfare computations since it neglects higher moments which are essential for measuring overall risk and, therefore, estimation of welfare gains. In order to

[1] See Ljungqvist and Sargent (2000) for a recent formulation of optimal monetary and fiscal policy.

capture these higher moments, this paper employs the projection method of Judd (1992, 1998) to solve the Euler conditions for the optimal Ramsey policy for money growth, taxes, and debt. While the baseline economy contains no debt, the low-debt economy uses the prevailing debt-to-income ratio in the United States and the high-debt economy uses twice this level.

Since the set of operator equations is nonlinear, the projection approach begins by defining the policy functions in terms of Chebyshev polynomials. Arouba et al. (2003) examine various solution methods for dynamic equilibrium economies and find that Chebyshev polynomials are stable, accurate, and easier to implement than other techniques when the policy functions are smooth.[2] The set of operator equations also contains conditional expectations which must be evaluated. Since the processes that govern the shocks to technology and government spending are assumed to be normally distributed, expectations are evaluated using Gauss-Hermite Quadrature. In this procedure, the form of the policy function is assumed to be independent of the realization of the shocks. Expectations are found by integrating over the possible realizations of uncertainty while treating the policy function as a constant.

Once properly specified and calibrated to match the general features of the U.S. economy, the system is solved using a nonlinear equation optimizer in Matlab. Based on an initial guess of the polynomial coefficients, the program computes the residual functions and uses standard computational techniques to iterate to the Ramsey equilibrium. Then using the optimal Ramsey plan that defines policy choices of the government, allocations by the household, and the resulting price system, each economy was simulated under the effects of technology and government spending shocks in order to examine the affects of debt on optimal policies and activity.

As suggested by Barro (1979, 1987), debt provides the government with an additional degree of policy freedom relative to an economy without debt. Initially, the government sets tax and monetary policy based on generating an expected level of revenues to cover expected government expenditures. As shocks to technology and government spending affect the government budget

[2] In the competitive economy with distortions considered here, the policy functions for the private sector are continuous and unique for a given specification of government policy. See Coleman (1991) for a demonstration of the conditions necessary for the policy functions to be continuous and unique in an economy with distortions. In our particular case we are able to solve for explicit continuously differentiable equilibrium for the competitive equilibrium. The Ramsey problem will not yield a unique strategy for the government and may be a discontinuous bang-bang solution (Ljungqvist and Sargent (2000, Chapter 16)). However, in this work we only consider the commitment equilibrium which is a well defined solution to a dynamic programming problem following Abreu, Pearce and Stacchetti (1990). As a result, we characterize the equilibrium strategy with Chebsyshev polynomials.

constraint, optimal policy responds by smoothing the impact of distortionary taxes and money growth with debt. Simulations of the model economies indicate that volatility of distortionary policy in the economies with debt, as measured by standard deviation in percent, are lower than the baseline economy without debt. For example, the standard deviation of the tax rate falls from 2.85 percent under no debt to 2.46 percent under high debt. Declines in the volatility of taxes, however, are offset by slight increases in the volatility of money growth. Given that revenue policy is distortionary and the system is nonlinear, the net gain from the introduction of debt will depend on whether the reduction in tax volatility outweighs the loss from the increase in monetary policy volatility.

A utility equivalence framework is used to measure the gain to households from policy smoothing with debt. The gain is measured in terms of a lump-sum present discounted value of utility. Moving from an economy with no debt to the economies with debt is welfare improving since increased utility of consumption of the credit good and leisure more than offset the decline in utility of consumption of the cash good. The calculated welfare gains from policy smoothing are similar in magnitude to the gains from reducing business cycle volatility (Lucas 1987) or eliminating the costs of moderate inflation (Cooley and Hansen 1991).

## 2. A Stochastic Monetary Economy

The model is a combination of a cash-in-advance model and a stochastic growth model.[3] The economy is populated by a representative household, a representative firm, and a government. The household derives utility from leisure and two consumption goods, a cash good and a credit good. Previously accumulated cash balances are needed to purchase units of the cash good. Output is produced according to a production function that combines capital, labor, and technology, where the process governing technology is assumed to be exogenous and stochastic. The government raises revenue with distortionary effects to finance its exogenous stochastic spending through a tax on labor income, printing money, or financing debt. The Ramsey equilibrium determines optimal fiscal and monetary policy given the equilibrium behavior of the private sector. This Ramsey equilibrium may be reduced to four conditions for money growth, taxes, equilibrium labor, and the multiplier on the government budget

[3] This model is similar to models employed in Cooley and Hansen (1995), Chari et al. (1991), and Lucas and Stokey (1983).

constraint given the equilibrium behavior of interest and prices.[4]

### 2.1. Production

Aggregate output, $Y_t$, is produced according to the following constant returns-to-scale production function,

$$Y_t = \exp(\theta_t) H_t^{\alpha} K_t^{1-\alpha}, \quad 0 < \alpha < 1, \tag{2.1}$$

where $K_t$ and $H_t$ are the aggregate capital stock and labor supply, respectively, and $\theta_t$ represents the available technology. Technology is assumed to be the realization of an exogenous stochastic process and evolves according to the following law of motion,

$$\theta_t = \rho_\theta \theta_{t-1} + \epsilon_{\theta,t}, \quad 0 < \rho_\theta < 1. \tag{2.2}$$

The random variable, $\epsilon_{\theta,t}$, is normally distributed with mean zero and standard deviation $\sigma_{\theta,t}$ and the realization of $\epsilon_{\theta,t}$ is known to all agents at the beginning of period $t$. The portion of output that is not consumed is invested into physical capital. Investment in period $t$ produces capital in period $t+1$,

$$K_{t+1} = (1-\delta) K_t + X_t, \quad 0 < \delta < 1, \tag{2.3}$$

where $X_t$ is the level of investment in period $t$ and $\delta$ is the rate of depreciation.

The capital stock is assumed to remain fixed throughout the analysis so that $X_t = X = \delta K$. A constant capital stock can be justified, in part, by the well established result from the literature on optimal taxation that tax rates on capital should be close to zero on average.[5] Furthermore, firms are assumed to take depreciation charges before taxes are applied at the household level. If the constant capital stock assumption were maintained but firms were not allowed

[4] Assumptions of a fixed capital stock and logarithmic preferences enable computation of closed form equilibrium solutions for the private sector given a particular government policy. If the private sector is made more complex, these four conditions would need to be augmented with equilibrium conditions for interest rates and prices. These additional conditions would limit the accuracy of the projection method since additional equations would limit the number of nodes the computer can solve. In order to focus on the distortionary effects of labor taxes, a fixed capital stock is assumed. Thus, the optimal government policy will account for its impact on interest rates and prices as well as the optimal behavior of the household and firms.

[5] For earlier work on optimal capital income taxes, see Atkinson (1971), Diamond (1973), Pestieu (1974), Atkinson and Sandmo (1980), Judd (1985), Chamley (1986), and Chari et al. (1991, 1994).

to take depreciation charges before taxes were applied, the government would find it optimal to tax inelastically supplied investment and use the proceeds to retire money balances.

Given a fixed capital stock, the representative firm seeks to maximize profit, equal to $Y_t - w_t H_t - r_t K$, by choosing labor supply. The first-order conditions for the firm's problem describe the wage rate in equilibrium as,

$$w_t(\theta_t, H_t) = \alpha \frac{Y_t}{H_t}, \tag{2.4}$$

which is simply the marginal product of labor. The rental rate on capital in equilibrium is,

$$r_t(\theta_t, H_t) = (1-\alpha)\frac{Y_t}{K}, \tag{2.5}$$

which is the marginal product of capital. Income from the production process is then returned to the household net of depreciation. Given that the capital stock is assumed to remain fixed and investment each period covers depreciation, income from the production process sent to the household each period is closely related to the concept of net national product.

### 2.2. Households

Household preferences are summarized by the following utility function,

$$E_t \sum_{t=0}^{\infty} \beta^t \left[\eta \ln C_{1t} + (1-\eta)\ln C_{2t} - \gamma H_t\right], \tag{2.6}$$

where $C_1$ is the cash good, $C_2$ is the credit good, $\gamma$ is a positive constant and $0 < \beta, \eta < 1$. The specification of linear labor supply is discussed in Hansen (1985) and Rogerson (1988) and is derived from the assumptions that labor is indivisible and allocation of labor is determined by employment lotteries.

The household enters period $t$ with previously accumulated assets equal to the stock of money holdings, $M_t$, and gross returns from government bonds, $B_t R_{t-1}$, where $B_t$ is the stock of real debt and $R_{t-1}$ is the gross real interest rate. These assets augment the income received from capital and the after-tax income from labor and are used to finance household expenditures during the period. The current shocks to technology and government spending are revealed at the beginning of the period. After the shocks are revealed, expectations are formed and the household then chooses labor supply, consumption, money balances for the next period, and bonds. Overall, household allocations must satisfy the following budget constraint,

$$C_{1t} + C_{2t} + \frac{M_{t+1}^{d}}{P_t} + B_{t+1} \leq (1 - \alpha\tau_t)(Y_t - X) + \frac{M_t}{P_t} + B_t R_{t-1}, \qquad (2.7)$$

where $P_t$ equals the price level and $\tau_t$ is the tax on labor income, $\alpha Y_t$. The term $M_{t+1}^{d}$ is the demand for money balances and is aggregated across households in relation to money supply in equilibrium. Previously accumulated money balances are used to purchase the cash good in the current period and must also satisfy the cash-in-advance constraint,

$$P_t C_{1t} \leq M_t. \qquad (2.8)$$

Assuming money supply equals money demand in equilibrium, or $M_{t+1} = M_{t+1}^{d}$, the first-order conditions from the household's problem can be combined to derive closed-form solutions for consumption which are as follows,[6]

$$C_{1t} = \frac{(Y_t - X - G_t)\,\beta\left(\frac{\eta}{1-\eta}\right)\exp(-\mu_{t+1})}{1 + \beta\left(\frac{\eta}{1-\eta}\right)\exp(-\mu_{t+1})}, \qquad (2.9)$$

$$C_{2t} = \frac{(Y_t - X - G_t)}{1 + \beta\left(\frac{\eta}{1-\eta}\right)\exp(-\mu_{t+1})}, \qquad (2.10)$$

where $\exp(\mu_{t+1}) = M_{t+1}/M_t$ is the gross growth rate of money and $G_t$ is the level of real government consumption. Inserting the solution for the cash good in (2.9) into the cash-in-advance constraint in (2.8), which holds with equality in equilibrium, produces the following closed-form equation for the price level,

$$P_t = \frac{M_t}{(Y_t - X - G_t)}\left[\frac{1 + \beta\left(\frac{\eta}{1-\eta}\right)\exp(-\mu_{t+1})}{\beta\left(\frac{\eta}{1-\eta}\right)\exp(-\mu_{t+1})}\right]. \qquad (2.11)$$

The closed-form solution for the interest rate is found by inserting the solution for the credit good in 2.10 into the equilibrium interest rate condition,

$$R_t = \frac{1}{\beta C_{2t}}\left[\frac{1}{E_t\left[\frac{1}{C_{2t+1}}\right]}\right]. \qquad (2.12)$$

[6]See Appendix for details.

The solution for the credit good in (2.10) along with the wage rate in (2.4) can be substituted into the Euler condition for labor to solve for the equilibrium quantity of labor. Doing so, and noting the specification for output in (2.1), defines an implicit function,

$$H_t = h\left(g_t, \theta_t, \mu_{t+1}, \tau_t\right). \tag{2.13}$$

This equation cannot be solved for $H_t$ explicitly, but the implicit function theorem allows for the construction of an implicit function which defines the explicit function. Defined derivatives can be obtained since an implicit function is known to exist under the implicit function theorem.[7]

Optimal household allocations and the equilibrium price system are all functions of contemporaneous government policy and the exogenous shocks to government spending and technology. However, the equilibrium price system is also dependent on past policy and expectations of future policy and uncertainty. The price level is dependent on the choice of money balances during the previous period which is a result of the cash-in-advance specification. The interest rate in period $t$ is also a function of the expectation over future government policy and labor supply decisions in period $t+1$ since the interest on the stock of debt chosen by the household in period $t$ will not be available for use until period $t+1$.

The nonlinearity of the equilibrium labor condition defines a loss function over government policy and shocks since the contemporaneous tax on labor income and money growth are determinants of optimal household labor supply. In addition, the shocks to technology and government spending also induce equilibrium responses by households. While not explicitly present in the equilibrium labor function, debt is indirectly present since the choices of taxes and money determine the level of debt as a residual in the government budget constraint, described in more detail in the next section. Equilibrium decisions by households, firms, and the government are then transmitted across time through the price level and the interest rate.

### 2.3. The Ramsey equilibrium

The goal of the government is to maximize the welfare of the household subject to raising revenues through distortionary means. Real government consumption, $G_t$, is assumed to follow an exogenous stochastic process. Government policy includes sequences of labor taxes and supplies of money, and bonds

[7] See Appendix for details.

which must satisfy the government's budget constraint,

$$\frac{M_t}{P_t} + B_t R_{t-1} = \alpha\tau_t\left(Y_t - X\right) - G_t + B_{t+1} + \frac{M_{t+1}}{P_t}, \tag{2.14}$$

where the initial stocks of money, $M_0$, and bonds, $B_0$, are given. The money supply and government spending in period $t$ are assumed to grow at the rate $\exp(\mu_{t+1}) - 1$ and $\exp(g_t) - 1$, respectively. Thus, the level of government spending and money stock are defined as,

$$G_t = \exp(g_t)G_{t-1}, \tag{2.15}$$

$$M_{t+1} = \exp(\mu_{t+1})M_t. \tag{2.16}$$

The random variable $g_t$ is assumed to evolve according to the following autoregressive process,

$$g_t = \rho_g g_{t-1} + \epsilon_{g,t}, \quad 0 < \rho_g < 1, \tag{2.17}$$

and $\epsilon_{g,t}$ is normally distributed with mean zero and standard deviation $\sigma_{g,t}$. Like the shock to technology, the realization of $\epsilon_{g,t}$ is known to all at the beginning of period $t$. After the shocks to the system are revealed, the government selects a policy profile and households respond with a set of allocations. The resulting equilibrium determines the state variables for the next period. Therefore, when choosing an optimal policy mix of taxes, money supply, and debt, the government must take into account the equilibrium reactions by households and firms to the chosen policy mix. The government is also constrained in its policy choices since it must choose a policy mix to maximize household utility while satisfying the government budget constraint.

The Ramsey problem in the general equilibrium dynamic programming setting incorporates many of the reputational mechanisms for credible government policies as discussed in Ljungqvist and Sargent (2000).[8] In general, the government would find it optimal to deviate from its original set of policies if allowed and some mechanism, reputational or otherwise, is needed to ensure credibility of government policy. Under the assumption that an institution or commitment technology exists through which the government can bind itself to a particular sequence of policies, the government solves for the commitment equilibrium by maximizing (2.6) subject to (2.14) while taking into account the equilibrium specification for the price system in equations (2.12) and (2.11)

[8] They use the analysis of Abreu, Pearce and Stacchetti (1990) to represent the government policy game as a dynamic programming problem.

and the optimal response by households and firms in equations (2.1), (2.9), (2.10), and (2.13).

Optimal policy is a mapping of state variables to labor taxes, money supply, and the amount of debt so that the government's budget constraint is satisfied. Like the household's problem, the government's problem can be set up as a dynamic programming problem whereby the government seeks to maximize,

$$V(s_t) = \underset{\Delta_t}{Max} \left\{ \begin{array}{c} \eta \ln C_{1t} + (1-\eta)\ln C_{2t} - \gamma H_t + \\ \lambda_{gt} \left( \begin{array}{c} \alpha\tau_t (Y_t - X) - G_t + B_{t+1} + \\ (\exp(\mu_{t+1}) - 1)\frac{M_t}{P_t} - B_t R_{t-1} \end{array} \right) + \beta E_t V(s_{t+1}) \end{array} \right\} \quad (2.18)$$

where $\Delta_t = (\tau_t,\, \mu_{t+1},\, B_{t+1})$ is the set of choice variables and $s_t$ represents the set of state variables $\left(B_t,\, \frac{M_t^d}{P_{t-1}},\, \theta_{t-1},\, g_{t-1},\, \tau_{t-1},\, R_{t-1}\right)$. Here the government takes consumption of the cash good as given by (2.9), consumption of the credit good by (2.10), labor is given by (2.13), output satisfies the production function in (2.1), the price level is given by (2.11) and the interest rate on debt is (2.12). $\lambda_{gt}$ is the Lagrange multiplier on the government budget constraint. The first-order conditions for the Ramsey problem are,[9]

$$\tau_t \ : \ \left\{ \begin{array}{c} \frac{\eta}{C_{1t}}\frac{\partial C_{1t}}{\partial \tau_t} + \frac{1-\eta}{C_{2t}}\frac{\partial C_{2t}}{\partial \tau_t} - \gamma\frac{\partial H_t}{\partial \tau_t} + \\ \lambda_{gt}\left[ \begin{array}{c} \alpha\tau_t\frac{\partial Y_t}{\partial \tau_t} + \alpha(Y_t - X) - B_t\frac{\partial R_{t-1}}{\partial \tau_t} \\ -\left(\exp(\mu_{t+1}) - 1\right)\frac{M_t}{P_t}\frac{1}{P_t}\frac{\partial P_t}{\partial \tau_t} \end{array} \right] \end{array} \right\} = \quad (2.19)$$
$$\beta E_t\left\{\lambda_{gt+1} B_{t+1}\frac{\partial R_t}{\partial \tau_t}\right\},$$

$$\mu_{t+1} \ : \ \left\{ \begin{array}{c} \frac{\eta}{C_{1t}}\frac{\partial C_{1t}}{\partial \mu_{t+1}} + \frac{1-\eta}{C_{2t}}\frac{\partial C_{2t}}{\partial \mu_{t+1}} - \gamma\frac{\partial H_t}{\partial \mu_{t+1}} + \\ \lambda_{gt}\left[ \begin{array}{c} \alpha\tau_t\frac{\partial Y_t}{\partial \mu_{t+1}} - \frac{M_{t+1}}{P_t}\exp(\mu_{t+1}) - B_t\frac{\partial R_{t-1}}{\partial \mu_{t+1}} + \\ -\left(\exp(\mu_{t+1}) - 1\right)\frac{M_t}{P_t}\frac{1}{P_t}\frac{\partial P_t}{\partial \mu_{t+1}} \end{array} \right] \end{array} \right\} = \quad (2.20)$$
$$\beta E_t\left\{\lambda_{gt+1}\left[B_{t+1}\frac{\partial R_t}{\partial \mu_{t+1}}\right]\right\},$$

$$B_{t+1} : \lambda_{gt} = \beta E_t\left\{\lambda_{gt+1} R_t\right\}, \quad (2.21)$$

where the Benveniste - Scheinkman condition (Benveniste and Scheinkman, 1979) is used repeatedly to replace $\partial V/\partial\Delta$.

[9]The first order condition for money shown here is $\partial/\partial\left(\exp(-\mu_{t+1})\right)$ for simplicity. The optimal government policy for money balances can then be found by taking the $-\ln(x)$ of the result.

The distortionary effects of money and tax policy are evident through examination of the first-order conditions to the government's problem. For example, the Euler condition in (2.19) describes the trade-off between taxation and issuing debt. The first terms on the left-hand side reflect the change in consumption of the cash and credit goods and provision of labor by the household from a change in taxes. A change in the tax rate enters consumption of the cash and credit good indirectly via the equilibrium labor condition. The partial derivative of the equilibrium labor condition with respect to taxes is negative, so that $\partial H_t/\partial \tau_t$, $\partial C_{1t}/\partial \tau_t$, $\partial C_{2t}/\partial \tau_t$ are all negative. Higher labor income taxes discourage labor supply, reduce output, and decrease consumption. The net impact on household utility is negative. The bracketed term in (2.19) describes the change in the government budget constraint from a change in taxes scaled by the multiplier. The first terms inside the bracket represent the direct change in tax revenue from a change in tax policy, the sign of which depends on the nonlinear response of labor supply to a change in taxes. The next term results from the commitment technology and details the change in the interest rate on debt during the previous period, respectively, from a change in the one-period ahead tax rate. The remaining term inside the bracket describes the price effect on nominal balances. In particular, an increase in taxes increases the price level today since consumption of the cash good falls, reducing the real value of money balances chosen during the previous period. These combined effects in the left-hand side of (2.19) must be equal to the alternative policy of issuing additional debt carried into the next period.

The trade-off between issuing money and debt is described in (2.20) and is more complicated since money enters (2.20) directly through the money growth term and indirectly through the equilibrium labor condition. The first terms on the left-hand side detail the effects of money growth on consumption and labor supply. Increases in money growth decrease equilibrium labor so that $\partial H_t/\partial \mu_{t+1}$ is negative. Combined with the direct effects of money on consumption, $\partial C_{1t}/\partial \mu_{t+1}$ and $\partial C_{2t}/\partial \mu_{t+1}$ are also negative. The bracketed term, as in the tax derivative, details the impact of changes in money on the government budget constraint scaled by the multiplier. The first term describes the change in labor tax revenue based on the change in equilibrium labor from changes in money growth. Increases in money growth that decrease equilibrium labor result in lower output and reduced labor tax revenue. The second term relates to seigniorage revenues. The next term arises from the commitment technology and the remaining term describes the price effect on nominal balances. Increases in money growth result in a higher price level, reducing the real value of nominal money balances chosen during the previous period. These combined effects on the left-hand side must be equal to the alternative policy of issuing debt which matures during the next period.

The set of Euler conditions from the Ramsey problem, the labor equation from the household's problem, and the government budget constraint can be generalized to a set of four operator equations $N(f)$ that define equilibrium,

$$N_\tau\left(H_t,\, \mu_{t+1},\, \tau_t,\, g_t,\, \theta_t,\, B_{t+1},\, R_t,\, s_t\right) = 0, \tag{2.22}$$

$$N_\mu\left(H_t,\, \mu_{t+1},\, \tau_t,\, g_t,\, \theta_t,\, B_{t+1},\, R_t,\, s_t\right) = 0, \tag{2.23}$$

$$N_H\left(H_t,\, \mu_{t+1},\, \tau_t,\, g_t,\, \theta_t,\, B_{t+1},\, R_t,\, s_t\right) = 0, \tag{2.24}$$

$$N_\lambda\left(H_t,\, \mu_{t+1},\, \tau_t,\, g_t,\, \theta_t,\, B_{t+1},\, R_t,\, s_t\right) = 0. \tag{2.25}$$

The solution procedure described in detail in the section below optimally solves for government policy, labor supply, and the multiplier on the government budget constraint as functions of state variables including uncertainty in the system. Consequently, the solution method applied in this paper differs from the more traditional primal approach. The primal approach recasts the problem of choosing optimal policy as a problem of choosing allocations subject to constraints which capture restrictions on those allocations.[10] In practice this means using an infinite horizon budget constraint with prices and policy substituted out using first-order conditions, commonly referred to as the implementability constraint. The use of the implementability constraint often requires a search procedure that iterates across candidate solutions for the multiplier (Chari et al., 1994) as opposed to endogenously solving for the multiplier as is done in this paper. Furthermore, the multiplier on the implementability constraint has a different interpretation than the multiplier in this paper. In this analysis, the multiplier on the government budget constraint represents the shadow value of reducing debt or, equivalently, the value of the ability to collect a lump-sum tax. A solution procedure which solves for the multiplier as an endogenous variables is preferred in this instance since the multiplier is a crucial element in determining the cost of distortionary policy.

## 3. Projection Method

The characterization of the policies that generate the Ramsey equilibrium theoretically is difficult since the system is nonlinear. Therefore, the system is characterized numerically. Following the process in Cooley and Hansen (1995), Hansen and Wright (1992), Christiano and Eichenbaum (1992), Chari et al.

[10] See Chari et al. (1994) and Chari and Kehoe (1999) for examples of the primal approach.

Table 3.1: Parameter Values Corresponding to U.S. Economy.

| Parameter Values | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $\beta$ | $\eta$ | $\gamma$ | $\delta$ | $\rho_\theta$ | $\sigma_\theta$ | $\rho_g$ | $\sigma_g$ |
| 0.6 | 0.991 | 0.44 | 2.47 | 0.017 | 0.95 | 0.007 | 0.96 | 0.021 |

(1991), and Hansen (1985), the model is calibrated to match the general features of the U.S. economy. Parameter values are chosen such that elements of the non-stochastic steady state match the average values from the post-Korean War U.S. time series and are summarized in Table 3.1. In estimating the government spending process and the share of output attributable to capital and labor, a gross capital concept is assumed so that the assumed constant investment includes government investment. Ratios relative to net national product are used instead of gross domestic product, although either procedure could be implemented. The ratio of the cash good to the credit good was estimated using first order conditions from the household problem.[11] The initial stock of debt was calibrated from the NIPA database using federal debt held in the hands of the public.[12] Non-marketable government debt was excluded from the sample. Based on this data, a ratio of debt to net national product of 0.49 is used to simulate the U.S. debt-to-income economy or "low" debt economy. The "high" debt economy is calibrated to twice this level.

The computational solution procedure is based on the projection approach as described in Judd (1992, 1998). The four operator equations in equations (2.22) - (2.25), are continuous maps where $N : B_1 \to B_2$, $B_1$ and $B_2$ are complete normed vector spaces of the functions $f : D \subset R^n \to R^m$. $D$ is a compact subset of $R$ and, as discussed below, will be defined over $D = [-1, 1]$. The solution procedure will solve for the optimal set of policies $(H_t,\, \mu_{t+1},\, \tau_t,\, \lambda_{gt})$ as functions of the exogenous shocks $(g_t,\, \theta_t)$ and state variables $s_t = \left(B_t,\, \frac{M_t^d}{P_{t-1}},\, \theta_{t-1},\, g_{t-1},\, \tau_{t-1},\, R_{t-1}\right)$ that set $N(f) = 0$ simultaneously and hence satisfy the Ramsey equilibrium. In addition to the various allocation rules, price systems, and government policy which are based on current or past information, the equilibrium interest rate function also includes condi-

[11] See Chari et al. (1991), for example.

[12] Based on the government budget constraint in this model, a more accurate measure of debt would include combined federal, state, and local debt. However, many states have balanced budget amendments which restrict the ability of states and local governments to issue debt for purposes other than infrastructure development and cash management operations, limiting the ability of states to issue debt for policy smoothing purposes. Furthermore, calibrating the level of debt is less important in this model since the initial level is set exogenously and can be adjusted as is done in this paper.

tional expectations over future government policy and the shocks to technology and government spending. Consequently, it is through the interest rate function that expectations over future policy are translated into current allocation decisions.

### 3.1. Basis and degree of approximation

The solution to the system, $N(f) = 0$, is represented by assuming that the approximation, $\widehat{f}$, to the true solution, $f$, is built up as a combination of some function of the basis $B_1$.[13] A good basis for the vector space is often found among the family of orthogonal polynomials.[14] The most useful of these polynomials have a recursive representation in terms of eigenfunctions and a weighting function such that the integral over $[a, b]$ forms a complete orthogonal set of functions. Orthogonal bases are important since nonorthogonal bases reduce numerical accuracy in the same manner in which multicollinearity widens confidence intervals in regression analysis.[15] It is for this reason that orthogonal bases are chosen.

In this paper Chebyshev polynomials are used to form the approximations to the true policy functions. Chebyshev polynomials, $T_n(x)$, are solutions to the Chebyshev differential equation and form a complete orthogonal set on the interval $[-1, 1]$ with weighting function $(1 - x^2)^{-1/2}$. A Chebyshev polynomial at one point can be expressed by neighboring Chebyshev polynomials at the same point using the following recursive formula,

$$T_0(x) = 0, \quad T_1(x) = x, \tag{3.1}$$

$$T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x), \tag{3.2}$$

where $T_n(x) = \cos(n \cos^{-1} x)$.

Applied to the Ramsey problem described in this paper, the true optimal

[13] A basis is a set of linearly independent vectors in a vector space such that any vector in the vector space can be expressed as a linear combination of them with appropriately chosen coefficients.

[14] Examples of such families of orthogonal polynomials are Legendre polynomials, Laguerre polynomials, Hermite polynomials, and Chebyshev polynomials. Each represents a complete orthogonal set under different circumstances.

[15] Standard solution procedures require an initial guess of the set of coefficients followed by an iterative method which converges to the optimal set of coefficients. The accuracy of the final solution to the system of equations depends directly on the non-colliniearity of the rows and columns of the matrix of coefficients used in the iterative solution procedure. In particular, the iterative process may rely on the Jacobian and an orthogonal basis will reduce the likelihood of a poorly conditioned Jacobian.

policy functions ($H_t$, $\mu_{t+1}$, $\tau_t$, $\lambda_{gt}$) are approximated by Chebyshev polynomial functions and the solution procedure will yield the set of coefficients $(b, d, q, v)$ that describe how policy will respond to the exogenous shocks to technology and government spending. Given the choice of polynomial function, the approximation of the set of policy functions are,

$$\widehat{\mu}_{t+1}(\theta_t, g_t, b) = \sum_{i=1}^{n_\theta} \sum_{j=1}^{n_g} b_{ij} \Psi_{ij}(\theta_t, g_t), \tag{3.3}$$

$$\widehat{\tau}_t(\theta_t, g_t, d) = \sum_{i=1}^{n_\theta} \sum_{j=1}^{n_g} d_{ij} \Omega_{ij}(\theta_t, g_t), \tag{3.4}$$

$$\widehat{H}_t(\theta_t, g_t, q) = \sum_{i=1}^{n_\theta} \sum_{j=1}^{n_g} q_{ij} \Phi_{ij}(\theta_t, g_t), \tag{3.5}$$

$$\widehat{\lambda}_{gt}(\theta_t, g_t, v) = \sum_{i=1}^{n_\theta} \sum_{j=1}^{n_g} v_{ij} \Gamma_{ij}(\theta_t, g_t). \tag{3.6}$$

The system is therefore reduced from an infinite-dimensional system to a finite-dimensional system; the number of coefficients can be varied and the candidate solutions can be tested using various standard tests of convergence until the "true" solution is found.

Since the special properties of Chebyshev polynomials apply their restrictions over $[-1, 1]$, a linear transformation is applied to the state space of $\theta$ and $g$. The initial state space is $[\theta_m, \theta_M] \times [g_m, g_M]$ where $m$ refers to the minimum value and $M$ to the maximum value in the range. It is hoped that the solution is a good approximation on $[\theta_m, \theta_M] \times [g_m, g_M]$. Since $\theta$ and $g$ are both exogenously defined, the interval for each is taken as a multiple of the standard deviation of the error process, $\epsilon_{\theta,t}$ and $\epsilon_{g,t}$, both of which are normally distributed with mean equal to zero and variances which are determined by historical data. This procedure adds four additional parameters to the problem $\theta_m$, $\theta_M$, $g_m$, and $g_M$. Using Chebyshev polynomials in (3.3) - (3.6) along with the transformation to the $[-1, 1]$ interval yields,

$$\Psi_{ij}(\theta_t, g_t) \equiv T_{i-1}\left(2\left(\frac{\theta - \theta_M}{\theta_M - \theta_m}\right) - 1\right) T_{j-1}\left(2\left(\frac{g - g_M}{g_M - g_m}\right) - 1\right), \tag{3.7}$$

$$\Omega_{ij}(\theta_t, g_t) \equiv T_{i-1}\left(2\left(\frac{\theta - \theta_M}{\theta_M - \theta_m}\right) - 1\right) T_{j-1}\left(2\left(\frac{g - g_M}{g_M - g_m}\right) - 1\right), \tag{3.8}$$

$$\Phi_{ij}(\theta_t, g_t) \equiv T_{i-1}\left(2\left(\frac{\theta-\theta_M}{\theta_M-\theta_m}\right)-1\right)T_{j-1}\left(2\left(\frac{g-g_M}{g_M-g_m}\right)-1\right), \quad (3.9)$$

$$\Gamma_{ij}(\theta_t, g_t) \equiv T_{i-1}\left(2\left(\frac{\theta-\theta_M}{\theta_M-\theta_m}\right)-1\right)T_{j-1}\left(2\left(\frac{g-g_M}{g_M-g_m}\right)-1\right). \quad (3.10)$$

### 3.2. Gauss-Hermite quadrature

The system of equations in (2.22) - (2.25) also contain conditional expectations which must be evaluated. Since the processes that define $\theta$ and $g$ are both assumed to be distributed $N(0, \sigma^2_{\theta,g})$, expectations are evaluated using Gauss-Hermite Quadrature. As the form of the policy function is independent of the realization of the shocks to $\theta$ and $g$, expectations are found by integrating over the possible realizations of $\theta$ and $g$ while treating the policy function as a constant.

If $\theta_t \sim N(0, \sigma^2_\theta)$, $g_t \sim N(0, \sigma^2_g)$, and $\rho_{\theta g} = 0$ where $\rho_{\theta g}$ is the covariance between $\theta$ and $g$, then Gauss-Hermite Quadrature specifies,

$$\begin{aligned} & E_{t-1}\left[f(\theta_t, g_t)\right] \\ = \; & (2\pi\sigma_\theta\sigma_g)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\theta_t, g_t)\exp\left(-\frac{(\theta-\mu_\theta)^2}{2\sigma^2_\theta}-\frac{(g-\mu_g)^2}{2\sigma^2_g}\right)d\theta dg. \end{aligned} \quad (3.11)$$

Next, implement a change of variables where,

$$x_\theta = \frac{(\theta-\mu_\theta)}{\sqrt{2}\sigma_\theta}, \quad x_g = \frac{(g-\mu_g)}{\sqrt{2}\sigma_g}, \quad (3.12)$$

and use the following identity with the assumption of a mean zero process,

$$\begin{aligned} & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\theta_t, g_t)\exp\left(-\frac{(\theta-\mu_\theta)^2}{2\sigma^2_\theta}-\frac{(g-\mu_g)^2}{2\sigma^2_g}\right)d\theta dg \quad (3.13) \\ = \; & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\sqrt{2}\sigma_\theta x_\theta, \sqrt{2}\sigma_g x_g\right)\exp\left(-x^2_\theta-x^2_g\right)2\sigma_\theta\sigma_g dx_\theta dx_g. \end{aligned}$$

Cancelling the appropriate terms yields the general Gauss-Hermite Quadrature rule to evaluate expectations of functions of two normal random variables,

$$E_{t-1}\left[f(\theta_t, g_t)\right] = \pi^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\sqrt{2}\sigma_\theta x_\theta, \sqrt{2}\sigma_g x_g\right)\exp\left(-x^2_\theta-x^2_g\right)dx_\theta dx_g \quad (3.14)$$

$$= \pi^{-1} \sum_{l_\theta=1}^{r_\theta} \sum_{l_g=1}^{r_g} w_{l_\theta} w_{l_g} f\left(\sqrt{2}\sigma_\theta x_{l_\theta}, \sqrt{2}\sigma_g x_{l_g}\right), \tag{3.15}$$

where $w_{l_\theta} = \exp\left(-x_{l_\theta}^2\right)$, $w_{l_g} = \exp\left(-x_{l_g}^2\right)$ are the weights and $x_{l_\theta}$, $x_{l_g}$ are the nodes under Gauss-Hermite Quadrature.[16] This procedure produces a $r_\theta \times r_g$ system of equations where the choice of $r_\theta$ defines the weights and nodes for the distribution of $\theta$ and $r_g$ defines the weights and nodes for the distribution of $g$. In addition, the choice of $r$ can be on the interval of $[-\infty, \infty]$. Therefore, a larger $r$ produces a more accurate estimation, but increases computation time of the solution procedure. Fortunately, since the weighting function is $\exp(-x^2)$, most of the density is captured at a relatively low order since $\exp(-x^2)$ approaches zero as $x$ passes 2. A modest selection of $r$ will enable an accurate solution to be reached without adding too much time to the computational procedure. For the simulations in this paper, $r_\theta = r_g = 11$ was used.

As an example of this procedure, the equilibrium solution for the interest rate in period $t$ is,

$$R_t = \frac{1}{\beta C_{2t}} \left[ \frac{1}{E_t\left[\frac{1+\beta\left(\frac{1}{1-\eta}\right)\exp\left(-\widehat{\mu}_{t+2}(\theta_{t+1}, g_{t+1}, b)\right)}{\exp(\theta_{t+1})\left(\widehat{H}_{t+1}(\theta_{t+1}, g_{t+1}, q)\right)^\alpha K^{1-\alpha} - X - \exp(g_{t+1})G_t}\right]} \right], \tag{3.16}$$

where the equilibrium value of $C_{2t}$ is given by (2.10). Following the steps outlined in (3.11) - (3.15), the computation of the expected value term in the denominator involves,

$$\begin{aligned} & E_t\left[f\left(\exp\left(-\widehat{\mu}_{t+2}\left(\theta_{t+1}, g_{t+1}, b\right)\right), \theta_{t+1}, \widehat{H}_{t+1}\left(\theta_{t+1}, g_{t+1}, q\right), g_{t+1}\right)\right] \\ = \quad & \pi^{-1} \sum_{l_\theta=1}^{r_\theta} \sum_{l_g=1}^{r_g} w_{l_\theta} w_{r_g} \left[\digamma\right], \end{aligned} \tag{3.17}$$

where $\digamma$ is a function of,

$$\left( \begin{array}{c} \exp\left(-\widehat{\mu}_{t+2}\left(\rho_\theta \theta_t + \sqrt{2}\sigma_\theta x_{l_\theta},\ \rho_g g_t + \sqrt{2}\sigma_g x_{l_g},\ b\right)\right), \rho_\theta \theta_t + \sqrt{2}\sigma_\theta x_{l_\theta}, \\ \widehat{H}_{t+1}\left(\rho_\theta \theta_t + \sqrt{2}\sigma_\theta x_{l_\theta},\ \rho_g g_t + \sqrt{2}\sigma_g x_{l_g},\ q\right), \rho_g g_t + \sqrt{2}\sigma_g x_{l_g} \end{array} \right) \tag{3.18}$$

and $\theta_t$ and $g_t$ are known when expectations are formed in period $t$.

[16] For a table on Gauss-Hermite Quadrature weights and nodes, see Stroud and Secrest (1966) or Judd (1998).

### 3.3. Galerkin and Chebyshev collocation

Inserting the Chebyshev polynomial approximations and Gauss-Hermite Quadrature for the expectation operators into the set of operator equations in (2.22) - (2.25) yields the following stacked set of residual functions,

$$R_\tau\left(\theta_t, g_t; b, d, q, v\right) = N\left(\widehat{f}_\tau\left(\theta_t, g_t\right)\right) = 0, \tag{3.19}$$

$$R_\mu\left(\theta_t, g_t; b, d, q, v\right) = N\left(\widehat{f}_\mu\left(\theta_t, g_t\right)\right) = 0, \tag{3.20}$$

$$R_H\left(\theta_t, g_t; b, d, q, v\right) = N\left(\widehat{f}_H\left(\theta_t, g_t\right)\right) = 0, \tag{3.21}$$

$$R_\lambda\left(\theta_t, g_t; b, d, q, v\right) = N\left(\widehat{f}_\lambda\left(\theta_t, g_t\right)\right) = 0. \tag{3.22}$$

These equations are referred to as residual functions since the object will be to find the coefficients that yield the zero function, or minimize the residual of each equation. The *Galerkin* approach uses the first $n$ elements of the basis, which determines the number of coefficients ($b$, $d$, $q$, and $v$), to form the projection directions. The only assumption made here is that these elements of the basis $B_1$ lie inside $B_2$. Therefore, the Galerkin approach forms the $n_\theta \times n_g$ projections, where $b$, $d$, $q$, and $v$ are chosen for $i = 1, ..., n_\theta$ and $j = 1, ..., n_g$ to solve the following stacked set of projection equations,

$$P_{1ij}\left(b, d, q, v\right) \equiv \int_{\theta_m}^{\theta_M}\int_{g_m}^{g_M} R_\tau\left(\theta, g; b, d, q, v\right) \Psi_{ij}(.)\Omega_{ij}(.)\Phi_{ij}(.)\Gamma_{ij}(.)dgd\theta, \tag{3.23}$$

$$P_{2ij}\left(b, d, q, v\right) \equiv \int_{\theta_m}^{\theta_M}\int_{g_m}^{g_M} R_\mu\left(\theta, g; b, d, q, v\right) \Psi_{ij}(.)\Omega_{ij}(.)\Phi_{ij}(.)\Gamma_{ij}(.)dgd\theta, \tag{3.24}$$

$$P_{3ij}\left(b, d, q, v\right) \equiv \int_{\theta_m}^{\theta_M}\int_{g_m}^{g_M} R_H\left(\theta, g; b, d, q, v\right) \Psi_{ij}(.)\Omega_{ij}(.)\Phi_{ij}(.)\Gamma_{ij}(.)dgd\theta, \tag{3.25}$$

$$P_{4ij}\left(b, d, q, v\right) \equiv \int_{\theta_m}^{\theta_M}\int_{g_m}^{g_M} R_\lambda\left(\theta, g; b, d, q, v\right) \Psi_{ij}(.)\Omega_{ij}(.)\Phi_{ij}(.)\Gamma_{ij}(.)dgd\theta. \tag{3.26}$$

While choice of $n$ specifies the number of coefficients in each polynomial and the accuracy of the approximation, it does not eliminate the need to integrate over the possible realizations of the government spending and productivity shock. Further specification is needed to define the solution procedure over the $[\theta_m, \theta_M] \times [g_m, g_M]$ space.

Given these $n_\theta \times n_g$ projections, a *collocation* method then proceeds by choosing the coefficients so that $P_{1ij}(.)$, $P_{2ij}(.)$, $P_{3ij}(.)$, and $P_{4ij}(.)$ equal zero simultaneously at a finite set of values for $\theta$ and $g$. Furthermore, *orthogonal collocation* is a method whereby the finite choices for this state space, $\theta_k$ and $g_k$ for $k = 1, ..., m$, are the $k$ zeros of the $k - th$ orthogonal polynomial basis element and the basis elements are orthogonal with respect to the inner product. Orthogonal collocation with Chebyshev polynomials as the basis is referred to as *Chebyshev collocation* and the Chebyshev interpolation theorem details the optimality of Chebyshev collocation.[17] Thus, Chebyshev collocation with the Galerkin method is used to solve the system of equations.

The Chebyshev collocation method starts with choosing $m_\theta$ productivity levels, $\{\theta_k\}_{k=1}^m$, and $m_g$ government spending levels, $\{g_k\}_{k=1}^m$, and uses these to compute the projections. For $i = 1, ..., n_\theta$, $j = 1, ..., n_g$, the solution is to find the values of $b$, $d$, $q$, and $v$ to solve the following system,

$$\widehat{P}_{1ij}(b,d,q,v) \equiv \sum_{k_\theta=1}^{m_\theta} \sum_{k_g=1}^{m_g} R_\tau\left(\theta_{k_\theta}, g_{k_g}; b,d,q,v\right) \Psi_{ij}(.)\Omega_{ij}(.)\Phi_{ij}(.)\Gamma_{ij}(.) = 0, \tag{3.27}$$

$$\widehat{P}_{2ij}(b,d,q,v) \equiv \sum_{k_\theta=1}^{m_\theta} \sum_{k_g=1}^{m_g} R_\mu\left(\theta_{k_\theta}, g_{k_g}; b,d,q,v\right) \Psi_{ij}(.)\Omega_{ij}(.)\Phi_{ij}(.)\Gamma_{ij}(.) = 0, \tag{3.28}$$

$$\widehat{P}_{3ij}(b,d,q,v) \equiv \sum_{k_\theta=1}^{m_\theta} \sum_{k_g=1}^{m_g} R_H\left(\theta_{k_\theta}, g_{k_g}; b,d,q,v\right) \Psi_{ij}(.)\Omega_{ij}(.)\Phi_{ij}(.)\Gamma_{ij}(.) = 0, \tag{3.29}$$

$$\widehat{P}_{4ij}(b,d,q,v) \equiv \sum_{k_\theta=1}^{m_\theta} \sum_{k_g=1}^{m_g} R_\lambda\left(\theta_{k_\theta}, g_{k_g}; b,d,q,v\right) \Psi_{ij}(.)\Omega_{ij}(.)\Phi_{ij}(.)\Gamma_{ij}(.) = 0, \tag{3.30}$$

where,

$$\theta_{k_\theta} = \theta_m + \frac{1}{2}(\theta_M - \theta_m)\left(z_{k_\theta}^{m_\theta} + 1\right), \; k_\theta = 1, ..., m_\theta, \tag{3.31}$$

$$g_{k_g} = g_m + \frac{1}{2}(g_M - g_m)\left(z_{k_g}^{m_g} + 1\right), \; k_g = 1, ..., m_g, \tag{3.32}$$

$$z_k^n = \cos\left(\frac{(2k-1)\pi}{2n}\right), \; k = 1, ..., n, \tag{3.33}$$

and $z_k^n$ are the $n$ zeros of the Chebyshev polynomials.

[17] The Chebyshev interpolation theorem states that if $R(x;a)$ is smooth in $x$ and $R(z;a) = 0$ where $z$ is the set of the $n$ zeros of the Chebyshev polynomial $T_n$, then the zero conditions force $R(x;a)$ to be close to zero for all $x \in D$, where $D = [-1, 1]$. See Judd (1998) for further details.

In sum, the projection approach begins by defining the policy functions for taxes, money growth, labor, and the multiplier in terms of Chebyshev polynomials. Each polynomial is a function of the two random elements in the system, government spending and technology. The evaluation of expectations operators is done using Gauss-Hermite Quadrature and the choice of $r_\theta$ and $r_g$ determine the accuracy of the approximation since this choice determines the number of nodes the function is integrated over. Fortunately, the weighting function is such that most of the density is captured at a relatively low order. The "value" at each node is based on the standard deviation of the error process and the optimally defined weight. The evaluation of each expectation operator requires the computation of an $r_\theta \times r_g$ system. The Galerkin approach forms the $n_\theta \times n_g$ projections while specifying the number of coefficients. The larger the choice of $n_\theta$ and $n_g$, the more accurate the solution. However, this numerical approach is computationally constrained and this procedure settles for the smallest number of coefficients where additional coefficients yields relatively little in terms of approximation. Chebyshev collocation methods then divide the state space over $\theta$ and $g$ into discrete grid points, where the larger the choice of $m_\theta$ and $m_g$ produces a more defined grid space, and use orthogonal Chebyshev polynomials to optimally search for the coefficients necessary to set the system of projection equations equal to zero simultaneously.

Expectation terms formed in period $t-1$, $E_{t-1}(.)$, are based on information available in $t-1$. Calibration determines the initial values of $\theta_{t-1}$ and $g_{t-1}$, and the Gauss-Hermite Quadrature routine evaluates the $r_\theta \times r_g$ possibilities given these initial values. Subsequently, in period $t$, the values for $\theta_t$ and $g_t$ are revealed based on the defined gridspace in $m_\theta \times m_g$. The system, however, must be able to solve for the policy functions based on all possible realizations of $\theta_t$ and $g_t$, given $\theta_{t-1}$ and $g_{t-1}$. The realizations of $\theta_t$ and $g_t$ determine the deviation from the expected values of $\theta_t$ and $g_t$ based on previous information, defining the nonlinearity in the system and creating the necessary responses by the remaining policy variables, including taxes, labor, and money growth. Overall, the Gauss-Hermite nodes define a state space over $\theta$ and $g$ in terms of expectations while the collocation method defines a state space over the realizations of $\theta$ and $g$.

### 3.4. A low-order example

In order to provide better insight into the projection method as applied to this problem, a low-order example is briefly described. Some of the output is displayed from the solution to the optimal policy functions derived under the low debt-to-income case which is calibrated to the debt-to-income ratio of the U.S. economy. Remaining parameter values were specified according to the calibration techniques discussed earlier in the paper.

First, the projection approach begins by defining the policy functions in terms of Chebyshev polynomials. For example, suppose that $n = 2$. This is often a good starting point for complex systems since low order polynomials will capture a large portion of the nonlinearity within the system without adding too much to the dimension of the system. Since each polynomial is a function of two random variables, technology and government spending, each policy function in (3.3) - (3.6) will have a total of $n_\theta \times n_g = 4$ coefficients. Consequently, under the Galerkin approach this defines 16 projection equations. The solution procedure will optimally solve for four policy functions ($H_t$, $\mu_{t+1}$, $\tau_t$, $\lambda_{gt}$) comprised of 16 total coefficients ($b_{11}$, $b_{12}$, $b_{21}$, $b_{22}$, $d_{11}$, ..., $v_{22}$). In order to ensure that the system is properly identified, the Chebyshev collocation methods dictate $m_\theta = m_g = 2$ which results in $m_\theta \times m_g = 4$ combinations of spending and technology shocks and uses these values to compute the projections. The evaluation of expectations operators is done using Gauss-Hermite Quadrature and the choice of $r_\theta$ and $r_g$ determines the weights and nodes of the function integrated over. Setting $r_\theta = r_g = 11$ requires the computation of 121 possible future combinations of technology and government spending values in order to evaluate the expectation operator.

The solution procedure begins with an initial guess for each coefficient and the residual functions are computed. Since the entire set of projection equations must equal zero simultaneously, the set of residual functions is simply the stacked vector of Euler conditions. The solution program evaluates whether the system is sufficiently "close" to zero by comparing the norm of the residual vector to some predefined criterion. In this case, a norm of $1.0x10^{-9}$ is used as a minimum convergence criterion. Thus the norm of the residual vector must be less than this before the solution procedure quits. If the norm of the residual vector is greater than this stopping criterion, the nonlinear solution program computes the Jacobian and uses the gradient information to adjust the coefficients further. If the Jacobian is nearly singular, the accuracy of any solution will be low and convergence will be slow, if not impossible. Choosing an orthogonal basis will greatly reduce the odds of having a poorly conditioned Jacobian matrix.

The nonlinear equation solver was run on a computer with 512 MB of RAM and a Pentium III processor with 2 Ghz of processing speed.[18] Each iteration

[18] The nonlinear equation solver is a variant of the routine available on the website of Sims (2000).

of the solution procedure took approximately 30 seconds and the number of iterations until the solution was found depends largely on the accuracy of the initial guess. The residual vector and coefficients listed below result from solving the low-debt economy with a debt-to-income ratio of 49 percent. In this case, the solution procedure required nearly 60 iterations for a total computing time of about 30 minutes. The resulting residual vector is,

$$1.0x10^{-8}* \begin{bmatrix} -0.08153149821943 \\ -0.16691329429719 \\ -0.04647413009984 \\ -0.13309029989195 \\ 0.33068698612748 \\ 0.40442434956489 \\ 0.38097114352098 \\ 0.45744165122485 \\ -0.08688465016893 \\ -0.15021170141072 \\ -0.11757701756654 \\ -0.18067903215235 \\ 0.28799174156546 \\ 0.10434484387645 \\ 0.08828937581029 \\ -0.07808245161556 \end{bmatrix},$$

with coefficients on the optimal policy functions of

$$b_{ij} = \begin{bmatrix} 1.14608002432578 \\ -0.16519570673541 \\ -0.11173241474005 \\ 0.14011942756373 \end{bmatrix}, \quad d_{ij} = \begin{bmatrix} 0.05724289844865 \\ 0.32378903289563 \\ -0.03995604069151 \\ -0.15297339348807 \end{bmatrix},$$

$$q_{ij} = \begin{bmatrix} 0.38438668734576 \\ -0.00108787280851 \\ -0.07471316249354 \\ 0.00078124074289 \end{bmatrix}, \quad v_{ij} = \begin{bmatrix} -0.42051817069223 \\ 0.96449152940428 \\ 0.34835466487284 \\ -0.75405756754003 \end{bmatrix}.$$

The selection of the stopping criterion is important since it also determines the accuracy of the final solution. Given the stopping criterion of $1x10^{-9}$, a mistake in the optimal policy function would cost the government less than \$1 per billion in nominal GDP. Consequently, the error in this solution is less than \$10,000 given the size of the current U.S. economy.

Given the complexity of the system, a move to higher order polynomials greatly increases the dimensionality of the problem and increases the computational time required, largely due to the need to compute the Jacobian matrix during each iteration. For example, moving from second to third order polynomials means each policy function now has $n_\theta \times n_g = 9$ coefficients. Given the four policy functions, this creates the need to solve for 36 coefficients across 36 projection equations. Each iteration in the solution procedure now requires approximately 90 seconds to complete, which is triple the time necessary under the lower order example above. Furthermore, given the size of the residual vector, it becomes more difficult to drive the norm of the residual vector "close" to zero, meaning that the nonlinear solution procedure may require more iterations to reach the stopping criterion used above or necessitate an increase in the stopping criterion to facilitate the higher order. Such a move would decrease the accuracy of the overall solution. Therefore, while the additional coefficients could capture additional nonlinearity within the model, they also greatly increase the computational demands on the solution procedure and may require a decrease in the overall accuracy of the solution.

## 4. Results

In order to test Barro's idea (Barro 1987, 1979) that optimal policy smooths the response of distortionary taxes and money growth to shocks by using debt, the Ramsey problem was solved in economies with and without debt. The baseline economy contains no debt while the low debt case uses the prevailing debt-to-income ratio in the U.S. and the high debt case uses twice this level. After finding the optimal coefficients of the polynomial approximations that describe the Ramsey plan, each economy was simulated under the effects of technology and government spending shocks. Statistics were computed by running multiple simulations of 5000 periods in length, taking logarithms, and filtering each simulated time series using the H-P filter as described in Hodrick and Prescott (1997).

### 4.1. The steady-state and the role of debt

The values in Table 4.1 represents the steady state Ramsey equilibrium in levels or growth rates.[19] Optimal household allocations smooth consumption and

[19] While not the main focus of this paper, the properties of each variable under each of the model simulations match some of the general characteristics of the overall U.S. economy as discussed in Stock and Watson (1999) and Hodrick and Prescott (1997). The economies

labor supply with the constant $\eta$, the relative importance of the cash good to the credit good in the utility function, determining the split between the two consumption goods. In each model economy, optimal government policy sets money growth equal to the rate of time preference as described in Friedman (1969). According to Friedman, optimal monetary policy satiates the economy with real balances to the extent that it is possible to do so. Government policy that follows the Friedman rule results in a real return on debt and money balances equal to the inverse of time preference in the steady state. In enacting this monetary policy rule, the government equates the real gross rate of return across money balances and debt in expectation, satisfying Euler conditions.[20] Monetary policy that follows the Friedman rule requires the government to run a gross-of-interest surplus by setting equilibrium labor income taxes high enough to cover government spending, interest on the debt, and the withdrawal of money balances from the economy. In addition to the optimality of the Friedman rule, a similar feature of each model is the low volatility of money growth. Almost all of the volatility in distortionary revenue generating policy is accounted for through the volatility of labor taxes, suggesting that preservation of the Friedman rule may take priority over distortionary impacts of labor taxes. This result is the opposite of Chari et al. (1991), who find that money growth should be more variable to preserve smoother taxes on labor income.

The construction of a stacked system of residual equations describing equilibrium as opposed to the more traditional primal approach is illustrative in that it allows for a complete examination of the trade-offs involved in government policy choices by maintaining the multiplier on the government budget constraint as an endogenous "policy" variable. In the economy with no debt, the multiplier reflects the fact that government spending has to be financed by distortionary means. In the economies with debt, the value of the multiplier increases since the interest cost of the debt is added to existing government

with and without debt are reported in Table 4.3. The models only generate about half of the standard deviation of output as found in the U.S. economy, a common shortcoming of most real business cycle models which is magnified here because of the fixed capital stock. However, the autocorrelation of output is slightly higher than found in other studies.

The volatility of prices and inflation more closely match features of U.S. data. Since the price level and the rate of inflation are determined by the cash-in-advance constraint in equilibrium, volatility of the cash good imparts volatility into prices and compensates for the lack of volatility in money growth. The correlations of inflation with the shocks to government spending and technology have the expected opposite signs, leading to low correlations between inflation and output.

[20] As discussed in Chari, et al. (1991, 1996), the so-called Friedman rule turns out to be optimal in a variety of monetary economies with distorting taxes.

Table 4.1: Selected Simulations: Steady-State Values.

| Steady-State Values | | | |
|---|---|---|---|
| | *Baseline* | *Debt/Income Ratio* | |
| Variable | No Debt | Low | High |
| | (in levels) | | |
| Output | 1.741 | 1.736 | 1.731 |
| Cash Good | 0.487 | 0.485 | 0.483 |
| Credit Good | 0.623 | 0.620 | 0.617 |
| Labor | 0.311 | 0.309 | 0.308 |
| Multiplier | 0.133 | 0.138 | 0.144 |
| | (in percent) | | |
| Inflation | -0.9% | -0.9% | -0.9% |
| Interest Rate | 0.9% | 0.9% | 0.9% |
| Money Growth Rate | -0.9% | -0.9% | -0.9% |
| Tax Rate 1/ | 18.3% | 18.8% | 19.3% |
| Tax Rate 2/ | 30.5% | 31.4% | 32.2% |

1/ In percent of income.

2/ In percent of labor income.

spending costs. As the debt-to-income ratio rises, the equilibrium tax rate increases to produce a gross-of-interest surplus necessary to cover the associated higher interest costs with the higher stock of government debt. As a result, the distortionary effects of taxation on utility increase as the equilibrium tax rate rises with the level of outstanding debt. The higher welfare costs can be seen in Table 4.1 through the value of the multiplier on the government budget constraint. Overall, a higher debt stock requires higher equilibrium labor taxes, which result in higher welfare costs.

While the ability to issue debt imposes welfare costs through higher steady-state taxes, debt has benefits through policy smoothing. As suggested by Barro (1979, 1987), the existence of debt provides the government with an additional degree of policy freedom which allows for a smoother path of distortionary policy over time, thereby affording the household a smoother stream of consumption and leisure. The simulations of the model economies displayed in the Table 4.2 indicate that volatility of government tax policy in the economies with debt, as measured by standard deviation in percent, are lower than the baseline economy without debt. Since the government is required to raise revenue from distortionary means, Ramsey policies smooth the response of fiscal and monetary policy to various shocks that affect the economy and the government budget constraint. Initially, the government sets tax and monetary policy based on generating an expected level of revenues to cover expected government expenditures. As shocks to technology and government spending affect

Table 4.2: Selected Simulations: Standard Deviations.

| Standard Deviation in Percent | | | |
|---|---|---|---|
| | *Baseline* | *Debt/Income Ratio* | |
| Variable | No Debt | Low | High |
| Output | 0.79 | 0.78 | 0.79 |
| Cash Good | 1.38 | 1.41 | 1.45 |
| Credit Good | 1.38 | 1.36 | 1.34 |
| Labor | 0.23 | 0.22 | 0.21 |
| Multiplier | 4.64 | 4.98 | 5.29 |
| Price Level | 1.38 | 1.41 | 1.49 |
| Inflation | 1.04 | 1.05 | 1.06 |
| Interest Rate 1/ | - | 0.07 | 0.06 |
| Debt | - | 0.07 | 0.07 |
| Money Growth Rate | 0.00 | 0.07 | 0.15 |
| Tax Rate 2/ | 2.85 | 2.66 | 2.46 |

1/ Gross rate.
2/ In percent of income.

the government budget constraint, optimal policy responds by smoothing the impact of distortionary taxes and money growth with debt. For example, the standard deviation of the tax rate falls from 2.85 percent under no debt to 2.66 percent and 2.46 percent in the low and high debt economies, respectively. The volatility of monetary policy however rises slightly from no volatility under the baseline economy without debt to 0.15 percent under the high-debt economy, contributing to similar increases in the volatility of inflation and consumption of the cash good. The volatility of labor also rises slightly, but its negative correlation with output implies that leisure is being used more optimally. Given that revenue policy is distortionary and the system is nonlinear, overall gains to the household are dependent on whether the gains from reductions in tax volatility outweigh the loss from the increase in monetary policy volatility.

A utility equivalence framework is used to measure the gain to households from the ability to issue debt. The gain is measured in terms of a lump-sum present discounted value of utility necessary to make the household indifferent between the baseline economy without debt and the economy with the selected debt-to-income ratio. Since gains from consumption of the credit good and labor more than offset losses in consumption of the cash good as shown in Table 4.4, debt issuance for policy smoothing purposes is welfare improving on the order of 0.4 to 0.7 percent of lifetime utility. Each of the three model economies with and without debt have different steady-states as shown in Table 4.1. Steady-state utility is highest under the no-debt economy since steady-

Table 4.3: Simulated Economies with Technology and Government Spending Shocks.

Economy without Debt

| | Cross-Correlation of Output with: | | | | | | | Cross-Correlation | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | Money | Tax | | Shocks | |
| Variable | x(-3) | x(-2) | x(-1) | x | x(+1) | x(+2) | x(+3) | Growth | Rate | Mult. | Tech. | Gov. |
| Output | 0.27 | 0.47 | 0.72 | 1.00 | 0.72 | 0.47 | 0.27 | 0.31 | -0.32 | -0.51 | 1.00 | 0.00 |
| Cash Good | 0.24 | 0.42 | 0.63 | 0.88 | 0.63 | 0.42 | 0.24 | -0.17 | -0.73 | -0.85 | 0.88 | -0.47 |
| Credit Good | 0.24 | 0.42 | 0.63 | 0.88 | 0.63 | 0.42 | 0.24 | -0.17 | -0.73 | -0.85 | 0.88 | -0.47 |
| Labor | -0.08 | -0.14 | -0.22 | -0.30 | -0.22 | -0.14 | -0.07 | -1.00 | -0.80 | -0.66 | -0.30 | -0.95 |
| Multiplier | -0.14 | -0.24 | -0.36 | -0.51 | -0.36 | -0.24 | -0.14 | 0.65 | 0.98 | 1.00 | -0.51 | 0.86 |
| Gov. Spending | 0.00 | -0.01 | 0.00 | 0.00 | -0.01 | -0.01 | -0.01 | 0.94 | 0.95 | 0.86 | 0.00 | 1.00 |
| Price Level | -0.24 | -0.42 | -0.63 | -0.88 | -0.63 | -0.42 | -0.24 | 0.17 | 0.73 | 0.85 | -0.88 | 0.47 |
| Inflation | -0.20 | -0.24 | -0.28 | -0.33 | 0.33 | 0.28 | 0.24 | 0.06 | 0.27 | 0.32 | -0.33 | 0.18 |
| Money Growth | 0.08 | 0.15 | 0.22 | 0.31 | 0.22 | 0.14 | 0.08 | 1.00 | 0.79 | 0.65 | 0.31 | 0.94 |
| Tax Rate | -0.09 | -0.16 | -0.23 | -0.32 | -0.23 | -0.16 | -0.10 | 0.79 | 1.00 | 0.98 | -0.32 | 0.95 |

Economy with Low Debt-to-Income Ratio

| | Cross-Correlation of Output with: | | | | | | | Cross-Correlation | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | Money | Tax | | Shocks | |
| Variable | x(-3) | x(-2) | x(-1) | x | x(+1) | x(+2) | x(+3) | Growth | Rate | Mult. | Tech. | Gov. |
| Output | 0.26 | 0.46 | 0.71 | 1.00 | 0.71 | 0.46 | 0.26 | -0.37 | -0.37 | -0.55 | 1.00 | -0.02 |
| Cash Good | 0.24 | 0.42 | 0.63 | 0.89 | 0.63 | 0.42 | 0.24 | -0.75 | -0.75 | -0.87 | 0.89 | -0.47 |
| Credit Good | 0.24 | 0.42 | 0.64 | 0.90 | 0.64 | 0.42 | 0.24 | -0.73 | -0.73 | -0.85 | 0.90 | -0.44 |
| Labor | -0.26 | -0.46 | -0.71 | -1.00 | -0.71 | -0.46 | -0.26 | 0.36 | 0.36 | 0.54 | -1.00 | 0.01 |
| Multiplier | -0.15 | -0.27 | -0.40 | -0.55 | -0.40 | -0.26 | -0.15 | 0.96 | 0.98 | 1.00 | -0.55 | 0.84 |
| Gov. Spending | -0.02 | -0.02 | -0.02 | -0.02 | -0.02 | -0.02 | -0.02 | 0.92 | 0.94 | 0.84 | -0.02 | 1.00 |
| Price Level | -0.20 | -0.38 | -0.61 | -0.88 | -0.64 | -0.44 | -0.27 | 0.73 | 0.73 | 0.85 | -0.88 | 0.45 |
| Inflation | -0.20 | -0.25 | -0.30 | -0.36 | 0.32 | 0.27 | 0.23 | 0.34 | 0.34 | 0.38 | -0.36 | 0.23 |
| Interest Rate | -0.25 | -0.44 | -0.66 | -0.93 | -0.66 | -0.44 | -0.25 | 0.68 | 0.67 | 0.81 | -0.93 | 0.37 |
| Debt | -0.10 | -0.24 | -0.42 | -0.63 | -0.88 | -0.63 | -0.41 | 0.47 | 0.46 | 0.54 | -0.63 | 0.26 |
| Money Growth | -0.11 | -0.18 | -0.27 | -0.37 | -0.27 | -0.18 | -0.11 | 1.00 | 0.98 | 0.96 | -0.37 | 0.92 |
| Tax Rate | -0.11 | -0.18 | -0.27 | -0.37 | -0.27 | -0.18 | -0.11 | 0.98 | 1.00 | 0.98 | -0.37 | 0.94 |

Economy with High Debt-to-Income Ratio

| | Cross-Correlation of Output with: | | | | | | | Cross-Correlation | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | Money | Tax | | Shocks | |
| Variable | x(-3) | x(-2) | x(-1) | x | x(+1) | x(+2) | x(+3) | Growth | Rate | Mult. | Tech. | Gov. |
| Output | 0.27 | 0.47 | 0.71 | 1.00 | 0.71 | 0.47 | 0.27 | -0.37 | -0.37 | -0.55 | 1.00 | 0.00 |
| Cash Good | 0.23 | 0.41 | 0.63 | 0.88 | 0.63 | 0.41 | 0.24 | -0.76 | -0.76 | -0.87 | 0.88 | -0.48 |
| Credit Good | 0.24 | 0.43 | 0.65 | 0.91 | 0.65 | 0.43 | 0.25 | -0.71 | -0.72 | -0.84 | 0.91 | -0.41 |
| Labor | -0.27 | -0.47 | -0.71 | -1.00 | -0.71 | -0.47 | -0.27 | 0.34 | 0.34 | 0.52 | -1.00 | -0.02 |
| Multiplier | -0.14 | -0.25 | -0.39 | -0.55 | -0.39 | -0.26 | -0.15 | 0.96 | 0.98 | 1.00 | -0.54 | 0.84 |
| Gov. Spending | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.01 | -0.01 | 0.91 | 0.93 | 0.84 | 0.00 | 1.00 |
| Price Level | -0.15 | -0.33 | -0.56 | -0.83 | -0.63 | -0.45 | -0.29 | 0.69 | 0.69 | 0.80 | -0.83 | 0.42 |
| Inflation | -0.20 | -0.26 | -0.32 | -0.38 | 0.29 | 0.25 | 0.21 | 0.40 | 0.40 | 0.44 | -0.38 | 0.28 |
| Interest Rate | -0.25 | -0.44 | -0.67 | -0.94 | -0.67 | -0.44 | -0.25 | 0.66 | 0.66 | 0.79 | -0.94 | 0.34 |
| Debt | -0.10 | -0.24 | -0.43 | -0.65 | -0.92 | -0.66 | -0.43 | 0.47 | 0.47 | 0.55 | -0.65 | 0.25 |
| Money Growth | -0.09 | -0.17 | -0.26 | -0.37 | -0.26 | -0.18 | -0.10 | 1.00 | 0.98 | 0.96 | -0.36 | 0.91 |
| Tax Rate | -0.10 | -0.17 | -0.26 | -0.37 | -0.26 | -0.18 | -0.10 | 0.98 | 1.00 | 0.98 | -0.36 | 0.93 |

state tax rates on labor are lower, resulting in higher steady-state labor supply, output, and consumption. In order to estimate the gains from allowing policy smoothing through the issuance of debt, an adjustment is made to each of the steady-state utility measures with debt to compensate for this differential. Therefore, the utility equivalence measures displayed in Table 4.4 should be viewed as an estimate of utility variations around similar steady-states.

A confidence interval for the gain in welfare can be found to test the significance of the estimated gains in Table 4.4. This is done by first forming the 95 percent confidence intervals around the standard deviations to technology and government spending.[21] These intervals were then used to re-estimate welfare gains. Based on this process, the 95 percent confidence intervals for the welfare gains are 0.3 to 0.5 percent of lifetime utility under low debt and 0.6 to 0.8 percent of lifetime utility under high debt. Therefore, while the lifetime gains in utility under low and high debt are relatively small in magnitude, they appear to be significant.

Furthermore, the welfare gains estimated from reducing policy volatility are similar to those reported in Lucas (1987), who estimates the cost of business cycles. Using logarithmic preferences and post World War II data series, Lucas (1987) reports that completely removing consumption variability entails a lifetime increase in utility equal to 0.2 percent of consumption. However, the author used a time preference parameter of $\beta = 0.95$. Using a calibrated value equal to $\beta = 0.991$ as done in this paper would result in a lifetime increase in utility equal to 0.9 percent. Furthermore, the welfare gains estimated here are similar in size to those reported by Cooley and Hansen (1991), who estimate the gains from eliminating moderate inflation. Employing a similar stochastic monetary economy as the one presented here, the authors report that tran-

[21] The calibrated values for the process governing technology and government spending are based on Cooley and Hansen (1995) and Christiano and Eichenbaum (1992), respectively. The confidence intervals are formed based on the chi-squared distribution with $(n-2)$ degrees of freedom where $n$ is the number of observations in the sample. With 120 degrees of freedom, the 95 percent confidence interval for $\sigma_\theta^2$ is,

$$\frac{(n-2)\,(0.0837)}{\chi^2_{0.975}} < \sigma_\theta^2 < \frac{(n-2)\,(0.0837)}{\chi^2_{0.025}},$$

or,

$$0.012 < \sigma_\theta < 0.004.$$

Following the same process, the 95 percent confidence interval for $\sigma_g^2$ is,

$$0.036 < \sigma_g < 0.013.$$

See Greene (1993) for additional information.

Table 4.4: Utility Equivalence.

| Utility Equivalent Over No-Debt Economy | | |
|---|---|---|
| | Debt-to-Income Ratio | |
| | Low | High |
| (Discounted present value in percent) | | |
| Total Utility | 0.4 | 0.7 |
| Consumption | -0.2 | -0.3 |
| Cash Good | -0.6 | -1.3 |
| Credit Good | 0.3 | 0.7 |
| Labor | 0.8 | 1.5 |

sitioning from 5 percent and 10 percent inflation to zero inflation results in gains in lifetime utility of 0.4 and 0.6 percent, respectively.

## 5. Conclusion

This paper has examined the usefulness of nonlinear projection methods in solving Ramsey problems by investigating welfare enhancing properties of debt. The presence of nonlinear distortions in the Ramsey problem requires the use of a solution procedure which captures these effects. The nonlinear projection method, even with low-order Chebyshev polynomials as employed in this paper, is able to capture a significant portion of the Jensen's inequality effects. We estimate welfare gains from debt issuance and associated policy smoothing. In an explicit monetary economy where cash and credit goods are valued nearly equally in the utility function, increases in the volatility of distortionary money supply are more than offset by declines in the volatility of distortionary labor taxes so that introduction of debt is welfare improving.

# 6. Appendix: Household Optimization Problem

The household chooses consumption of the cash and credit goods, the amount of money to be carried into the next period, labor supply, and stock debt to maximize (2.6) subject to the cash-in-advance constraint in (2.8) and the budget constraint in (2.7). This can be set up as a dynamic programming problem where,

$$V(s_t) = \underset{\chi_t}{Max} \left\{ \begin{array}{c} \eta \ln C_{1t} + (1-\eta)\ln C_{2t} - \gamma H_t + \lambda_{1t}\left(\frac{M_t^d}{P_t} - C_{1t}\right) + \\ \lambda_{2t}\left( \begin{array}{c} (1-\alpha\tau_t)(Y_t - X) + \frac{M_t^d}{P_t} + B_t R_{t-1} \\ -C_{1t} - C_{2t} - \frac{M_{t+1}^d}{P_t} - B_{t+1} \end{array} \right) + \beta E_t V(s_{t+1}) \end{array} \right\}, \tag{6.1}$$

where $\chi_t = \left(C_{1t}, C_{2t}, M_{t+1}^d, B_{t+1}, H_t\right)$ is the vector of household choice variables and $s_t$ represents the set of state variables, $\left( B_t, \frac{M_t^d}{P_t}, \theta_{t-1}, g_{t-1}, \tau_{t-1}, R_{t-1}\right)$. Here, $\lambda_{1t}$ and $\lambda_{2t}$ are the Lagrange multipliers for the cash-in-advance constraint and household budget constraint, respectively. The resulting first-order conditions are,

$$C_{1t} : \frac{\eta}{C_{1t}} = \lambda_{1t} + \lambda_{2t}, \tag{6.2}$$

$$C_{2t} : \frac{1-\eta}{C_{2t}} = \lambda_{2t}, \tag{6.3}$$

$$M_{t+1}^d : \frac{\lambda_{2t}}{P_t} = \beta E_t \left\{ \frac{\lambda_{1t+1}}{P_{t+1}} + \frac{\lambda_{2t+1}}{P_{t+1}} \right\}, \tag{6.4}$$

$$B_{t+1} : \lambda_{2t} = \beta E_t \left\{ \lambda_{2t+1} R_t \right\}, \tag{6.5}$$

$$H_t : \gamma = \lambda_{2t} (1-\alpha\tau_t)\, \alpha \frac{Y_t}{H_t}. \tag{6.6}$$

Combining the first-order condition on the cash good in (6.2) with the first-order condition on labor in (6.6) yields,

$$\lambda_{1t} = \frac{\eta}{C_{1t}} - \frac{\gamma}{(1-\alpha\tau_t)} \frac{H_t}{\alpha Y_t}. \tag{6.7}$$

The multiplier on the cash-in-advance constraint is equal to the marginal utility of consumption of the cash good reduced by the marginal disutility of having to supply additional hours of labor for the equal amount of consumption of the credit good.

The first-order conditions above can be combined to yield the following

Euler equations,

$$M_{t+1}^d : \frac{1-\eta}{C_{2t}} = \beta E_t \left\{ \frac{\eta}{C_{1t+1}} \frac{P_t}{P_{t+1}} \right\}, \tag{6.8}$$

$$B_{t+1} : \frac{1-\eta}{C_{2t}} = \beta E_t \left\{ \frac{1-\eta}{C_{2t+1}} R_t \right\}, \tag{6.9}$$

$$H_t : \gamma = \frac{1-\eta}{C_{2t}} \left(1 - \alpha\tau_t\right) \alpha \frac{Y_t}{H_t}. \tag{6.10}$$

Here the Benveniste-Scheinkman condition is used repeatedly (Benveniste and Scheinkman, 1979). For example, $\partial V / \partial \frac{M_t}{P_t} = \lambda_{1t} + \lambda_{2t}$. The Euler condition on bonds can be used to derive the interest rate condition as,

$$R_t = \frac{1}{\beta C_{2t}} \left[ \frac{1}{E_t \left[ \frac{1}{C_{2t+1}} \right]} \right]. \tag{6.11}$$

Maximization of expression (2.6) is subject to $M^d \geq 0$ for all $t \geq 0$, given the initial stock of money, $M_0$. There is no similar restriction on debt since a negative stock of government bonds would indicate household indebtedness to the government, although transversality conditions will prevent debt from growing without bound in either direction. Transversality conditions can be derived by consolidating two consecutive household budget constraints yielding,

$$C_{1t} + C_{2t} + \frac{1}{R_t}\left(C_{1t+1} + C_{2t+1}\right) + \frac{M_{t+1}^d}{P_t}\left(1 - \frac{1}{R_t}\frac{P_t}{P_{t+1}}\right) \leq \left(1 - \alpha\tau_t\right)\left(Y_t - X\right) +$$
$$\frac{M_t^d}{P_t} + B_t R_{t-1} + \frac{1}{R_t}\left[\left(1 - \alpha\tau_{t+1}\right)\left(Y_{t+1} - X\right) - \frac{M_{t+2}^d}{P_{t+1}} - B_{t+2}\right]. \tag{6.12}$$

To ensure a bounded budget set, the term multiplying $M_{t+1}^d / P_t$ must be greater than or equal to zero. If this was not the case, households could make infinitely large profits by increasing money balances financed by issuing bonds. Since money balances earn no interest, the gross real return on money from $t$ to $t+1$ is just the inverse of the inflation rate, or $R_t^M = P_t / P_{t+1}$. The result is that real return on money must be less than or equal to the return on bonds,

$$1 - \frac{1}{R_t}\frac{P_t}{P_{t+1}} = 1 - \frac{R_t^M}{R_t} \geq 0, \tag{6.13}$$

or the net nominal interest rate cannot be negative.

If the process of recursively using successive household budget constraints to eliminate successive indexed bond terms is continued, the present-value budget constraint of the household can be derived as,

$$\sum_{i=0}^{\infty} q_i[C_{1t+i} + C_{2t+i} + \frac{M^d_{t+i+1}}{P_{t+i}}\left(1 - \frac{1}{R^L_{t+i}}\frac{P_{t+i}}{P_{t+i+1}}\right) - \tag{6.14}$$

$$(1-\alpha\tau_{t+i})\,(Y_{t+i} - X)\,] \leq \frac{M^d_t}{P_t} + B_t R_{t-1},$$

where

$$q_0 = 1 \text{ and } q_i = q_0 \prod_{n=1}^{i} \frac{1}{R_{t+n-1}}, \tag{6.15}$$

and where the following transversality conditions have been imposed,

$$\lim_{I\to\infty}(q_I B_{t+I+1}) = 0, \tag{6.16}$$

$$\lim_{I\to\infty}\left(q_I \frac{M^d_{t+I+1}}{P_{t+I}}\right) = 0. \tag{6.17}$$

Households would not find it optimal to accumulate levels of money balances or bonds that violate these conditions because alternative allocations exists that would afford higher levels of consumption and higher lifetime utility.

The specification of log preferences causes income and substitution effects to cancel, allowing equilibrium household allocations to be characterized for a given set of government policy. Output can either be consumed by households or used by the government resulting in the economy-wide resource constraint,

$$C_{1t} + C_{2t} + X + G_t = Y_t. \tag{6.18}$$

The resource constraint can be used with (2.8) and (2.16) in the Euler condition for money balances to yield closed-form solutions for consumption, prices, and interest rates. Assuming money supply equals money demand in equilibrium, or $M_{t+1} = M^d_{t+1}$, the closed-form solutions for consumption are as follows,

$$C_{1t} = \frac{(Y_t - X - G_t)\,\beta\left(\frac{\eta}{1-\eta}\right)\exp(-\mu_{t+1})}{1+\beta\left(\frac{\eta}{1-\eta}\right)\exp(-\mu_{t+1})}, \tag{6.19}$$

$$C_{2t} = \frac{(Y_t - X - G_t)}{1+\beta\left(\frac{\eta}{1-\eta}\right)\exp(-\mu_{t+1})}. \tag{6.20}$$

Inserting the solution for the cash good in (6.19) into the cash-in-advance constraint, which holds with equality in equilibrium, produces the following closed-form equation for the price level,

$$P_t = \frac{M_t}{(Y_t - X - G_t)} \left[ \frac{1 + \beta \left(\frac{\eta}{1-\eta}\right) \exp(-\mu_{t+1})}{\beta \left(\frac{\eta}{1-\eta}\right) \exp(-\mu_{t+1})} \right]. \tag{6.21}$$

The closed-form solution for the interest rate is found by inserting the solution for the credit good in (6.20) at time $t$ and $t+1$ into the Euler condition in (6.11).

Finally, the solution in (6.20) can be substituted into the Euler condition for labor in (6.10) to solve for optimal labor supply. Doing so yields,

$$\frac{\gamma \left(Y_t - X - G_t\right)}{1 + \beta \left(\frac{\eta}{1-\eta}\right) \exp(-\mu_{t+1})} = (1-\eta)\left(1 - \alpha\tau_t\right) \alpha \frac{Y_t}{H_t}, \tag{6.22}$$

which, noting the specification for output, defines an implicit function,

$$H_t = h(g_t, \theta_t, \mu_{t+1}, \tau_t). \tag{6.23}$$

This equation cannot be solved for $H_t$ explicitly, but the implicit function theorem will allow for the construction of an implicit function which defines the explicit function. The defined derivatives can be obtained as long as an implicit function is known to exist under the implicit function theorem.

**Proposition 1.** *The function* $F\left(H_t,\, g_t,\, \theta_t,\, \mu_{t+1},\, \tau_t\right) = 0$ *defines an implicit function* $H_t = h(g_t,\, \theta_t,\, \mu_{t+1},\, \tau_t)$.

The implicit function theorem states that given $F\left(H_t,\, g_t,\, \theta_t,\, \mu_{t+1},\, \tau_t\right) = 0$, if (a) the function $F$ has continuous partial derivatives $F_H$, $F_g$, $F_\theta$, $F_\mu$, and $F_\tau$ and, (b) at a point $(H_0,\, g_0,\, \theta_0,\, \mu_0,\, \tau_0)$ satisfying $F\left(H_t,\, g_t,\, \theta_t,\, \mu_{t+1},\, \tau_t\right) = 0$, $F_H$ is non-zero except when $H = 0$, then there exists a 4-dimensional neighborhood of $(g_0,\, \theta_0,\, \mu_0,\, \tau_0)$, $N$, in which $h$ is an implicitly defined function of the variables $g$, $\theta$, $\mu$, and $\tau$ in the form of $h(g_t,\, \theta_t,\, \mu_{t+1},\, \tau_t)$.[22]

The continuous partial derivatives of (6.23) are,

$$F_H : \frac{\alpha Y_t}{H_t} \left[ \frac{\gamma}{1 + \beta \left(\frac{\eta}{1-\eta}\right) \exp(-\mu_{t+1})} + \frac{(1-\alpha)\left(1 - \alpha\tau_t\right)}{H_t} \right], \tag{6.24}$$

[22] See Sydsaeter (1981, p. 81)

$$F_g : \frac{-\gamma G_t}{1 + \beta\left(\frac{\eta}{1-\eta}\right)\exp(-\mu_{t+1})}, \tag{6.25}$$

$$F_\theta : \frac{-\gamma}{1 + \beta\left(\frac{\eta}{1-\eta}\right)\exp(-\mu_{t+1})}, \tag{6.26}$$

$$F_\mu : \frac{\gamma C_{1t}}{1 + \beta\left(\frac{\eta}{1-\eta}\right)\exp(-\mu_{t+1})}, \tag{6.27}$$

$$F_\tau : (1-\alpha)\,\alpha^2 \frac{Y_t}{H_t}. \tag{6.28}$$

Given that $0 < \alpha, \beta < 1$, and $\gamma$ is defined as a positive constant, $F_H$ is non-zero except when $H = 0$, where $F_H$ becomes undefined. Thus, around any point on the function, except $H = 0$, a neighborhood, $N$, can be constructed in which $F\left(H_t, g_t, \theta_t, \mu_{t+1}, \tau_t\right) = 0$ defines an implicit function $H_t = h(g_t, \theta_t, \mu_{t+1}, \tau_t)$.

Further examination of the labor supply function shows that optimal labor supply will be bounded away from zero and unique over the interval examined. Equation (6.23) acts as the difference function between the left and right-hand sides of equation (6.22). The left-hand side of equation (6.22) is upward sloping in labor supply while the right-hand side is downward sloping in labor supply. The left-hand side contains the term for overall consumption, $(Y_t - X - G_t)$ and when calibrated to match the features of the U.S. economy and examined over the interval $[0, 1]$ in labor supply, begins below zero and slowly increases. At low levels of labor supply, output is less than government spending. As additional labor supply is added, output quickly outpaces government spending. The function is always upward sloping over the interval in question. The term on the right-hand side contains the marginal product of labor and is downward sloping in labor supply. The calibrated function begins at higher levels with low labor supply since marginal productivity of labor is high and slowly decreases as labor is increased. Consequently, the difference function begins negative at low levels of labor supply (low total consumption relative to high marginal product of labor) and turns positive as labor supply is increased (high total consumption relative to low marginal product of labor). Since the difference function is continuous and maintains a positive slope over the interval in question, the optimal labor supply which equates the two sides and satisfies the Euler condition is strictly greater than zero and is unique over the $[0, 1]$ interval.

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