Appendix for Solving Asset Pricing Models when the Price-Dividend Function is Analytic

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This appendix provides proofs of some results stated in our paper. First, we prove the existence and uniqueness of the price-dividend function (Proposition 1) to the integral equation (6) in the vector space $\mathcal{S}$ (Definition 1). Second, we calculate the system of linear equations (22) for the coefficients of the power series for the price-dividend function. Finally, we use Cauchy's integral formula to bound all the derivatives of the price-expected dividend function (23) so that we can calculate the error terms (24), (25) and (26).

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APPENDIX

We use the following notation in the proofs: $C(n) = \sum_{i=1}^{n} |\phi|^i = \frac{|\phi|(1-|\phi|^n)}{1-|\phi|}$ for $n = 0, 1, 2, \ldots$, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{s^2}{2}}\, ds$ for all $x \in \mathbb{R}$, $\theta = x_0 + \sigma^2(1+\phi-\alpha)(1-\gamma)$, $K_3 = K_0 e^{x_0 K_1 + \frac{\sigma^2(1-\gamma)^2(\phi-\alpha)(2+\phi-\alpha)}{2}}$, $K_4 = K_3 \left[1 + 2\Phi\left(\frac{|\phi K_1|\sigma}{1-|\phi|}\right)\right]$, $K_5 = K_4 e^{\frac{\phi^2 K_1^2 \sigma^2}{2(1-|\phi|)^2} + \frac{|\phi K_1 \theta|}{1-|\phi|}}$ and $K_6 = \frac{1}{\sqrt{\pi}} K_0 e^{-\frac{\sigma^2}{2}(1-\gamma)^2 + \frac{\sigma^2}{2}[1-\gamma+K_1]^2}$. Q is defined in (18) which satisfies the integral equation (19).

A series of lemmas are now proved which are used in the proof of Proposition 1.

Lemma 2: *Let $a > 0$ and $\varphi(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-x-a}^{-x+a} e^{-\frac{s^2}{2\sigma^2}}\, ds$ for all $x \in \mathbb{R}$. Then $\varphi(x) \leq 2\Phi\left(\frac{a}{\sigma}\right)$.*

Proof: This follows from the fact that φ has a unique global maximum at $x = 0$.

Q.E.D.

Lemma 3: *For any real numbers A and k with $k \geq 0$, we have*

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(s-A)^2 + k|s|}\, ds \leq e^{\frac{k^2\sigma^2}{2} + k|A|} \left[1 + 2\Phi(k\sigma)\right].$$

Proof: Note that $-\frac{1}{2\sigma^2}(s-A)^2 + ks = -\frac{1}{2\sigma^2}(s-A-k\sigma^2)^2 + \frac{k^2\sigma^2}{2} + kA$.

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(s-A)^2 + k|s|}\, ds &= \frac{1}{\sqrt{2\pi}\sigma} \left(\int_{-\infty}^{0} e^{-\frac{1}{2\sigma^2}(s-A)^2 - ks}\, ds + \int_{0}^{\infty} e^{-\frac{1}{2\sigma^2}(s-A)^2 + ks}\, ds \right) \\
&= \frac{e^{\frac{k^2\sigma^2}{2}}}{\sqrt{2\pi}\sigma} \left(e^{-kA} \int_{-\infty}^{0} e^{-\frac{1}{2\sigma^2}(s-A+k\sigma^2)^2}\, ds + e^{kA} \int_{0}^{\infty} e^{-\frac{1}{2\sigma^2}(s-A-k\sigma^2)^2}\, ds \right) \\
&\leq \frac{e^{\frac{k^2\sigma^2}{2} + k|A|}}{\sqrt{2\pi}\sigma} \left(\int_{-\infty}^{0} e^{-\frac{1}{2\sigma^2}(s-A+k\sigma^2)^2}\, ds + \int_{0}^{\infty} e^{-\frac{1}{2\sigma^2}(s-A-k\sigma^2)^2}\, ds \right) \\
&= \frac{e^{\frac{k^2\sigma^2}{2} + k|A|}}{\sqrt{2\pi}\sigma} \left(\int_{-\infty}^{-A+k\sigma^2} e^{-\frac{s^2}{2\sigma^2}}\, ds + \int_{-A-k\sigma^2}^{\infty} e^{-\frac{s^2}{2\sigma^2}}\, ds \right) \\
&= e^{\frac{k^2\sigma^2}{2} + k|A|} \left(1 + \frac{1}{\sqrt{2\pi}\sigma} \int_{-A-k\sigma^2}^{-A+k\sigma^2} e^{-\frac{s^2}{2\sigma^2}}\, ds \right)
\end{aligned}$$

By Lemma 2, we achieve $\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(s-A)^2 + k|s|}\, ds \leq e^{\frac{k^2\sigma^2}{2} + k|A|}[1 + 2\Phi(k\sigma)]$.

Q.E.D.

Lemma 4: *For any $f \in \mathcal{S}$, the function Tf defined by*

$$(Tf)(x) = \frac{K_3 e^{\phi K_1 x}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(s) e^{-\frac{1}{2\sigma^2}[s-(\phi x+\theta)]^2}\, ds \tag{1}$$

lies in the vector space $\mathcal{S}$. Thus, T defines a linear transformation from the vector space $\mathcal{S}$ to itself.

Proof: By the definition of $\mathcal{S}$, we can find positive constants M and k such that $|f(x)| \leq Me^{k|x|}$. Using Lemma 3, we deduce

$$\begin{aligned}
|(Tf)(x)| &\leq \frac{K_3 e^{\phi K_1 x}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} |f(s)| e^{-\frac{1}{2\sigma^2}[s-(\phi x+\theta)]^2}\, ds \\
&\leq \frac{MK_3 e^{\phi K_1 x}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[s-(\phi x+\theta)]^2 + k|s|}\, ds \\
&\leq MK_3[1+2\Phi(k\sigma)] e^{\phi K_1 x + \frac{k^2\sigma^2}{2} + k(|\phi||x|+|\theta|)} \\
&\leq MK_3 e^{\frac{k^2\sigma^2}{2} + k|\theta|}[1+2\Phi(k\sigma)] e^{|\phi|(k+|K_1|)|x|}\, .
\end{aligned}$$

The continuity of Tf can be easily checked by applying the standard argument in real analysis; for example, see Theorem 56 in Kaplan (1956).

Q.E.D.

Now construct a sequence of functions $\{\, Q_n \mid n = 0, 1, 2, \ldots \}$ by setting $Q_0 = 0$ and

$$Q_{n+1}(x) = K_0 + \frac{K_3 e^{\phi K_1 x}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} Q_n(s) e^{-\frac{1}{2\sigma^2}[s-(\phi x+\theta)]^2}\, ds \tag{2}$$

and set $Q = \sum_{n=0}^{\infty}[Q_{n+1} - Q_n]$.

Recall that $C(n) = \sum_{i=1}^{n} |\phi|^i = \frac{|\phi|(1-|\phi|^n)}{1-|\phi|}$ satisfies (a) $C(n+1) = |\phi|(C(n)+1)$, (b) $C(n) < \frac{|\phi|}{1-|\phi|}$, and (c) $\lim_{n\to\infty} C(n) = \frac{|\phi|}{1-|\phi|}$.

Lemma 5: $0 \leq Q_{n+1}(x) - Q_n(x) \leq K_0 K_4^n e^{\frac{\sigma^2 K_1^2}{2}\sum_{i=1}^{n-1} C(i)^2 + |K_1\theta| \sum_{i=1}^{n-1} C(i)} e^{C(n)|K_1||x|}$ *for* $n = 0, 1, 2, \ldots$.

Proof: These inequalities can be easily proven by induction on n.

Q.E.D.

Lemma 6: *The series* $Q = \sum_{n=0}^{\infty}[Q_{n+1} - Q_n]$ *is uniformly convergent on any bounded closed interval.*

Proof: Fix a bounded closed interval $[a, b]$. For any $x \in [a, b]$, by Lemma 5 we see

$$0 \le \sum_{n=0}^{\infty}[Q_{n+1}(x) - Q_n(x)] \le K_0 \sum_{n=0}^{\infty} K_4^n e^{\frac{\sigma^2 K_1^2}{2}\sum_{i=1}^{n-1} C(i)^2 + |K_1\theta| \sum_{i=1}^{n-1} C(i)} e^{C(n)|K_1|(|a|+|b|)} .$$

The ratio test together with Weierstrass M-test (see Theorem 30 Kaplan (1956)) implies the uniform convergence of the series $Q = \sum_{n=0}^{\infty}[Q_{n+1} - Q_n]$ on $[a, b]$, provided

$$\lim_{n\to\infty} K_4 e^{\frac{\sigma^2 K_1^2}{2} C(n)^2 + |K_1\theta| C(n)} e^{[C(n+1)-C(n)]|K_1|(|a|+|b|)} = K_4 e^{\frac{\sigma^2 K_1^2 \phi^2}{2(1-|\phi|)^2} + \frac{|\phi K_1 \theta|}{1-|\phi|}} = K_5 < 1 .$$

Q.E.D.

Lemma 7: *The function* Q *lies in the vector space* $\mathcal{S}$*.*

Proof: The continuity of Q follows from Theorem 31 in Kaplan (1956) and Lemma 6. By Lemma 5, we also have

$$\begin{aligned} |Q(x)| &\le K_0 \sum_{n=0}^{\infty} K_4^n e^{\frac{\sigma^2 K_1^2}{2}\sum_{i=1}^{n-1} C(i)^2 + |K_1\theta| \sum_{i=1}^{n-1} C(i)} e^{C(n)|K_1||x|} \\ &\le \left(K_0 \sum_{n=0}^{\infty} K_4^n e^{\frac{\sigma^2 K_1^2}{2}\sum_{i=1}^{n-1} C(i)^2 + |K_1\theta| \sum_{i=1}^{n-1} C(i)} \right) e^{\frac{|\phi K_1|}{1-|\phi|}|x|} \end{aligned}$$

Q.E.D.

Lemma 8: *For any* $x \in \mathbb{R}$*, we have* $\lim_{n\to\infty} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} [Q(s) - Q_n(s)] e^{-\frac{1}{2\sigma^2}[s-(\phi x+\theta)]^2}\, ds = 0$.

Proof: This is an immediate consequence of Theorem 32 in Kaplan (1956) and Lemma 6.

Q.E.D.

Proof of Proposition 1:

Equivalently, we are going to show that the equation (19) has a unique solution Q in the vector space $\mathcal{S}$.

Existence

Applying the limit as n approaches ∞ to the equation (2) shows that the function $Q = \sum_{n=0}^{\infty}[Q_{n+1} - Q_n] = \lim_{n\to\infty} Q_n$ is a solution to the equation (19) in the vector space $\mathcal{S}$.

Uniqueness

Suppose that $\tilde{Q} \in \mathcal{S}$ is another solution to the equation (19). Then Q and $\tilde{Q}$ satisfy the functional equation $Q - \tilde{Q} = T(Q - \tilde{Q})$. By the definition of $\mathcal{S}$, we can also find positive constants M and k such that $|Q(x) - \tilde{Q}(x)| \le Me^{k|x|}$. Using the results in the proof of Lemma 4, we see

$$|Q(x) - \tilde{Q}(x)| \le (T|Q - \tilde{Q}|)(x) \le MK_3 e^{\frac{k^2\sigma^2}{2} + k|\theta|}[1 + 2\Phi(k\sigma)]e^{(|\phi|k + C(1)|K_1|)|x|}$$

for all $x \in \mathbb{R}$. Applying this computation n times gives rise to

$$\begin{aligned} &|Q(x) - \tilde{Q}(x)| \\ &\le MK_3^n \left\{ \prod_{i=0}^{n-1} e^{\frac{(|\phi|^i k + C(i)|K_1|)^2\sigma^2}{2} + (|\phi|^i k + C(i)|K_1|)|\theta|}[1 + 2\Phi(\sigma|\phi|^i k + \sigma C(i)|K_1|)] \right\} e^{(|\phi|^n k + C(n)|K_1|)|x|} . \end{aligned}$$

Note that

$$\begin{aligned} &\lim_{n\to\infty} K_3 e^{\frac{(|\phi|^n k + C(n)|K_1|)^2\sigma^2}{2} + (|\phi|^n k + C(n)|K_1|)|\theta|}[1 + 2\Phi(\sigma|\phi|^n k + \sigma C(n)|K_1|)] \\ &= K_3 e^{\frac{\phi^2 K_1^2 \sigma^2}{2(1-|\phi|)^2} + \frac{|\phi K_1 \theta|}{1-|\phi|}} \left[1 + 2\Phi\left(\frac{|\phi K_1|\sigma}{1-|\phi|}\right)\right] = K_5 < 1 . \end{aligned}$$

We can find a positive integer N and a positive real number $\delta < 1$ so that for any $n \ge N$,

$$K_3 e^{\frac{(|\phi|^n k + C(n)|K_1|)^2\sigma^2}{2} + (|\phi|^n k + C(n)|K_1|)|\theta|}[1 + 2\Phi(\sigma|\phi|^n k + \sigma C(n)|K_1|)] \le \delta .$$

Then for all $n \ge N$, we have

$$\begin{aligned} &|Q(x) - \tilde{Q}(x)| \\ &\le MK_3^N \left\{ \prod_{i=0}^{N-1} e^{\frac{(|\phi|^i k + C(i)|K_1|)^2\sigma^2}{2} + (|\phi|^i k + C(i)|K_1|)|\theta|}[1 + 2\Phi(\sigma|\phi|^i k + \sigma C(i)|K_1|)] \right\} \delta^{n-N} e^{(|\phi|^n k + C(n)|K_1|)|x|} . \end{aligned}$$

Since $\lim_{n\to\infty} \delta^{n-N} = 0$ and $\lim_{n\to\infty} e^{(|\phi|^n k + C(n)|K_1|)|x|} = e^{\frac{|\phi||K_1|}{1-|\phi|}|x|}$, we obtain $Q(x) = \tilde{Q}(x)$ for all $x \in \mathbb{R}$. This completes the proof of Proposition 1

Q.E.D.

Solving for the coefficients in the analytic Price-Dividend Function

The integral equation (6) may be written as

$$P(x) = K_0 e^{K_1 x} + K_0 e^{K_1 x - \frac{\sigma^2}{2}(1-\gamma)^2} I(x).$$

where $I(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{(1-\gamma)s - \frac{s^2}{2\sigma^2}} P(x_0 + \phi x + s) ds$. To solve the integral equation we posit that $P(x)$ is analytic in which case it has the functional form (21). Consequently we may write $P(x_0 + \phi x + s)$ as

$$P(x_0 + \phi x + s) = e^{K_1(x_0+\phi x+s)} \sum_{k=0}^{n} b_k (x_0 + \phi x + s - x_*)^k.$$

Using the Binomial theorem[26] we may rewrite this equation as

$$P(x_0 + \phi x + s) = e^{K_1(x_0+\phi x+s)} \Big(\sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} s^i \Big).$$

Substituting this result into $I(x)$ we obtain

$$I(x) = e^{K_1(x_0+\phi x)} \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} \gamma_i$$

where $\gamma_i = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{(1-\gamma+K_1)s - \frac{s^2}{2\sigma^2}} s^i \, ds$. The evaluation of the integral γ_i entails the use of a change of variable followed by the evaluation of all the moments of the normal distribution. $\gamma_i = K_3 a_i$ where $K_7 = \frac{1}{\sqrt{\pi}} e^{\frac{\sigma^2}{2}[1-\gamma+K_1]^2}$ and $a_i \equiv \sum_{j=0}^{i} \binom{i}{j} (\sigma^2[1-\gamma+K_1])^{i-j} (\sqrt{2}\sigma)^j (1 + (-1)^j)(1/2)\Gamma[\frac{j+1}{2}]$.

We can now write $I(x)$ as $K_7 e^{K_1(x_0+\phi x)} \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} a_i$ so that the integral equation becomes

$$e^{-K_1 x} P(x) = K_0 + K_6 e^{K_1(x_0+\phi x)} \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} a_i = \sum_{k=0}^{n} b_k (x - x_*)^k.$$

Next we find the undetermined coefficients b_i, $i = 0, \dots n$. In order to do this, we collect all the functions of x on the left hand side of this equation and use Taylor's theorem to take an n^{th} order Taylor expansion around x_*. This produces a system of $(n+1)$ linear equations in the variables b_i, $i = 0, \dots n$.

Define the following functions $q(x) = K_1(x_0 + \phi x)$, $r_{i,k}(x) = (x_0 - x_* + \phi x)^{k-i}$, and $w_{i,k}(x) = \exp(q(x))r_{i,k}(x) \approx \sum_{l=0}^{n} \frac{1}{l!} w_{i,k}^{(l)}(x_*)(x - x_*)^l$. Substituting for $w_{i,k}(x)$ into the integral equation gives us

$$K_0 + K_6 \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} \left(\sum_{l=0}^{n} \frac{1}{l!} w_{i,k}^{(l)}(x_*)(x - x_*)^l \right) a_i = \sum_{k=0}^{n} b_k (x - x_*)^k.$$

Finally, equate the coefficients on the left and right hand side of this equation to yield

$$b_0 = K_0 + K_6 \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} w_{i,k}^{(0)}(x_*) a_i \text{ and } b_l = K_6 \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} \frac{1}{l!} w_{i,k}^{(l)}(x_*) a_i,$$

where $l = 1, \dots, n$. These Equations are an $n+1$ system of linear equations in the b_i's.

Analytic error of the Taylor polynomial approximation

The complex function $Q(z) = e^{-K_1 z} P(z)$ is analytic and it is expressible as a Taylor series

$$Q(z) = \sum_{k=0}^{\infty} \frac{Q^{(k)}(z_*)}{k!} (z - z_*)^k \quad \text{for any complex number } z,$$

where $z_* = x_*$. We may use Cauchy's integral formula to estimate the error of the Taylor polynomial approximation $Q(z) \approx \sum_{k=0}^{n} \frac{Q^{(k)}(z_*)}{k!}(z - z_*)^k$ near $z = z_*$.

Write C_r for the circle of radius $r > 0$ centered at $z = z_*$ in the complex plane. Cauchy's integral formula (see Corollary 5.9 in Conway (1973)) gives

$$Q^{(k)}(z_*) = \frac{k!}{2\pi i} \oint_{C_r} \frac{Q(z)}{(z - z_*)^{k+1}} \, dz \quad \text{for } k = 0, 1, 2, \dots .$$

Each point $z = x + iy$ on C_r satisfies $x_* - r \le x \le x_* + r$ and $-r \le y \le r$. Set

$$A = K_0 \sum_{n=0}^{\infty} K_4^n e^{\frac{\sigma^2 K_1^2}{2} \sum_{i=1}^{n-1} C(i)^2 + |K_1 \theta| \sum_{i=1}^{n-1} C(i)}.$$

Using the result in the proof of Lemma 7, we obtain $0 \le Q(s) \le Ae^{\frac{|\phi K_1|}{1-|\phi|}|s|}$ for all $s \in \mathbb{R}$, and

$$\begin{aligned}
|Q(z)| &\le K_0 + \frac{K_3|e^{\phi K_1(x+iy)}|}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} Q(s)\left|e^{-\frac{1}{2\sigma^2}[s-\phi x-\theta-i\phi y]^2}\right| ds \\
&= K_0 + \frac{K_3 e^{\phi K_1 x+\frac{\phi^2 y^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} Q(s)e^{-\frac{1}{2\sigma^2}[s-\phi x-\theta]^2} ds \\
&\le K_0 + \frac{AK_3 e^{|\phi K_1 x|+\frac{\phi^2 y^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[s-\phi x-\theta]^2+\frac{|\phi K_1|}{1-|\phi|}|s|} ds \\
&\le K_0 + AK_3 e^{|\phi K_1 x|+\frac{\phi^2|K_1 x|}{1-|\phi|}+\frac{\phi^2 y^2}{2\sigma^2}+\frac{\phi^2K_1^2\sigma^2}{2(1-|\phi|)^2}+\frac{|\phi K_1\theta|}{1-|\phi|}}\left[1+2\Phi\left(\frac{|\phi K_1|\sigma}{1-|\phi|}\right)\right] \\
&= K_0 + AK_3 e^{\frac{|\phi K_1|(|x|+|\theta|)}{1-|\phi|}+\frac{\phi^2 y^2}{2\sigma^2}+\frac{\phi^2K_1^2\sigma^2}{2(1-|\phi|)^2}}\left[1+2\Phi\left(\frac{|\phi K_1|\sigma}{1-|\phi|}\right)\right] \\
&\le K_0 + AK_3 e^{\frac{|\phi K_1|(r+|x_*|+|\theta|)}{1-|\phi|}+\frac{\phi^2 r^2}{2\sigma^2}+\frac{\phi^2K_1^2\sigma^2}{2(1-|\phi|)^2}}\left[1+2\Phi\left(\frac{|\phi K_1|\sigma}{1-|\phi|}\right)\right].
\end{aligned}$$

Define $B_r = K_0 + AK_3 e^{\frac{|\phi K_1|(r+|x_*|+|\theta|)}{1-|\phi|}+\frac{\phi^2 r^2}{2\sigma^2}+\frac{\phi^2K_1^2\sigma^2}{2(1-|\phi|)^2}}\left[1+2\Phi\left(\frac{|\phi K_1|\sigma}{1-|\phi|}\right)\right]$. Cauchy's integral formula for $Q^{(k)}(z_*)$ together with the above estimate to $|Q(z)|$ gives rise to

$$\begin{aligned}
|Q^{(k)}(z_*)| &\le \left|\frac{k!}{2\pi i}\oint_{C_r}\frac{Q(z)}{(z-z_*)^{k+1}}\,dz\right| = \frac{k!}{2\pi}\left|\int_0^{2\pi}\frac{Q(z_*+re^{i\theta})}{(re^{i\theta})^{k+1}}ire^{i\theta}\,d\theta\right| \\
&\le \frac{k!}{2\pi}\int_0^{2\pi}\frac{|Q(z_*+re^{i\theta})|}{r^k}\,d\theta \le \frac{B_r k!}{r^k}.
\end{aligned}$$

This bound allows us to estimate the analytic error (24).

FOOTNOTES

[26]The binomial theorem states that we may write $(x_0+\phi x+s-x_*)^k = \sum_{i=0}^k \binom{k}{i}(x_0+\phi x-x_*)^{k-i}s^i$.