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Analytic solving of asset pricing models: The by force of habit case

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ABSTRACT

Analytic methods for solving asset pricing models are developed to solve asset pricing models. Campbell and Cochrane's [1999. By force of habit, a consumption-based explanation of aggregate stock market behavior. Journal of Political Economy 107, 205–251] habit persistence model provides a prototypical example to illustrate this method. When the parameters involved satisfy certain conditions, the integral equation of this model has a solution in the space of continuous functions that grows exponentially at infinity. However, the parameters advocated by Campbell and Cochrane do not satisfy one of these conditions. The existence problem is removed by restricting the price–dividend function to avoid values of dividend growth that are extreme. Thus, existence and uniqueness of the solution in the space of continuous and bounded functions is proved. Using complex analysis the price–dividend function is also shown to be analytic in a region large enough to cover all relevant values of dividend growth. Next, a numerical method is presented for computing higher order polynomial approximations of the solution. Finally, a uniform upper bound on the error of these approximations is derived. An intensive search of the parameter space results in no parameter values for which the solution matches the historic equity premium and Sharpe ratio within Campbell and Cochrane's model.

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1. Introduction

Various asset pricing models have been proposed to explain the equity premium puzzle, i.e., the high return on stocks relative to the return on risk-free bonds.[1] Each of these models produces an integral equation whose solution determines the price–dividend function and hence also the equilibrium return on a risky asset held by a representative investor. Although there is limited information about the essential properties of the solutions to these integral equations, economists and mathematicians employ numerical methods to approximate them. The recent work of CCCH and CCH shows how to obtain the essential mathematical properties of the solutions to the well-known asset pricing models of Mehra and Prescott (1985) and a generalized version of Abel (1990).[2] They go on to show how these properties can lead to improvements in the numerical algorithms developed to represent the solutions to these models. This paper lays out how these analytic methods can be used to identify the mathematical properties of more general asset pricing problems. Subsequently, it shows how to develop numerical algorithms exploiting these properties.

To illustrate these analytic methods we use the asset pricing model of Campbell and Cochrane (CC) (1999). A shortcoming of early asset pricing models is that they require unrealistically large levels of relative risk aversion and/or high risk free interest rates in order for their price–dividend functions to yield equity premiums consistent with empirical evidence.[3] CC seek to reconcile this gap between theory and evidence by introducing a surplus consumption ratio into the pricing kernel. This surplus consumption ratio compares the investor's consumption with their external habit. They also develop an equation of motion for this surplus consumption ratio, which is dependent on random shocks to dividend growth. In this equation of motion the log-normally distributed random shocks to dividend growth are amplified by a sensitivity function that places larger (smaller) weight on small (large) random shocks to dividend growth. This sensitivity function is dependent on the surplus consumption ratio so that the surplus consumption ratio is the state variable for the determination of the price–dividend ratio. This specification creates a precautionary savings which keeps the risk free interest rate low.[4]

The CC model leads to an integral equation for the price–dividend function which depends on the surplus consumption ratio. Moreover, the surplus consumption ratio depends on future dividend growth. In this model it is assumed that dividend growth follows a log-normal distribution. Therefore, it is natural to seek a solution to the integral equation in the space of continuous functions that grow exponentially. That is, the real vector space $C(\mathbb{R}, e^{\gamma x})$ of all continuous functions $f(x)$ with domain $\mathbb{R}$ such that $|f(x)| \leqslant m_1 e^{\gamma x} + m_2$ for all x, where $m_1, m_2 \geqslant 0$ may depend on $f(x)$. When the parameters involved satisfy certain conditions, it is shown that the integral equation of the CC model has a unique solution in the space $C(\mathbb{R}, e^{\gamma x})$ (see Proposition 1).

Unfortunately, the condition for the existence of a unique solution to Campbell and Cochrane's model in the space $C(\mathbb{R}, e^{\gamma x})$ is not satisfied. Given the procedure for choosing the parameter values espoused by CC, it is shown that the coefficient of risk aversion must be larger than 76 for existence of the solution, which is even higher than the value Mehra and Prescott (1985) needed to explain the equity premium. The reason for this failure is that dividend growth is too low relative to the risk free interest rate. Thus, there is no known proof of existence for the original specification of the model by Campbell and Cochrane.

To deal with the existence problem the price–dividend function is restricted to avoid values of the surplus consumption ratio that are extreme. Then, existence and uniqueness of the solution is proved in the space of continuous and bounded functions for a larger set of the parameters, including the ones used by CC. Also, a uniform upper bound for the solution (see Theorem 1) is found. This

[1] See Mehra and Prescott (2003) for a recent survey of these models.

[2] Throughout the paper, we will write CCCH for the paper by Calin et al. (2005), write CCH for the paper by Chen et al. (2008), and write CC for the paper by Campbell and Cochrane (1999).

[3] Weil (1989) refers to this phenomenon as the risk free rate puzzle.

[4] Cecchetti et al. (2000) provide an alternative motive for precautionary savings to explain the equity premium with a low risk free interest rate.

provides a firm basis for further study of the mathematical properties of the Campbell and Cochrane model.

Indeed, using complex analysis it is shown that the price–dividend function is analytic near the steady state surplus consumption ratio and the region of convergence of its Taylor series is computed (see Theorem 2). The radius of convergence is large enough so that the region of convergence contains the consumption growth in the interval $[x_0 - 25\%, x_0 + 25\%]$ where x_0 is the steady state level of consumption growth. This range of consumption growth would include any fluctuation in dividend growth around the world in the 20th century (see Barro, 2006). Thus, the price–dividend function is analytic for any feasible level of consumption growth.

Taking advantage of the established analyticity of the price–dividend function and the fact that its Taylor series has a large enough region of convergence, we develop a numerical algorithm that computes a nth order polynomial approximation for the solution. In addition, the uniform bound on the price–dividend function, along with it being an analytic function, yields a uniform bound on the approximation error for dividend growth in the interval $[x_0 - 25\%, x_0 + 25\%]$.[5] Thus, the numerical solution for the price–dividend function may be made as accurate as desired by increasing the order of the polynomial approximation.

In summary, this paper provides an overview of the analytic method for solving asset pricing models, and applies it to solve rigorously the interesting model of Campbell and Cochrane. It is worth mentioning that the value of the price–dividend ratio around the steady state surplus consumption ratio is very sensitive to its values far away from the steady state due to the global nature of the model (integral equation). This becomes apparent in calibrating and simulating the model (see Section 6). For example using the CC parameters the upper bound of the price–dividend function is below $\frac{187}{12} = 15.6$ when dividend growth is restricted to the interval $[x_0 - 25\%, x_0 + 25\%]$. As a result, it is impossible to obtain the historic value of 18.6 at the steady state under these circumstances. After an intensive search of the parameter space using available computational resources, there is no combination of parameters in which the equity premium and Sharpe ratio match their historic values.

In Section 2 we provide an overview of our analytic method for solving asset pricing models. In Section 3 we summarize CC's asset pricing model and prove existence and uniqueness of the price–dividend function when the parameters involved satisfy certain conditions. However, the parameters used by Campbell and Cochrane do not satisfy one of these conditions. We remedy this problem by restricting the price–dividend function to avoid big tails and prove existence and uniqueness of solution in the space of continuous and bounded functions. In Section 4, using complex analysis, we show that the price–dividend function is analytic at the steady state level of dividend growth and find the radius of convergence for its Taylor series. In Section 5 we present a numerical method for computing a higher order polynomial approximation of the solution and provide an error analysis of the numerical solution. In Section 6 we summarize the results from our numerical simulation of the CC asset pricing model. The conclusion is provided in Section 7. Finally, in the appendix we provide complete proofs of the mathematical results used in this paper.

2. Overview of analytic method

In this paper, we consider the properties of the solutions of integral equations, which are found in asset pricing models, of the form:

$$P(x) = \int_{\mathbb{R}^m} M(\varphi(x,v),v)(1 + P(\varphi(x,v)))f(v)\,\mathrm{d}v, \tag{1}$$

where $P : \mathbb{R}^m \to \mathbb{R}$ is an unknown price–dividend function of the state variables x, $\varphi(x,v)$ is a function for the motion of state variables which depends on the random shocks v with probability density function $f(v)$, and $M(\varphi(x,v),v)$ is the pricing kernel which represents the investor's evaluation of future cash flows from an investment. $M(\varphi(x,v),v)$ depends on the future state variables as well as the random changes in these state variables.

[5] Santos (1991, 1992, 1993, 1999, 2000) also provides error analysis for his numerical procedures.

The asset pricing models of Mehra and Prescott (1985), Abel (1990), and Campbell and Cochrane (1999) fit into this integral equation with one state variable. Epstein and Zin's (1989, 1990, 1991) asset pricing model also fits into this framework with an appropriate definition of the unknown function as a function of the price–dividend function, i.e.,

$$\mathrm{N}(x) \equiv (P(x) + 1)^{\gamma}.$$

The extensions of the Campbell and Cochrane model by Wachter (2002, 2006) yield a two-dimensional asset pricing function, since the consumption growth and inflation are added as additional state variables to Campbell and Cochrane's price–dividend function. The empirical habit based model of Chen and Ludvigson (2008) would have four state variables. The models of exchange rates, which treat foreign exchange as an asset,[6] may also be placed into this rubric. Although, most of the models of exchange rates treat the stochastic discount factor as a constant, exchange rates are subject to anomalies similar to asset prices so that this literature may benefit from the use of more sophisticated pricing kernels.[7] Stochastic growth models such as Brock and Mirman (1972) would not fit into this setup, since the pricing kernel would be a function of the unknown equilibrium function. As a result, stochastic growth models would yield a non-linear integral equation which we do not address in this paper.

What makes the integral equation (1) atypical is the dependence of the unknown price–dividend function on the equation of motion. Standard mathematical analysis of integral equations like (1) would have the unknown function on the right-hand side dependent only on the random shock, i.e.,

$$P(x) = g(x) + \int_{\mathbb{R}^m} K(x, y) P(y)\, \mathrm{d}y,$$

where $K : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is the kernel of the integral equation. The properties of the solution $P(x)$ depend on those of $g(x)$ and $K(x, y)$. Tauchen and Hussey (1991) use the quadrature procedure to solve numerically such integral equations, assuming the existence and uniqueness of the solution. In our work, we go beyond this to provide a polynomial approximation scheme by taking advantage of the fact that in our atypical integral equation the resulting (after some transformations) $g(x)$ and $K(x, y)$ are analytic, and therefore we can prove that $P(x)$ is analytic, too.[8]

The first step in the analysis of the integral equation (1) is to identify a vector space of functions in which a unique solution to the price–dividend function exists, e.g., see Definition 2 and Proposition 1 for the Campbell and Cochrane case.

Next, we simplify the analysis by using a change of variables which makes the future price–dividend function dependent only on the random shock(s). We introduce:

Assumption 1. $\det(\partial\varphi/\partial v) \neq 0$, so that we can make the change of variables $s = \varphi(x, v)$, or equivalently, $v = \psi(x, s)$.[9]

Then the integral equation (1) becomes

$$P(x) = \int_{\mathbb{R}^m} M(s, \psi(x, s))(1 + P(s)) f(\psi(x, s)) |\det(\partial\psi/\partial s)|\, \mathrm{d}s. \tag{2}$$

In this form, the integral equation is a Fredholm equation of the second type in which the kernel is $M(s, \psi(x, s)) f(\psi(x, s)) |\det(\partial\psi/\partial s)(x, s)|$.

To solve this integral equation we rely on analytic methods following CCCH. Before describing our solution method, we discuss briefly the basic properties of holomorphic functions in one complex variable. These properties are also valid in several complex variables with some additional technical complications. Let $f(z)$ be a complex-valued function of z defined in an open set (domain) D of the

[6] See Engel and West (2005) for a recent discussion of this topic.

[7] For example, see Choi et al. (2008).

[8] In CCCH, we provide a comparison between the quadrature method and the polynomial method for the Mehra and Prescott (1985) model.

[9] A similar assumption can be found in Blume et al. (1982).

complex plane $\mathbb{C}$. Suppose that this function is smooth and it can be expressed as

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \quad \text{for } |z - z_0| < r \text{ and } z_0 = x_0 + iy_0, \tag{3}$$

where $f^{(k)}(z_0)$ denotes the kth order complex derivative of $f(z)$ at $z = z_0$ (the usual calculus formulas for real differentiation hold true for complex differentiation, too). Separating this function into its real and imaginary parts $f(z) = f_1(x, y) + if_2(x, y)$, we obtain

$$\frac{\partial f}{\partial x} = \frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x} = \sum_{k=1}^{\infty} \frac{f^{(k)}(z_0)}{k!} k(z - z_0)^{k-1}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial f_1}{\partial y} + i\frac{\partial f_2}{\partial y} = \sum_{k=1}^{\infty} \frac{f^{(k)}(z_0)}{k!} ik(z - z_0)^{k-1}.$$

Multiplying the second equation by i and adding it to the first yields

$$\bar{\partial} f \equiv \frac{\partial f}{\partial \bar{z}} \equiv \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.^{10} \tag{4}$$

This equation is called the Cauchy–Riemann equation, which can also be written in the form:

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} \quad \text{and} \quad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}.$$

This motivates the following:

Definition 1. A function $f(z)$ is said to be *holomorphic* in an open set D in the complex plane if f is in $C^1(D)$ and satisfies the Cauchy–Riemann equation in D.

In fact, it can be shown that f is holomorphic in D if $\bar{\partial} f = 0$ in the sense of distribution theory (weak derivatives). A proof of this fact can be found in Hörmander (1983, p. 110, Theorem 4.4.1). The basic theory of holomorphic functions can be found in the classical text of Ahlfors (1979) or Hörmander (1979), where it is presented in the more general context. There, it is proved that a holomorphic function has derivatives of any order, and therefore its Taylor series can be formed. Moreover, for each $z_0 \in D$ there is (a maximal) positive number r, called the radius of convergence for $f(z)$ at z_0, such that $f(z)$ can be represented by its Taylor series, like (3), for $|z - z_0| < r$.

A key fact, which is very useful in our work here, is that r is equal to the distance of z_0 to the boundary of the domain of holomorphicity of f. For example, if $f(z) = 1/(1 + z^2)$, then the domain of holomorphicity of f is $\mathbb{C}\backslash\{\pm i\}$, and its radius of convergence r, say about 0, is equal to $|i - 0| = 1$.

The 'calculus' definition of an analytic function $f(x)$ of a real variable x in an open interval (a, b) is that it is in $C^\infty(a, b)$ and at each point $x_0 \in (a, b)$ the remainder $R_n(x)$ of the nth degree Taylor polynomial approximation of $f(x)$ tends to zero as $n \to \infty$ when $|x - x_0| < r$. By complexifying x to $z = x + iy$, we obtain a holomorphic function $f(z)$ in an open set D in $\mathbb{C}$, and we may think of $f(x)$ as the restriction of $f(z)$ from D to (a, b).

We now continue with our procedure. For proving the analyticity of the price–dividend function, we need:

Assumption 2. $M(s, \psi(x, s))$, $f(\psi(x, s))$, and $|\det(\partial\psi/\partial s)(x, s)|$ are analytic in x, and therefore can be complexified to give holomorphic functions with respect to the complex variable $z = x + iy$.

Consequently, we expect the price–dividend function to be holomorphic as well.

Assumption 2 gives

$$\bar{\partial}(M(s, \psi(z, s))f(\psi(z, s))|\det(\partial\psi/\partial s)(z, s)|) = 0,$$

[10] See Hörmander (1983, p. 62).

since the product of holomorphic functions is holomorphic. Verifying that we can pass differentiation $\bar{\partial}$ inside the integral sign of Eq. (2), we find that the integral

$$\int_S M(s,\psi(z,s))(1+P(s))f(\psi(z,s))|\det(\partial\psi/\partial s)(z,s)|\,\mathrm{d}s$$

is holomorphic for z in an open set D in $\mathbb{C}$. So the radius of convergence for the price–dividend function at $z_0 = x_0 + i0$ is determined by the distance of z_0 to the boundary of the domain D.

When there is a bound P_0 on the price–dividend function, we can use the Cauchy integral formula (see Conway, 1973) to place a bound on its derivatives. Denote by C_r for the circle of radius $r>0$ centered at $z=z_0$ in the complex plane.

$$P^{(k)}(z_0) = \frac{k!}{2\pi i}\oint_{C_r}\frac{P(z)}{(z-z_0)^{k+1}}\,\mathrm{d}z \quad \text{for } k=0,1,2,\ldots. \tag{5}$$

Therefore,

$$|P^{(k)}(z_0)| \leqslant \frac{k!}{2\pi}\oint_{C_r}\frac{|P(z)|}{|z-z_0|^{k+1}}\,|\mathrm{d}z| \leqslant \frac{P_0 k!}{r^k} \quad \text{for } k=0,1,2,\ldots. \tag{6}$$

Motivated by the fact that the price–dividend function has a power series expansion like (3), it makes good sense to use the following numerical procedure. For any given integer $n\geqslant 0$, we look for a polynomial approximation of the price–dividend function of the form:

$$P_n^c(x) = \sum_{k=0}^{n} b_k(x-x_0)^k, \tag{7}$$

where the coefficients b_k are determined by substituting the polynomial $P_n^c(x)$ into (1).

In CCCH, we use the bound (6) on the derivatives of the price–dividend function to find a bound on the error in our approximate solution (7).

$$\text{error} \equiv \max_{|x-x_0|<\mu r}|P(x)-P_n^c(x)|, \tag{8}$$

where $0<\mu<1$.

3. The Campbell and Cochrane model

Campbell and Cochrane (1999) assume that individual's preferences are represented by

$$\mathrm{E}_t\left(\sum_{j=t}^{\infty}\beta^j\frac{(S_j^{\mathrm{a}}C_j)^{1-\gamma}-1}{1-\gamma}\right),$$

where C_j is the individual's consumption at time $j=t,t+1,\ldots$, $S_j^{\mathrm{a}} = (C_j^{\mathrm{a}} - X_j)/C_j^{\mathrm{a}}$ is the surplus consumption ratio at time j, C_j^{a} is the average consumption of all individuals at time j and X_j is their habitual level of consumption at time j, and $\gamma>0$ is the relative risk aversion.[11] Finally, $\mathrm{E}_t(x)$ refers to the expectation of x conditional on the investor's information at time t.

The surplus consumption ratio S_t^{a} satisfies the AR(1) process with $0<\phi<1$:

$$s_{t+1}^{\mathrm{a}} = (1-\phi)\bar{s} + \phi s_t^{\mathrm{a}} + \lambda(s_t^{\mathrm{a}})(c_{t+1}^{\mathrm{a}} - c_t^{\mathrm{a}} - g),$$

where the lower case letters refer to the natural logarithms of the variables, e.g.,

$$s_t^{\mathrm{a}} = \ln(S_t^{\mathrm{a}}).$$

The consumption growth follows a normal distribution such that

$$c_{t+1}^{\mathrm{a}} - c_t^{\mathrm{a}} = g + v_{t+1}, \quad v_{t+1} \sim \text{i.i.d. } \mathrm{N}(0,\sigma^2).$$

[11] See Wachter (2005) for a recent survey of models based on CC's model. Korniotis (2005) provides estimation of CC's model under complete and incomplete markets.

To introduce heteroscedasticity of the random shock to the consumption growth, Campbell and Cochrane introduce the sensitivity function:

$$\lambda(s_t^a) \equiv \begin{cases} \sqrt{1-2(s_t^a-\bar{s})}/\bar{S}-1 & \text{if } s_t^a \leqslant s_{\max}, \\ 0 & \text{if } s_t^a > s_{\max}. \end{cases}$$

Here,

$$\bar{S} \equiv \sigma\sqrt{\frac{\gamma}{1-\phi}} \quad \text{and} \quad s_{\max} = \bar{s} + \frac{1-\bar{S}^2}{2}. \tag{9}$$

This sensitivity function was derived from four properties: (1) the domain of the pricing kernel is $\mathbb{R}_+$; (2) the natural logarithm of the risk free interest rate, r^f, is constant;[12] (3) the derivative of s_t^a with respect to c is 0 at $\bar{S}$; and (4) the second derivative of s_t^a with respect to c is 0 at $\bar{S}$.

The price–dividend function satisfies the Euler condition:

$$P(s_t) = \mathrm{E}_t\left(M_{t+1}\frac{D_{t+1}}{D_t}(1+P(s_{t+1}))\right), \tag{10}$$

where $P:\mathbb{R}\to\mathbb{R}_+$ is the price–dividend function that pays the dividend stream D_t and $M_{t+1} \equiv \beta \mathrm{e}^{-\gamma g-\gamma(\phi-1)(s_t-\bar{s})-\gamma(1+\lambda(s_t))v_{t+1}}$ is the pricing kernel. In the most general model, Campbell and Cochrane assume that dividend growth follows:

$$d_{t+1}^a - d_t^a = g + w_{t+1}, \quad w_{t+1} \sim \text{i.i.d. } \mathrm{N}(0,\sigma_w^2) \quad \text{and} \quad \mathrm{corr}(w_t, v_t) = \rho.$$

Consequently, the random shocks to dividends follow a log-normal distribution.

To find the price–dividend function $P(x)$, we rewrite the Euler condition (10) as

$$P(x) = \frac{K_0 \mathrm{e}^{K_1 x}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \mathrm{e}^{(K_2-\gamma(1+\lambda(x)))v}\cdot(1+P(\phi x+\lambda(x)v))\cdot \mathrm{e}^{-v^2/2\sigma^2}\,\mathrm{d}v, \tag{11}$$

where $x = s_t^a - \bar{s}$, $v = v_{t+1}$, and

$$K_0 \equiv \beta \mathrm{e}^{g(1-\gamma)+1/2(1-\rho^2)\sigma_w^2}, \quad K_1 \equiv (1-\phi)\gamma, \quad K_2 \equiv \frac{\rho\sigma_w}{\sigma}. \tag{12}$$

Now the sensitivity function is written as

$$\lambda(x) = \begin{cases} \sqrt{1-2x}/\bar{S}-1 & \text{if } x \leqslant x^*, \\ 0 & \text{if } x > x^*, \end{cases} \tag{13}$$

where $x^* = s_{\max} - \bar{s} = (1-\bar{S}^2)/2$. Consequently, we obtain an integral equation in the form (1). In addition, the equation of motion for the state variable $\varphi(x,v) = \phi x + \lambda(x)v$ satisfies Assumption 1 as long as $\lambda(x) > 0$. Thus, the interval of convergence for the price–dividend function must be limited to a subset of $(-\infty, x^*)$.

3.1. Existence and uniqueness of the price– dividend function

We follow CCCH and seek a solution in the following vector space.

Definition 2. Let $C(\mathbb{R}, \mathrm{e}^{\gamma x})$ denote the real vector space that consists of all continuous functions $f(x)$ such that $|f(x)| \leqslant m_1 \mathrm{e}^{\gamma x} + m_2$ for all $x \in \mathbb{R}$, where the constants $m_1 \geqslant 0$ and $m_2 \geqslant 0$ may depend on $f(x)$.

By completing the square (see Appendix A), we can write the Eq. (11) in the form:

$$P(x) = M(x) + \frac{M(x)}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \mathrm{e}^{-(1/2\sigma^2)(v-\sigma^2(K_2-\gamma(1+\lambda(x))))^2} P(\phi x+\lambda(x)v)\,\mathrm{d}v, \tag{14}$$

[12] If $r_f = r_0^f - B(s_t - \bar{s})$, then $\bar{S} = \sigma\sqrt{\gamma/(1-\phi-B/\gamma)}$. This does not influence any of the analysis of the integral equation for the price–dividend function.

where

$$M(x) \equiv K_0 e^{K_1 x + \sigma^2 (K_2 - \gamma(1+\lambda(x)))^2/2}. \tag{15}$$

Note that $|M(x)| \leqslant m_1 e^{\gamma x} + m_2$ for all $x \in \mathbb{R}$, where

$$m_1 \equiv K_0 e^{(\sigma^2 (K_2-\gamma)^2 - \gamma\phi(1-\bar{S}^2))/2} \quad \text{and} \quad m_2 \equiv K_0 e^{\sigma^2 (K_2^2 + \gamma^2/\bar{S}^2)/2 - \sigma^2 \gamma K_2}. \tag{16}$$

So $M(x)$ is in $C(\mathbb{R}, e^{\gamma x})$.

Definition 3. For any $f(x) \in C(\mathbb{R}, e^{\gamma x})$, we define the transformation

$$(Tf)(x) \equiv \frac{M(x)}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-(1/(2\sigma^2))(v - \sigma^2(K_2 - \gamma(1+\lambda(x))))^2} f(\phi x + \lambda(x) v)\, dv. \tag{17}$$

The next result shows that the transformation T maps the space $C(\mathbb{R}, e^{\gamma x})$ into itself.

Lemma 1. *If* $f(x) \in C(\mathbb{R}, e^{\gamma x})$, *then* $(Tf)(x) \in C(\mathbb{R}, e^{\gamma x})$.

The proof of this lemma is provided in Appendix B.

Now Eq. (14) is equivalent to $P(x) = M(x) + (TP)(x)$. The following assumption will guarantee that this equation has a unique solution in $C(\mathbb{R}, e^{\gamma x})$,

Assumption 3. $0 < \bar{S} < 1$, $0 \leqslant m_1 < 1$, and $0 < m_2 < 1$.

Proposition 1. *Under Assumption* 3, *the integral equation* (14) *has a unique solution in the space* $C(\mathbb{R}, e^{\gamma x})$.

We prove this proposition in Appendix C. Next, we discuss the validity of Assumption 3.

The parameters used by Campbell and Cochrane satisfy the first two conditions in Assumption 3. However, they do *not* satisfy the last condition. More precisely we have the following remark.

Remark. The condition $m_2 < 1$ is not satisfied for the Campbell and Cochrane parameters.

In fact, in the consumption claim case we have $K_2 = 1$, $\rho = 1$ and $\sigma_w = \sigma$. Combining these conditions with the definition of K_0 in (12) and m_2 in Eqs. (16) yields

$$m_2 = \beta e^{g(1-\gamma)} e^{\sigma^2(1+\gamma^2/\bar{S}^2)/2 - \sigma^2\gamma}. \tag{18}$$

Campbell and Cochrane set the parameters with a monthly time frame, so that $g = 0.0118/12 = 0.00157$, $\gamma = 2$, $\sigma = 0.0112/\sqrt{12} = 0.00323$, $\bar{S} = 0.0448$, and $\beta = 0.9894$. In this case, $m_2 = 1.0008$, so that the Campbell and Cochrane model does not satisfy the condition for existence in Proposition 1. Similar calculations for the dividend claim case yield the same result. In the calibration section, we show that this condition is not satisfied for most circumstances.

To remedy this problem we shall restrict the price–dividend function when the surplus consumption ratio gets too big. But first, we transform integral equation (11) for convenience.

3.2. Transformed integral equation

As we shall see below it is more convenient to work with the following transformation of the price–dividend function.

Definition 4. Define the transformation:

$$Q(x) = e^{-\gamma x}(1 + P(x)). \tag{19}$$

Also, using the change of variables $y = \phi x + \lambda(x) v$, (11) is transformed to

$$Q(x) = e^{-\gamma x} + \frac{K_3}{\sqrt{2\pi}\sigma\lambda(x)} \int_{-\infty}^{\infty} Q(y) e^{-(1/2\sigma^2\lambda(x)^2)(y - \psi(x))^2}\, dy, \tag{20}$$

where

$$K_3 \equiv K_0 e^{\sigma^2 (K_2-\gamma)^2/2}, \tag{21}$$

and

$$\psi(x) \equiv \phi x + \sigma^2 (K_2 - \gamma)\lambda(x). \tag{22}$$

Furthermore, let

$$\tilde{\lambda}(x) \equiv \sigma\lambda(x) = \begin{cases} c\sqrt{1-2x} - \sigma & \text{if } x \leqslant (1-\bar{S}^2)/2, \\ 0 & \text{if } x > (1-\bar{S}^2)/2, \end{cases} \tag{23}$$

where $c = \sigma/\bar{S} = \sqrt{(1-\phi)/\gamma}$. Then $\psi(x) = \phi x + \sigma(K_2 - \gamma)\tilde{\lambda}(x)$ and (20) takes the form:

$$Q(x) = e^{-\gamma x} + \frac{K_3}{\sqrt{2\pi}\tilde{\lambda}(x)} \int_{-\infty}^{\infty} Q(y) e^{-(1/(2(\tilde{\lambda}(x))^2))(y-\psi(x))^2}\, dy. \tag{24}$$

Choosing an r such that

$$0 < r < (1-\bar{S}^2)/2, \tag{25}$$

we have that the denominator $\tilde{\lambda}(x)$ in (24) never vanishes for $x < r$. So we modify the integral equation (24) as follows (see Appendix D):

$$Q(x) = \begin{cases} e^{-\gamma x} + \dfrac{K_3}{\sqrt{2\pi}\tilde{\lambda}(x)} \displaystyle\int_{-\infty}^{\infty} Q(y) e^{-(1/(2(\tilde{\lambda}(x))^2))(y-\psi(x))^2}\, dy & \text{if } -r \leqslant x \leqslant r, \\ Q(r) & \text{if } x > r, \\ Q(-r) & \text{if } x < -r. \end{cases} \tag{26}$$

Now we have the following result.

Theorem 1. *If the condition* (25) *holds and* $K_3 < 1$, *then the Eq.* (26) *has a unique solution* $Q(x)$ *in the vector space of all continuous and bounded functions defined in* $\mathbb{R}$. *Moreover, we have*

$$\|Q\| \equiv \sup_{x\in\mathbb{R}} |Q(x)| \leqslant \frac{e^{\gamma r}}{1-K_3}. \tag{27}$$

We prove this theorem in Appendix E. Next, we discuss the analyticity of $Q(x)$.

4. Analyticity of the price–dividend function

Since $Q(x) = e^{-\gamma x}[1 + P(x)]$, to prove that the price–dividend function $P(x)$ is analytic near $x = 0$ and its Taylor series has radius of convergence r_c it suffices to prove it for the function $Q(x)$. For this, we shall apply basic properties of holomorphic functions. First, we will show that $Q(x)$ makes sense when x is replaced by $w = u + iv$. (Here, we use the complex variable $w = u + iv$ instead of the traditional notation $z = x + iy$.) By (20) it suffices to do this for the function

$$F(w) \equiv \int_{-\infty}^{\infty} Q(t) \cdot e^{-(1/(2\sigma^2\lambda(w)^2))(t-\psi(w))^2}\, dt, \tag{28}$$

since the other pieces in the formula defining Q are well-known elementary functions.

Second, we need to show that

$$\bar{\partial}F(w) = \int_{-\infty}^{\infty} Q(t) \cdot \bar{\partial}(e^{-(1/(2\sigma^2\lambda(w)^2))(t-\psi(w))^2})\, dt = 0; \tag{29}$$

that is, the Cauchy–Riemann equation holds in an open set D in $\mathbb{C}$. We already know that $e^{-(1/2\sigma^2\lambda(w)^2)[t-\psi(w)]^2}$ is holomorphic at the origin. Consequently, we must verify that the integrals (28) and (29) exist. Both of these conditions are satisfied when

$$\mathrm{Re}\left\{\frac{1}{2\sigma^2\lambda(w)^2}(t-\psi(w))^2\right\}>0. \tag{30}$$

This method is implemented in Appendix F. Here, it is summarized in the following lemma.

Lemma 2. *Let*

$$f(x)=\int_{-\infty}^{\infty}Q(t)\mathrm{e}^{-(1/(2\sigma^2\lambda(x)^2))(t-\psi(x))^2}\,\mathrm{d}t, \tag{31}$$

where $Q(x)$ is a bounded and continuous function and $\psi(x)$ is given by (22). Then $f(x)$ is analytic for $x<x^*=(1-\bar{S}^2)/2$, and the radius of convergence of its Taylor series around the origin is equal to

$$r_\mathrm{c}=\frac{\bar{S}^2R_z}{2}, \tag{32}$$

where

$$R_z\equiv\sqrt{(x_0-\bar{S}^{-2})^2+\frac{1}{4}(x_0^2-1)^2}\quad and \quad x_0\equiv\frac{-3^{1/3}\bar{S}^{4/3}+(9+\sqrt{3}\sqrt{27+\bar{S}^4})^{2/3}}{3^{2/3}\bar{S}^{2/3}(9+\sqrt{3}\sqrt{27+\bar{S}^4})^{1/3}}. \tag{33}$$

The next theorem, which is the main result of this section, follows immediately from the relation between $P(x)$ and $Q(x)$ given in Definition 4.

Theorem 2. *The price–dividend function $P(x)$ is analytic for $x<x^*=(1-\bar{S}^2)/2$, and the radius of convergence of its Taylor series about the origin is $r_\mathrm{c}=\bar{S}^2R_z/2$.*

Theorem 2 gives the best estimate for the radius of convergence for the Taylor series of the price–dividend function $P(x)$ about the origin. In the next section, we exploit the analyticity of the price–dividend function in the interval determined by Theorem 2 for the purpose of approximating the price–dividend function numerically.

5. Numerical solving for the price–dividend function

Based on Lemma 2, the solution of (26) is analytic at the origin and the radius of convergence for the Taylor series of $Q(x)$ around the origin is r_c. Choosing r to satisfy the additional condition:

$$0<r<r_\mathrm{c}, \tag{34}$$

we have that the Taylor series of $Q(x)$ about the origin converges to $Q(x)$ for $-r\leqslant x\leqslant r$, and therefore $Q(x)$ satisfies equation:

$$Q(x)=\begin{cases}\displaystyle\sum_{n=0}^{\infty}a_nx^n & \text{if } -r\leqslant x\leqslant r,\\ \displaystyle\sum_{n=0}^{\infty}a_nr^n & \text{if } x>r,\\ \displaystyle\sum_{n=0}^{\infty}(-1)^na_nr^n & \text{if } x<-r.\end{cases} \tag{35}$$

In Appendix G, we show that the coefficients a_n satisfy the linear equations

$$a_l=\frac{(-1)^l\gamma^l}{l!}+\sum_{n=0}^{\infty}a_n\left\{\delta_{l,0}\frac{K_3(1+(-1)^n)r^n}{2}+\sum_{k=0}^{\infty}\frac{K_3}{\sqrt{2\pi}2^kk!}\right.$$
$$\times\left(\sum_{j=0}^{n}\sum_{i=0}^{2k+j+1}\binom{n}{j}\binom{2k+j+1}{i}\frac{[(-1)^{j+k}+(-1)^{i+j+k+1}]r^i}{2k+j+1}\cdot b_{2k+1+n-i,2k+1,l}$$

$$+\sum_{j=0}^{2k+1}\binom{2k+1}{j}\frac{[(-1)^{j+k}+(-1)^{n+k+1}]r^{n+j}}{2k+1}\cdot b_{2k+1-j,2k+1,l}\Bigg)\Bigg\}, \tag{36}$$

where

$$\delta_{l,0}=\begin{cases}1 & \text{if } l=0\\ 0 & \text{if } l\neq 0\end{cases}$$

and the $b_{s,t,l}$ are the coefficients of the power series:

$$\frac{\psi(x)^r}{\tilde{\lambda}(x)^t}=\sum_{l=0}^{\infty} b_{s,t,l}x^n. \tag{37}$$

The computer cannot solve a system of infinitely many equations as in (36). We consider the approximate solution:

$$T_n(x)=\sum_{k=0}^{n} a_k x^k. \tag{38}$$

The a_k are the solutions to the system (36) of linear equations which is truncated at n. The approximation error for $Q(x)$ which solves Eq. (35) is

$$\text{error}\equiv\max_{|x|<\mu r}|Q(x)-T_n(x)|. \tag{39}$$

Here, $0<\mu<1$ is to be chosen later. Write

$$Q(x)=Q_n(x)+R_n(x), \tag{40}$$

where

$$Q_n(x)=\sum_{k=0}^{n}\frac{Q^{(k)}(0)}{k!}x^k \quad\text{and}\quad R_n(x)=\sum_{k=n+1}^{\infty}\frac{Q^{(k)}(0)}{k!}x^k.$$

Then

$$|Q(x)-T_n(x)|\leqslant|R_n(x)|+|Q_n(x)-T_n(x)|.$$

Next, we derive an estimate for $R_n(x)$. For this, we need the following result.

Lemma 3. *If* $r<\frac{1}{2}+\bar{S}\left(\frac{3}{2}\bar{S}-\sqrt{2}\sqrt{\bar{S}^2+1}\right)$, *then* $1-2r+\bar{S}^2-2\bar{S}\sqrt{1+2r}>0$.

Thanks to this lemma which is proved in Appendix H, we can give the following:

Definition 5. Set

$$B_r\equiv\max\left\{\|Q\|, e^{\gamma r}+K_3\|Q\|\sqrt{\frac{1+4r+\bar{S}^2-2\bar{S}\sqrt{1+2r}}{1-2r+\bar{S}^2-2\bar{S}\sqrt{1+2r}}}\right\}. \tag{41}$$

Theorem 3. *If* $r<\frac{1}{2}+\bar{S}\left(\frac{3}{2}\bar{S}-\sqrt{2}\sqrt{\bar{S}^2+1}\right)$, *then*

$$|R_n(x)|\leqslant\frac{B_r\mu^{n+1}}{1-\mu}\quad\text{for } |x|\leqslant\mu r. \tag{42}$$

We prove this theorem in Appendix I. As a result, we obtain an upper-bound for the error of the numerical solution $T_n(x)$:

$$\text{error}\leqslant\frac{B_r\mu^{n+1}}{1-\mu}+\max_{|x|<\mu r}|Q_n(x)-T_n(x)|. \tag{43}$$

6. Calibration and numerical solution of Campbell and Cochrane model

In this section, the asset pricing model of Campbell and Cochrane (1999) is calibrated and simulated using our numerical algorithm. An important part of this simulation is the procedure, which CC use to calibrate the parameters for obtaining a reasonable match with the data. They start with the following approximation for the Sharpe ratio:

$$SR(x) \equiv \frac{\mathrm{E}_t(R^e_{t+1}) - R_{t+1}}{\Sigma(R^e_{t+1} - R_{t+1})} \approx \gamma\sigma(1 + \lambda(s)), \tag{44}$$

where $\mathrm{E}_t(R^e_{t+1}) - R_{t+1}$ is the conditional expectation of the equity premium and $\Sigma(R^e_{t+1} - R_{t+1})$ is its standard deviation. They want the Sharpe ratio to be equal to the historic average 0.5 for the consumption claim model, and 0.45 for the dividend claim model. As a result, they evaluate the Sharpe ratio approximation (44) at the steady state $x = 0$, and find that

$$\gamma = \frac{SR(0)^2}{1 - \phi}. \tag{45}$$

Next, Campbell and Cochrane want to choose parameters so that the risk free interest rate is equal to its historic value 0.0094. They use the expression for the risk free interest rate given by

$$\ln(R_{t+1}) = -\ln(\beta) + \gamma g + \gamma(\phi - 1)x - \frac{\sigma^2\gamma^2}{2}(1 + \lambda(x))^2. \tag{46}$$

The parameter β is chosen so that this relation holds at the steady state value $x = 0$. As a result,

$$\beta = \exp\left\{\frac{SR(0)^2}{1 - \phi} g - \frac{SR(0)^2}{2} - \ln(R_{t+1})\right\}. \tag{47}$$

Using this procedure for choosing the parameters, we can check the condition for Proposition 1 for the existence of a solution to Campbell and Cochrane's asset pricing model in the space $C(\mathbb{R}, e^{\gamma x})$. Substituting (44) and (47) into (16) the definition of m_2, we have

$$m_2 = \exp\left\{g - \ln(R_{t+1}) + \frac{1}{2}(1 - \rho^2)\sigma_w^2 + \frac{\sigma^2}{2} K_2^2 - \sigma^2 K_2 \frac{SR(0)^2}{1 - \phi}\right\}. \tag{48}$$

All the terms in this equation are determined by the data except ϕ and $SR(0)$. In the consumption claim model Campbell and Cochrane simulate the model using a monthly time interval so that their parameters are $\ln(R_{t+1}) = 0.0094/12 = 0.00078$, $g = 0.0188/12 = 0.00157$, $\sigma = 0.0112/\sqrt{12} = 0.00323$, $\sigma_w = \sigma$, and $\rho = 1$.

In Fig. 1, we look at the possible values of m_2 for various values of $SR(0)$ and ϕ, given the other parameters chosen by Campbell and Cochrane. All points in the shaded area satisfy the existence condition from Proposition 1, $m_2 < 1$. As a result, only with the persistence of the surplus consumption ratio, ϕ, close to 1 is this existence condition satisfied in the Campbell and Cochrane model. In particular, for the Sharpe ratio found in the data, $SR(0) = 0.5/\sqrt{12} = 0.1443$, ϕ must be greater than 0.9998. However, for the Campbell and Cochrane parameter choices $SR(0) = 0.5/\sqrt{12} = 0.1443$, and $\phi = 0.87^{1/12} = 0.9885$, the condition of Proposition 1 is not satisfied. This point, labeled as *CC*, is highlighted in Fig. 1, which is outside the feasible region. Similar results are found for the dividend claim case. Thus, there is no known proof for the existence of a solution to the CC model with the CC parameters.

To see why the condition $m_2 < 1$ fails, we substitute into (48) the condition (45) for choosing γ under the consumption claim case and obtain

$$g - \ln(R_{t+1}) + \frac{\sigma^2}{2} - \sigma^2\gamma < 0, \tag{49}$$

since $K_2 = 1$. Consequently, the problem is that for the average, $g = 0.00157$, and standard deviation, $\sigma = 0.00323$, of consumption growth this condition is satisfied when $\gamma > 76.24$, which is even bigger than the coefficient of risk aversion in the Mehra and Prescott (1985) simulation for the equity

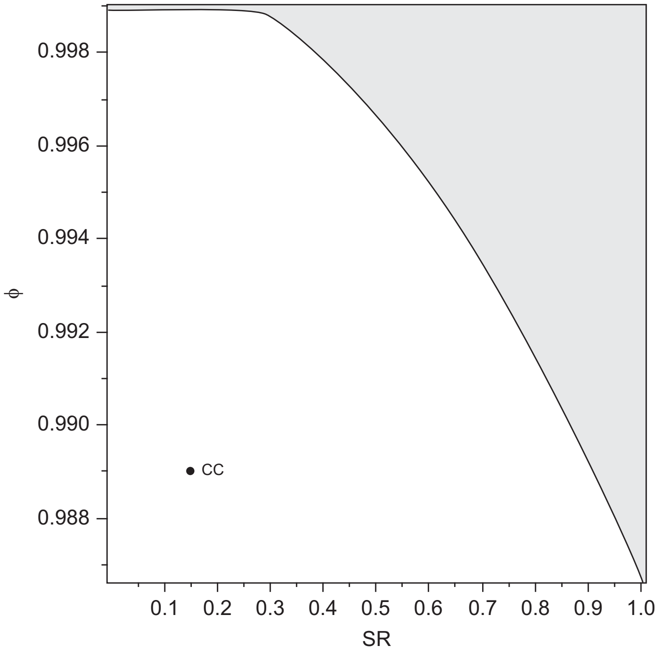

Fig. 1

premium puzzle.[13] The basic problem is that the growth rate of consumption is too big relative to the risk free interest rate and the low standard deviation of consumption growth magnifies this difference.

6.1. Calibration of the asset pricing equation

We now consider the truncated model (26) for the transformed price–dividend function $Q(x)$ in the Campbell and Cochrane model. For the original parameters in the Campbell and Cochrane paper, the upper bound on the solution for the price–dividend ratio is too low for the theoretical solution to match the historic price–dividend ratio, when dividend growth is restricted to reasonable levels.

In Theorem 1 it is shown that the supremum of $Q(x)$ is

$$\|Q\| \leqslant \frac{e^{\gamma r}}{1-K_3}, \tag{50}$$

where

$$K_3 \equiv K_0 e^{\sigma^2 (K_2-\gamma)^2/2} < 1.$$

If we use Campbell and Cochrane's conditions for choosing parameters (45) and (47) along with the definition of the constant K_0 in (12), then the supremum is bounded by

$$\|Q\| \leqslant \frac{e^{(SR(0)^2/(1-\phi))r}}{1-K_4 e^{-SR(0)^2/2} e^{(\sigma^2 (K_2 - SR(0)^2/(1-\phi)^2)/2)}}, \tag{51}$$

[13] We looked at different time period's using Shiller's data set for the U.S. and found similar results. The data comes from Shiller (1989), obtained from http://www.econ.yale.edu/~shiller/data.htm. The only data set which satisfies this condition is the one found in Mehra and Prescott (2003) which was for 1880–2003. In their data set the risk free interest rate is above consumption growth so that any $\gamma > 0$ would satisfy $m_2 < 1$.

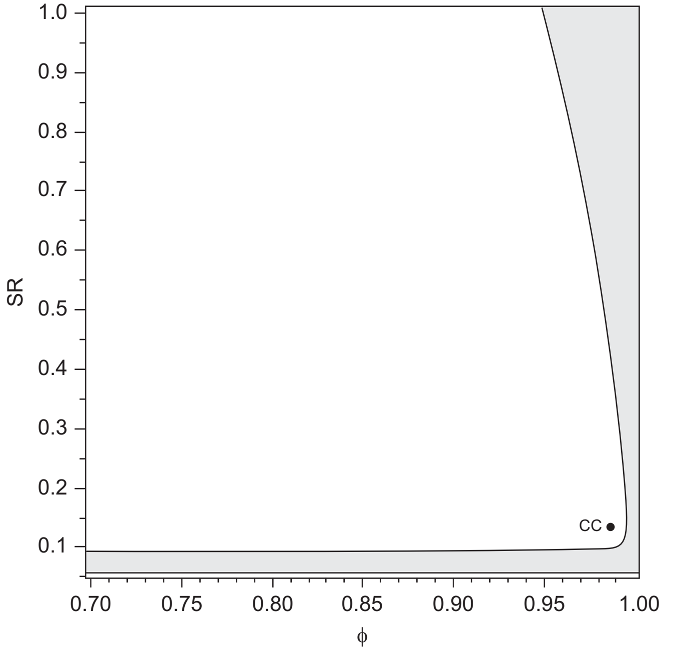

Fig. 2

where

$$K_4 \equiv \mathrm{e}^{-\ln(R_{t+1})} G \mathrm{e}^{(1-\rho^2)\sigma_w^2/2}.$$

We examine this upper bound (51) on the solution to the Campbell and Cochrane model in Fig. 2. Here an implicit plot of $\|Q\| = 357$, using (51), is provided for a Sharpe ratio, $SR(0) \in [0.0, 1.0]$, and the persistence of the surplus consumption ratio, $\phi \in [0.7, 1.0]$.[14] The other parameters are kept the same as in Fig. 1. We also need the value of the cutoff surplus consumption ratio r, which is chosen to be $0.25 < r_c$. As a result, the dividend growth is within the interval $[x_0 - 25\%, x_0 + 25\%]$ per month, where x_0 is its historic average value. In Fig. 2 the supremum of the price–dividend function is above the critical value in the shaded area. We also place the label CC for the combination of parameters used by Campbell and Cochrane which is outside this area. The critical value for the supremum is exceeded when ϕ is above about 0.99 for all values of the Sharpe ratio. Alternatively, the Sharpe ratio must be in the narrow band (at the bottom of Fig. 2) about $[0.05, 0.10]$ for any value of ϕ. Thus, an historically accurate price–dividend ratio is feasible in a limited parameter space which does not include the parameter values of Campbell and Cochrane.

It turns out that the supremum of $Q(x)$ under the Campbell and Cochrane parameters is only 187 when $SR(0) = 0.5/\sqrt{12} = 0.1443$ and $\phi = 0.882^{1/12} = 0.9895$. Therefore, the point CC in Fig. 2 is outside the feasible range. As a result, the price–dividend cannot be $18.3 \times 12 = 219.6$ when dividend growth is restricted to the interval $[x_0 - 25\%, x_0 + 25\%]$ per month. Even if the cutoff for dividend growth was increased to 0.45 the supremum of $Q(x)$ is still only 279, so that there is not enough variation in the price–dividend ratio.[15] Thus, the simulations presented in Campbell and Cochrane's paper cannot be a true representation of the price–dividend function in their model.

[14] The value of $\|Q\| = 357$ is chosen since it tends to yield a price–dividend ratio of $P(0)$, which is approximately $18.3 \times 12 = 219.6$. This level of the price–dividend is the value simulated by Campbell and Cochrane.

[15] This value of dividend growth is too close to the radius of convergence to obtain an accurate value of the true price–dividend function with our procedure.

Table 1
Approximation for consumption claim

Statistic	CC	CCH	Postwar sample
$\mathrm{E}(\ln[C_{t+1}/C_t])$	1.88[a]	1.88[a]	1.89
$\sigma(\ln[C_{t+1}/C_t])$	1.12[a]	1.12[a]	1.22
$\mathrm{E}(R_{t+1})$	0.094[a]	0.094[a]	0.094
$\frac{\mathrm{E}(R^e_{t+1}-R_{t+1})}{\sigma(R^e_{t+1}-R_{t+1})}$	0.2565[a]	0.1831[a]	0.50
$\mathrm{E}(R^e_{t+1}-R_{t+1})$	17.17	9.13	6.69
$\sigma(R^e_{t+1}-R_{t+1})$	66.93	49.85	15.7
$\exp[\mathrm{E}(\ln P)]$	6.83[a]	18.23[a]	24.7
$\sigma[\ln P]$	0.099	0.068	0.26
$Skewness(R^e_{t+1}-R_{t+1})$	2.227	3.246	−2.112
$Kurtosis(R^e_{t+1}-R_{t+1})$	3118.647	10132.765	27.167

Notes: R^e_{t+1} is the return to the stock and R_{t+1} is the return to the risk free bond. C_t is per capita consumption at time t and P is the price–dividend ratio. E represents the conditional expectation and σ is the conditional standard deviation. Campbell and Cochrane's Postwar sample comes from their Table 2. The Skewness and Kurtosis is calculated for the time period 1946–2003 with monthly data on the S & P 500 index and T-Bill rate which are taken from CRSP. The CC and CCH simulations are calibrated for monthly time periods with the parameters given by $\ln(R_{t+1}) = 0.0094/12 = 0.00078$, $g = 0.0188/12 = 0.00157$, $\sigma = 0.0112/\sqrt{12} = 0.00323$, $\sigma_w = \sigma$, and $\rho = 1$. In the CC simulation $\phi = 0.882^{1/12} = 0.9895$ and $SR(0) = 0.5/\sqrt{12} = 0.144$, so that $\gamma = 2$ and $\bar{S} = 0.0448$. In the CCH simulation all the parameters are the same except $\phi = 0.885^{1/12} = 0.9899$ and $SR(0) = 0.353/\sqrt{12} = 0.1019$, so that $\gamma = 1.025$ and $\bar{S} = 0.0325$. The simulations sets $0 \leqslant l, n \leqslant 15$. The expected values and standard deviation in the first column are found by using the price–dividend function and the risk free interest rate, converting the stock price to a return and integrating over the random shock to consumption growth given the surplus consumption variable, $\bar{S}$. The data was then annualized and expressed as a percentage when appropriate.

[a] Refers to statistics the model parameters were chosen to replicate.

As Fig. 2 demonstrates, there are limited feasible parameters, which will lead to a price–dividend function consistent with the data. To increase the supremum of $Q(x)$ to 357 while keeping the steady state Sharpe ratio at 0.1443, it is necessary to increase the persistence of the surplus consumption ratio to $\phi = 0.9469^{1/12} = 0.9955$ which corresponds to a coefficient of risk aversion of $\gamma = 4.59$. By moving the persistence of the surplus consumption ratio so close to one, the numerical scheme becomes ill conditioned, so that a more sophisticated method must be used to determine the Taylor polynomial approximation (38) for the analytic function.[16]

By reducing the Sharpe ratio to $SR(0) = 0.353/\sqrt{12} = 0.1019$, we can keep the persistence lower at $\phi = 0.885^{1/12} = 0.9899$, so that a solution can be accurately calculated. In this case the supremum of $Q(x)$ is 357 and the coefficient of relative risk aversion is $\gamma = 1.025$, so that the simulation below yields the correct price–dividend ratio but the Sharpe ratio is below its historic value.[17] Thus, we cannot match the equity premium and Sharpe ratio in the Campbell and Cochrane model.

6.2. *Simulation of consumption claim model*

With this understanding of the feasible parameter space for the Campbell and Cochrane model, we now provide some simulation results. These simulations were conducted using Maple on a standard PC. In the simulation, we choose $0 \leqslant l, n \leqslant 15$ in (36), since the computer program cannot evaluate an infinite number of coefficients. This choice of l and n leads to stable solutions for the price–dividend function in the cases considered below.

The parameters are set for a monthly time interval, while the results in Table 1 are reported on an annual basis. These parameters are given by $\ln(R_{t+1}) = 0.0094/12 = 0.00078$,

[16] While theoretically the number of coefficients could be raised sufficiently to find an accurate price–dividend function, rounding error in the computer program limits the number of coefficients to about 25 coefficients. To increase the number of coefficients in the Taylor polynomial approximation (38) above 20 a Fortran program to implement the procedure was developed. However, the rounding error becomes too big to go beyond 25 coefficients.

[17] We find similar results for the dividend claim model.

$g = 0.0188/12 = 0.00157$, $\sigma = 0.0112/\sqrt{12} = 0.00323$, $\sigma_w = \sigma$, and $\rho = 1$. To get the Campbell and Cochrane parameters we set $\phi = 0.882^{1/12} = 0.9895$ and $SR(0) = 0.5/\sqrt{12} = 0.144$, so that $\gamma = 2$ and $\bar{S} = 0.0448$.[18] To implement the polynomial approximation the restriction of $Q(x)$ in (26) must be chosen. In this case the restriction on dividend growth is $r = 0.25 < r_c = 0.4998$, so that the range of dividend growth is $[x_0 - 25\%, x_0 + 25\%]$ per month. In Table 1 column 2, the price–dividend at the steady state is only $P(0) = 6.83$ for the Campbell and Cochrane's case which is 63% below the value, $P(0) = 18.3$, Campbell and Cochrane were trying to match. As a result, the equity premium is 17.17 which is 2.5 times bigger than the historic average. Finally, the standard deviation of stock returns is 66.93%, which is 4.5 times above the historic average. The Sharpe ratio is only 0.25, which is half of the value targeted by the rule (44). Thus, there is substantial deviation of the simulations from what Campbell and Cochrane reported when dividend growth is restricted to $\pm 25\%$ per month.

For Campbell and Cochrane's parameters the supremum of $Q(x)$ is 187, so that the failure of the Campbell and Cochrane is not surprising when $r = 0.25$. One way to increase the supremum of $Q(x)$ is to raise the persistence of the surplus consumption ratio, however, the polynomial approximation is not very stable when $0 \leqslant l,\ n \leqslant 15$ in (36). We found that by lowering the Sharpe ratio to $SR(0) = 0.353/\sqrt{12} = 0.1019$, we could hold the persistence of the surplus consumption ratio to the value of $\phi = 0.885^{1/12} = 0.9899$, yet still keep an accurate simulation of the model. In this case the supremum of $Q(x)$ is now 357. We kept the other parameters the same, however $\gamma = 1.025$ and $\bar{S} = 0.033$. In this case, we get $P(0)$ is approximately $18.3 \times 12 = 219.6$. Consequently, we match the price–dividend ratio. In Table 1 column 3, we see that the standard deviation of this price–dividend ratio is high compared to the post war sample of Campbell and Cochrane. The equity premium has declined to 9.13% relative to the Campbell and Cochrane case, however it is still 2.4% above its historic value. In addition, the standard deviation has fallen to 49.8% but it is still three times bigger than its historic value. Thus, the Sharpe ratio turns out to be too low for this simulation.

This leaves open the possibility of increasing the persistence of the surplus consumption ratio to $\phi = 0.9469^{1/12} = 0.9955$, which corresponds to a coefficient of risk aversion of $\gamma = 4.59$. In this case, the supremum of $Q(x)$ is 357, so that one has a chance of matching $P(0) = 219.6$. The current computer programs in Maple and Fortran cannot handle this case, since the problem becomes ill conditioned with high ϕ. There are several possible ways to explore this possibility. First, use a programming language which allows for higher precision and speedier calculations. This would allow for the increase of the number of coefficients in (36). In addition, the higher precision would lead to an ability to multiply the large Taylor polynomial coefficients with the small x^k in the Taylor polynomial approximation (38). The alternative would be to use more efficient polynomial representations as discussed in Judd (1992, 1996, 1998). These approaches would fit into our numerical scheme and allow for more accurate solutions to Campbell and Cochrane's model, however the numerical procedures should be guided by the mathematical understanding of the Campbell and Cochrane model developed here.

6.3. *Understanding the simulations*

The inability of our numerical solution to match the price–dividend ratio $18.3 \times 12 = 219.6$ found in Campbell and Cochrane begs the question about how their numerical solution yields this answer. It turns out that their numerical solution is highly dependent on excessively large negative values for dividend growth. One can see this property by increasing r to 45% dividend growth per month which increases the supremum of $Q(x)$ to 279. Of particular importance is the lowest value of dividend growth.[19] Such a low level of the surplus consumption ratio is inconsistent with any plausible consumption growth. In particular, the data reported by Barro (2006) demonstrates that the worse

[18] There is a small difference between Cambell and Cochrane's parameters and these values which we could not account for given their description of how the parameters are set. This difference does not material effect our conclusions.

[19] This result is consistent with the simulations of Wachter (2005, Fig. 2) in which a substantial difference between hers and CC's simulation procedure exists when $-x$ is a high value. We would argue that the two procedures report the same price–dividend ratios when $-x$ is a high value since they have effectively imposed a higher left extension when they choose their grid points. Wachter provides a detailed description of her and CC's procedure.

downturns in the 20th century was −64% over a few years for Germany and Greece during World War II. Therefore, the lower bound on the surplus consumption ratio needed to reproduce the results of Campbell and Cochrane is too low relative to historic observations.

The dependence of Campbell and Cochrane's numerical solution on large negative values for consumption growth and the substantial error in their numerical simulations can be traced to the non-linear sensitivity function and the log-normal distribution of shocks to dividend growth. This problem arises because the sensitivity function increases to infinity as x declines. This sensitivity multiplies the standard deviation of $Q(x)$ in Eq. (24). Thus, the volatility of the surplus consumption ratio tends to infinity as the current surplus consumption ratio tends to zero.[20] Consequently, the integral equation (24) for $Q(x)$ is dependent on its behavior over the entire support of the probability distribution for dividend growth. This leads to a significant impact of low values of the surplus consumption ratio, even though $Q(x)$ is small, since the measure placed on $Q(x)$ in this range can be quite large. To obtain the results of Campbell and Cochrane with their parameters, the dividend growth must be reduced below $x_0 - 25\%$ per month. As a result, low values for the surplus consumption ratio, resulting from the tail of the distribution for dividend growth, magnifies the volatility of consumption growth enough to significantly alter the expected value of the price–dividend ratio. Our procedure is not dependent on these extreme negative values since the $Q(x)$ is kept constant for dividend growth below $x_0 - 25\%$ per month. Thus, the parameters, or the procedure for choosing the parameters, for the Campbell and Cochrane model must be altered in order to adequately represent the behavior of the equity premium and Sharpe ratio.

7. Conclusion

This paper develops a systematic procedure for accurately solving discrete time asset pricing models. Identifying the analytic properties of the integral equation is the key for developing a numerical algorithm which yields an accurate solution. The Campbell and Cochrane (1999) asset pricing model is used for illustrating the method. Below we outline the main steps of this method, which can also be applied to a broader collection of asset pricing models.

(1) Write the integral equation as a mapping of the unknown solution to the model and identify the space in which the solution is expected to live. In the CC model it was found that the solution is in the space of functions that grow exponentially (see Theorem Definition 2).
(2) Identify the conditions under which this mapping converges to a unique solution and make sure that the chosen parameters satisfy these conditions (see Proposition 1). In the CC model these conditions are not satisfied. By restricting the range of the price–dividend function for the extreme values of dividend growth it is shown that the solution converges with a uniform bound on the error (see Theorem 1).
(3) In the integral defining the mapping for the price–dividend function (28), replace the independent real variable with a complex variable (complexify). Then, take the complex derivative of this integral (29). Usually, the pricing kernel is analytic so that its complex derivative is zero. As a result, the complex derivative of the integral is also zero when that integral exist.
(4) Next, find the largest domain of the complex variable where these integrals exist. It follows immediately that the solution to the integral equation is analytic within this domain. The radius of convergence is the minimum distance from a particular point to the boundary of this domain. In the CC model this radius of convergence is larger than 49% dividend growth per month (see Theorem 2).
(5) Having developed the mathematical properties of the solution, the numerical procedure follows from these properties. Since the solution is known to be analytic within a well-defined region, it

[20] Samuelson (1970) first recognized this issue. Geweke (2001) finds a similar problem with asset pricing models based on CRRA utility, a lognormal distribution for consumption growth and Bayesian updating. See also Jin and Judd (2002) for a discussion of this issue when using the perturbation method.

is natural to express the solution as a polynomial approximation (38) within this region. Substitute this expression into the integral equation and solve for the linear system of equations (36) that yields the coefficients of the polynomial approximation.

(6) Find a uniform bound for the solution to the price–dividend function on a circle inside the domain and centered at a particular point with radius less than the radius of convergence for the solution. Using the Cauchy integral formula calculate a bound on the derivatives of the solution at the center of this circle for all orders. Thus, a bound on the Taylor series remainder can be estimated for all values of the independent variable within a certain fraction of the radius of convergence (see Theorem (3)). Consequently, the polynomial approximation can be used to accurately represent the solution, as long as the radius of convergence covers the values of interest.

This is a general procedure which can be applied to most asset pricing models, since researchers generally choose pricing kernels, dividend growth processes, and probability distributions which are analytic. In the appendix all the steps of this method are provided for the case of the CC model. The techniques developed here can also be extended to include several variables. Using them it is feasible to accurately solve higher dimensional asset pricing models such as the Wachter model (2002).

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Appendix A. Derivation of Eq. (14)

Rewrite Eq. (11) in the form:

$$P(x) = \frac{K_0 \mathrm{e}^{K_1 x}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \mathrm{e}^{-(v^2/2\sigma^2)+(K_2-\gamma(1+\lambda(x)))v}(1 + P(\phi x + \lambda(x)v))\,\mathrm{d}v.$$

Complete the square for

$$\begin{aligned} -\frac{v^2}{2\sigma^2} + (K_2 - \gamma(1+\lambda(x)))v = & -\frac{1}{2\sigma^2}\{v - \sigma^2(K_2 - \gamma(1+\lambda(x)))\}^2 \\ & + \frac{\sigma^2}{2}(K_2 - \gamma(1+\lambda(x)))^2. \end{aligned}$$

Substituting this equation into the integral equation (11) yields

$$\begin{aligned} P(x) = & \frac{K_0 \mathrm{e}^{K_1 x + (\sigma^2/2)(K_2-\gamma(1+\lambda(x)))^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \mathrm{e}^{-(1/(2\sigma^2))\{v-\sigma^2(K_2-\gamma(1+\lambda(x)))\}^2} \\ & \times (1 + P(\phi x + \lambda(x)v))\,\mathrm{d}v. \end{aligned}$$

Set $M(x) = K_0 \mathrm{e}^{K_1 x + (\sigma^2/2)(K_2-\gamma(1+\lambda(x)))^2}$. Since

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \mathrm{e}^{-(1/(2\sigma^2))\{v-\sigma^2(K_2-\gamma(1+\lambda(x)))\}^2}\,\mathrm{d}v = 1,$$

we get

$$P(x) = M(x) + \frac{M(x)}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \mathrm{e}^{-(1/(2\sigma^2))\{v-\sigma^2(K_2-\gamma(1+\lambda(x)))\}^2} P(\phi x + \lambda(x)v)\,\mathrm{d}v,$$

which corresponds to Eq. (14).

Appendix B. Proof of Lemma 1

Let $f(x) \in C(\mathbb{R}, e^{\gamma x})$. We can find $m_1' \geqslant 0$ and $m_2' \geqslant 0$ such that $|f(x)| \leqslant m_1' e^{\gamma x} + m_2'$ for all x. Recall that $0 < M(x) = K_0 e^{K_1 x + (\sigma^2/2)(K_2 - \gamma(1+\lambda(x)))^2} \leqslant m_1 e^{\gamma x} + m_2$ and

$$(Tf)(x) = \frac{M(x)}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-(1/(2\sigma^2))\{v - \sigma^2(K_2 - \gamma(1+\lambda(x)))\}^2} f(\phi x + \lambda(x) v)\, dv.$$

We have

$$\begin{aligned} |(Tf)(x)| &\leqslant \frac{M(x)}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-(1/(2\sigma^2))\{v - \sigma^2(K_2 - \gamma(1+\lambda(x)))\}^2} (m_1' e^{\gamma\phi x + \gamma\lambda(x) v} + m_2')\, dv \\ &\leqslant m_1' e^{\gamma\phi x + (\sigma^2/2)\,(2\gamma(K_2-\gamma)\lambda(x) - \gamma^2\lambda(x)^2)} M(x) + m_2' M(x). \end{aligned}$$

Since $e^{\phi\gamma x + (\sigma^2/2)(2\gamma(K_2-\gamma)\lambda(x) - \gamma^2\lambda(x)^2)} M(x) = K_0 e^{\sigma^2(K_2-\gamma)^2/2} e^{\gamma x}$, we get

$$|(Tf)(x)| \leqslant (m_1' K_0 e^{\sigma^2(K_2-\gamma)^2/2} + m_1 m_2') e^{\gamma x} + m_2' m_2. \tag{52}$$

Next, we prove that $(Tf)(x)$ is continuous at every $x_0 \in \mathbb{R}$. Set

$$\begin{aligned} g(x) &\equiv \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-(1/(2\sigma^2))\{v - \sigma^2(K_2 - \gamma(1+\lambda(x)))\}^2} f(\phi x + \lambda(x) v)\, dv \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\sigma\lambda(x) v + \zeta(x))\, e^{-v^2/2}\, dv, \end{aligned}$$

where

$$\zeta(x) \equiv \phi x + \sigma^2 \lambda(x)(K_2 - \gamma(1+\lambda(x))).$$

Since both $\lambda(x)$ and $M(x)$ are continuous at x_0, it suffices to show that $g(x)$ is continuous at x_0. For any $A > 0$, we have

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-A} |f(\sigma\lambda(x) v + \zeta(x))| e^{-v^2/2}\, dv \\ &\leqslant \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-A} (m_1' e^{\gamma(\sigma\lambda(x) v + \zeta(x))} + m_2') e^{-v^2/2}\, dv \\ &= \frac{m_1' e^{\gamma\zeta(x) + \gamma^2\sigma^2\lambda(x)^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{-A} e^{-(v - \gamma\sigma\lambda(x))^2/2}\, dv + \frac{m_2'}{\sqrt{2\pi}} \int_{-\infty}^{-A} e^{-v^2/2}\, dv \\ &= \frac{m_1' e^{\gamma\zeta(x) + \gamma^2\sigma^2\lambda(x)^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{-A - \gamma\sigma\lambda(x)} e^{-v^2/2}\, dv + \frac{m_2'}{\sqrt{2\pi}} \int_{-\infty}^{-A} e^{-v^2/2}\, dv. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int_{A}^{\infty} |f(\sigma\lambda(x) v + \zeta(x))| e^{-v^2/2}\, dv \\ &\leqslant \frac{m_1' e^{\gamma\zeta(x) + \gamma^2\sigma^2\lambda(x)^2/2}}{\sqrt{2\pi}} \int_{A - \gamma\sigma\lambda(x)}^{\infty} e^{-v^2/2}\, dv + \frac{m_2'}{\sqrt{2\pi}} \int_{A}^{\infty} e^{-v^2/2}\, dv. \end{aligned}$$

Let $\varepsilon > 0$. Since both $\lambda(x)$ and $\zeta(x)$ are bounded on the interval $[x_0 - 1, x_0 + 1]$ and $(1/\sqrt{2\pi}) \int_{-\infty}^{\infty} e^{-v^2/2}\, dv = 1$, we can find $A > 0$ such that for all $x \in [x_0 - 1, x_0 + 1]$,

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{-A} + \int_{A}^{\infty} \right) |f(\sigma\lambda(x) v + \zeta(x)) - f(\sigma\lambda(x_0) v + \zeta(x_0))| e^{-v^2/2}\, dv \\ &\leqslant \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{-A} + \int_{A}^{\infty} \right) [|f(\sigma\lambda(x) v + \zeta(x))| + |f(\sigma\lambda(x_0) v + \zeta(x_0))|] e^{-v^2/2}\, dv < \frac{\varepsilon}{2}. \end{aligned}$$

Since $f(\sigma\lambda(x) v + \zeta(x))$ is a continuous function of the variables v and x on the rectangle $[-A, A] \times [x_0 - 1, x_0 + 1]$, there exists $0 < \delta < 1$ such that for all $x_0 - \delta < x < x_0 + \delta$ and $-A \leqslant v \leqslant A$, we have $|f(\sigma\lambda(x)v+\zeta(x))-f(\sigma\lambda(x_0)v+\zeta(x_0))|<\frac{\varepsilon}{2}$, and therefore

$$\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}|f(\sigma\lambda(x)v+\zeta(x))-f(\sigma\lambda(x_0)x+\zeta(x_0))|\mathrm{e}^{-v^2/2}\,\mathrm{d}v<\frac{\varepsilon}{2}.$$

This proves that $|g(x)-g(x_0)|<\varepsilon$ for $x_0-\delta<x<x_0+\delta$. Hence $(Tf)(x)\in C(\mathbb{R},\mathrm{e}^{\gamma x})$.

$T:C(\mathbb{R},\mathrm{e}^{\gamma x})\to C(\mathbb{R},\mathrm{e}^{\gamma x})$ is a well-defined linear transformation.

Appendix C. Proof of Proposition 1

By (17), the price–dividend function $P(x)$ in the Campbell and Cochrane model satisfies

$$P(x)=M(x)+(TP)(x). \tag{53}$$

Set $K_3=K_0\mathrm{e}^{\sigma^2(K_2-\gamma)^2/2}$.

Lemma 4. *If* $0<\bar{S}<1$, $0<m_2<1$, *and* $f(x)\in C(\mathbb{R},\mathrm{e}^{\gamma x})$ *satisfies* $(Tf)(x)=f(x)$, *then* $f(x)=0$ *for all* $x\in\mathbb{R}$.

Proof. Find $m_1'\geqslant 0$ and $m_2'\geqslant 0$ such that $|f(x)|\leqslant m_1'\mathrm{e}^{\gamma x}+m_2'$ for all x. *By* (52), we get

$$|(Tf)(x)|\leqslant(m_1'K_3+m_1m_2')\mathrm{e}^{\gamma x}+m_2'm_2.$$

By mathematical induction, we can show that for all $n=1,2,3,\ldots$, we have

$$|f(x)|=|(T^nf)(x)|\leqslant\left(m_1'K_3^n+m_1m_2'\sum_{i=0}^{n-1}K_3^im_2^{n-1-i}\right)\mathrm{e}^{\gamma x}+m_2'm_2^n. \tag{54}$$

Since $K_3/m_2=\mathrm{e}^{(\sigma^2/2)(K_2-\gamma)^2-(\sigma^2/2)(K_2^2+(\gamma^2/\bar{S}^2))+\sigma^2\gamma K_2}=\mathrm{e}^{-\gamma^2\sigma^2(1-\bar{S}^2)/(2\bar{S}^2)}<1$, we get

$$\sum_{i=0}^{n-1}K_3^im_2^{n-1-i}=m_2^{n-1}\sum_{i=0}^{n-1}(K_3/m_2)^i\leqslant\frac{m_2^{n-1}}{1-K_3/m_2}=\frac{m_2^n}{m_2-K_3}$$

and

$$|f(x)|\leqslant\{[m_1'(K_3/m_2)^n+m_1m_2'/(m_2-K_3)]\mathrm{e}^{\gamma x}+m_2'\}m_2^n.$$

Since $\lim_{n\to\infty}m_2^n=0$, we must have $f(x)=0$ for all x. $\square$

Now we can prove Proposition 1. Define the functions $P_n(x)\in C(\mathbb{R},\mathrm{e}^{\gamma x})$ by setting

$$P_0(x)\equiv 0 \quad\text{and}\quad P_{n+1}(x)\equiv M(x)+(TP_n)(x).$$

Then $P_1(x)=M(x)+(TP_0)(x)=M(x)$ and $0\leqslant P_1(x)-P_0(x)=M(x)\leqslant m_1\mathrm{e}^{\gamma x}+m_2$. Since $0\leqslant P_{n+1}(x)-P_n(x)=(T^n(P_1-P_0))(x)$, (54) yields

$$0\leqslant P_{n+1}(x)-P_n(x)\leqslant m_1\left(\sum_{i=0}^{n}K_3^im_2^{n-i}\right)\mathrm{e}^{\gamma x}+m_2^{n+1}\leqslant\left(\frac{m_1\mathrm{e}^{\gamma x}}{m_2-K_3}+1\right)m_2^{n+1}.$$

Set $P(x)=\sum_{n=0}^{\infty}[P_{n+1}(x)-P_n(x)]=\lim_{n\to\infty}P_{n+1}(x)$. Then

$$0\leqslant P(x)\leqslant\left(\frac{m_1\mathrm{e}^{\gamma x}}{m_2-K_3}+1\right)\sum_{n=0}^{\infty}m_2^{n+1}=\left(\frac{m_1\mathrm{e}^{\gamma x}}{m_2-K_3}+1\right)\frac{m_2}{1-m_2}.$$

On the other hand, since

$$0\leqslant P(x)-P_n(x)=\sum_{k=n}^{\infty}(P_{k+1}(x)-P_k(x))\leqslant\left(\frac{m_1\mathrm{e}^{\gamma x}}{m_2-K_3}+1\right)\frac{m_2^{n+1}}{1-m_2}, \tag{55}$$

$\sum_{n=0}^{\infty}[P_{n+1}(x)-P_n(x)]$ uniformly converges to $P(x)$ on any finite closed interval. So $P(x)$ is continuous; that is, $P(x)\in C(\mathbb{R},\mathrm{e}^{\gamma x})$.

Next, by (52) and (55) we get

$$0 \leqslant (TP)(x) - (TP_n)(x) = (T(P-P_n))(x) \leqslant \frac{m_2^{n+1}}{1-m_2}\left[\left(\frac{m_1K_3}{m_2-K_3}+m_1\right)\mathrm{e}^{\gamma x}+m_2\right]$$

and

$$P(x) = \lim_{n\to\infty} P_{n+1}(x) = M(x) + \lim_{n\to\infty}(TP_n)(x) = M(x) + (TP)(x).$$

Thus, $P(x)$ is a solution of the Eq. (53) in $C(\mathbb{R}, \mathrm{e}^{\gamma x})$.

Finally, we prove the uniqueness of the solution. Suppose that $\tilde{P}(x)$ is another solution of (53) in $C(\mathbb{R}, \mathrm{e}^{\gamma x})$. Then $P(x) - \tilde{P}(x) \in C(\mathbb{R}, \mathrm{e}^{\gamma x})$ and $(T(P-\tilde{P}))(x) = (P-\tilde{P})(x)$. By Lemma 4, we obtain $P(x) = \tilde{P}(x)$ for all x.

Appendix D. Derivation of Eq. (26) from (11)

For $-r \leqslant x \leqslant r$, $Q(x) = (1+P(x))\mathrm{e}^{-\gamma x}$ satisfies

$$\begin{aligned}
Q(x) &= \mathrm{e}^{-\gamma x} + \mathrm{e}^{-\gamma x}P(x) \\
&= \mathrm{e}^{-\gamma x} + \frac{K_0\mathrm{e}^{K_1x-\gamma x}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(1+P(\phi x+\lambda(x)v))\mathrm{e}^{(K_2-\gamma(1+\lambda(x)))v}\mathrm{e}^{-v^2/(2\sigma^2)}\,\mathrm{d}v \\
&= \mathrm{e}^{-\gamma x} + \frac{K_0\mathrm{e}^{(K_1-\gamma)x}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}Q(\phi x+\lambda(x)v)\mathrm{e}^{\gamma(\phi x+\lambda(x)v)}\mathrm{e}^{(K_2-\gamma(1+\lambda(x)))v}\mathrm{e}^{-v^2/(2\sigma^2)}\,\mathrm{d}v \\
&= \mathrm{e}^{-\gamma x} + \frac{K_0\mathrm{e}^{-\phi\gamma x}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}Q(\phi x+\lambda(x)v)\mathrm{e}^{\phi\gamma x+K_2v-\gamma v}\mathrm{e}^{-v^2/(2\sigma^2)}\,\mathrm{d}v \\
&= \mathrm{e}^{-\gamma x} + \frac{K_0}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}Q(\phi x+\lambda(x)v)\mathrm{e}^{(K_2-\gamma)v}\mathrm{e}^{-v^2/(2\sigma^2)}\,\mathrm{d}v.
\end{aligned}$$

Complete the square for

$$(K_2-\gamma)v - \frac{v^2}{2\sigma^2} = -\frac{1}{2\sigma^2}(v-\sigma^2(K_2-\gamma))^2 + \frac{\sigma^2(K_2-\gamma)^2}{2}.$$

Set $K_3 = K_0\mathrm{e}^{\sigma^2(K_2-\gamma)^2/2}$. Then

$$\begin{aligned}
Q(x) &= \mathrm{e}^{-\gamma x} + \frac{K_3}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}Q(\phi x+\lambda(x)v)\mathrm{e}^{-(1/(2\sigma^2))(v-\sigma^2(K_2-\gamma))^2}\,\mathrm{d}v \\
&= \mathrm{e}^{-\gamma x} + \frac{K_3}{\sqrt{2\pi}\sigma\lambda(x)}\int_{-\infty}^{\infty}Q(y)\mathrm{e}^{-(1/(2\sigma^2\lambda(x)^2))(y-\phi x-\sigma^2(K_2-\gamma)\lambda(x))^2}\,\mathrm{d}y \\
&= \mathrm{e}^{-\gamma x} + \frac{K_3}{\sqrt{2\pi}\tilde{\lambda}(x)}\int_{-\infty}^{\infty}Q(y)\mathrm{e}^{-(1/(2(\tilde{\lambda}(x))^2))(y-\psi(x))^2}\,\mathrm{d}y,
\end{aligned}$$

where $\psi(x) = \phi x + \sigma^2(K_2-\gamma)\lambda(x)$ and $\tilde{\lambda}(x) = \sigma\lambda(x)$, which corresponds to (26).

Appendix E. Proof of Theorem 1

Define the functions $Q_n(x)$ by setting $Q_0(x) \equiv 0$ for all x and

$$Q_{n+1}(x) \equiv \begin{cases} \mathrm{e}^{-\gamma x} + \dfrac{K_3}{\sqrt{2\pi}\sigma\lambda(x)}\displaystyle\int_{-\infty}^{\infty}Q_n(y)\mathrm{e}^{-(1/(2(\tilde{\lambda}(x))^2))(y-\psi(x))^2}\,\mathrm{d}y & \text{if } -r \leqslant x \leqslant r, \\ Q_{n+1}(r) & \text{if } x > r, \\ Q_{n+1}(-r) & \text{if } x < -r. \end{cases}$$

Then $\|Q_1 - Q_0\| = \|Q_1\| = \mathrm{e}^{\gamma r}$. By mathematical induction, we can show that

$$\|Q_{n+1} - Q_n\| \leqslant K_3^n\mathrm{e}^{\gamma r} \quad \text{for } n = 0, 1, 2, \ldots.$$

Set $Q(x) = \sum_{n=0}^{\infty}[Q_{n+1}(x) - Q_n(x)] = \lim_{n\to\infty} Q_n(x)$. Since $K_3 < 1$, we get

$$\|Q - Q_n\| \leqslant \sum_{k=n}^{\infty} \|Q_{k+1} - Q_k\| \leqslant \sum_{k=n}^{\infty} K_3^k e^{\gamma r} = \frac{e^{\gamma r} K_3^n}{1 - K_3}. \tag{56}$$

$\sum_{n=0}^{\infty}[Q_{n+1}(x) - Q_n(x)]$ uniformly converges to the continuous function $Q(x)$. By (56),

$$\|Q\| = \|Q - Q_0\| \leqslant \frac{e^{\gamma r}}{1 - K_3}.$$

Next, for $-r \leqslant x \leqslant r$, we have

$$\begin{aligned} Q(x) = \lim_{n\to\infty} Q_{n+1}(x) &= e^{-\gamma x} + \frac{K_3}{\sqrt{2\pi}\tilde{\lambda}(x)} \int_{-\infty}^{\infty} \lim_{n\to\infty} Q_n(y) e^{-(1/(2(\tilde{\lambda}(x))^2))(y-\psi(x))^2}\, dy \\ &= e^{-\gamma x} + \frac{K_3}{\sqrt{2\pi}\,\tilde{\lambda}(x)} \int_{-\infty}^{\infty} Q(y) e^{-(1/(2(\tilde{\lambda}(x))^2))(y-\psi(x))^2}\, dy. \end{aligned}$$

So $Q(x)$ is a solution of the Eq. (26).

Finally, we prove the uniqueness of the solution. Suppose that $\tilde{Q}(x)$ is another continuous and bounded solution of (26). Then for $-r \leqslant x \leqslant r$,

$$Q(x) - \tilde{Q}(x) = \frac{K_3}{\sqrt{2\pi}\tilde{\lambda}(x)} \int_{-\infty}^{\infty} [Q(y) - \tilde{Q}(y)] e^{-(1/(2(\tilde{\lambda}(x))^2))(y-\psi(x))^2}\, dy,$$

and

$$\|Q - \tilde{Q}\| \leqslant K_3 \|Q - \tilde{Q}\|. \tag{57}$$

Since $K_3 < 1$, (57) forces $Q(x) = \tilde{Q}(x)$ for all x.

Appendix F. Proof of Lemma 2

We complexify $\lambda(x)$ by letting $w = u + iv$. The domain of holomorphicity of $\lambda(w)$ is the complex plane $\mathbb{C}$ with the ray $\{x + iy : x \geqslant \frac{1}{2}\}$ deleted, which removes the points where $1 - 2w$ is non-positive. To avoid the zero of $\lambda(w)$ so that $1/\lambda(w)$ in (28) exists, we restrict the domain of holomorphicity to the complex plane $\mathbb{C}$ with the ray $\{x + iy : x \geqslant x^*\}$ deleted, where $x^* = (1 - \bar{S}^2)/2$. Write

$$\lambda(w)^2 = \frac{1}{\bar{S}^2}(1 - 2w - 2\bar{S}\sqrt{1 - 2w} + \bar{S}^2) = a(w) + ib(w),$$

where $a(w) = \mathrm{Re}\ \lambda(w)^2$ and $b(w) = \mathrm{Im}\ \lambda(w)^2$. Then

$$\frac{1}{\lambda^2} = \frac{1}{a + ib} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}.$$

Complexifying the function $\psi(x) = \phi x + \sigma^2(K_2 - \gamma)\lambda(x)$ and writing $\psi(w) = c(w) + id(w)$, where $c(w) = \mathrm{Re}\psi(w)$ and $d(w) = \mathrm{Im}\psi(w)$, gives

$$(t - \psi(w))^2 = (t - c - id)^2 = (t - c)^2 - 2i(t - c)d - d^2,$$

and

$$\mathrm{Re}\frac{(t-\psi)^2}{\lambda^2} = \frac{a(t-c)^2}{a^2+b^2} - \frac{2bd(t-c)}{a^2+b^2} - \frac{ad^2}{a^2+b^2} = \frac{a}{a^2+b^2}\left((t-c)^2 - \frac{2bd(t-c)}{a}\right) - \frac{ad^2}{a^2+b^2}.$$

Completing the square, we obtain

$$\mathrm{Re}\frac{(t-\psi)^2}{\lambda^2} = \frac{a}{a^2+b^2}\left(t - \frac{ac+bd}{a}\right)^2 - \frac{d^2}{a}.$$

Using the last formula, we see that for the integral defining $f(u + iv)$ to make sense and to be able to differentiate under the integral sign, using the Dominated Convergence Theorem, we must restrict

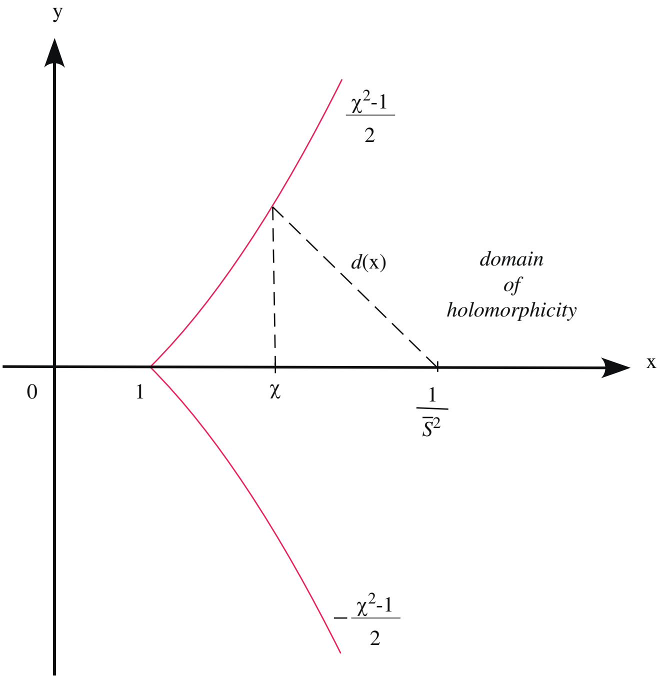

Fig. 3

$u+iv$ in a region Ω where $a>0$. Thus, we must determine when

$$a(w)=\mathrm{Re}\left(\sqrt{\frac{1-2w}{\bar{S}^2}}-1\right)^2>0, \quad w\in\Omega.$$

To simplify the calculations, we use the transformation $z=x+iy=(1-2w)/\bar{S}^2$. Then $w=x^*$ is mapped to $z=1$, and $\lambda(w)$ is transformed into $\lambda(z)=\sqrt{z}-1$, which is holomorphic and not zero for $z\in\mathbb{C}\backslash\{x+iy:z<1\}$. Using polar coordinates $z=re^{i\theta}$, we write

$$\begin{aligned}\lambda(z)^2 &= z-2\sqrt{z}+1=re^{i\theta}-2\sqrt{r}e^{i\theta/2}+1\\ &= r(\cos\theta+i\sin\theta)-2\sqrt{r}[\cos(\theta/2)+i\sin(\theta/2)]+1.\end{aligned}$$

Thus,

$$a(r,\theta)=\mathrm{Re}\,\lambda(r,\theta)^2=r\cos\theta-2\sqrt{r}\cos(\theta/2)+1.$$

Next, we shall prove the following:

Claim. *$a(z)>0$ for all $z\in D$, where*

$$D=\left\{x+iy:-\frac{x^2-1}{2}<y<\frac{x^2-1}{2},x>1\right\}.$$

This domain is displayed in Fig. 3.

Since $a(r,-\theta)=a(r,\theta)$, it suffices to consider the case where $\theta>0$. Solving the quadratic equation $r\cos\theta-2\sqrt{r}\cos(\theta/2)+1=0$ for $\sqrt{r}$, we find

$$\begin{aligned}\sqrt{r} &= \frac{\cos(\theta/2)\pm\sqrt{\cos^2(\theta/2)-\cos\theta}}{\cos\theta}=\frac{\cos(\theta/2)\pm\sin(\theta/2)}{\cos\theta}\\ &= \frac{\cos(\theta/2)\pm\sin(\theta/2)}{[\cos(\theta/2)+\sin(\theta/2)][\cos(\theta/2)-\sin(\theta/2)]}=\frac{1}{\cos(\theta/2)\mp\sin(\theta/2)}.\end{aligned}$$

Therefore,

$$r = \frac{1}{\cos^2(\theta/2) + \sin^2(\theta/2) \pm 2\cos(\theta/2)\sin(\theta/2)} = \frac{1}{1 \pm \sin\theta},$$

which gives $r(1 \pm \sin\theta) = 1$. Since $r > 1$ and $\sin\theta > 0$, we must have $r(1 - \sin\theta) = 1$, or $r = r\sin\theta + 1$, or $\sqrt{x^2 + y^2} = y + 1$, or

$$y = \frac{x^2 - 1}{2}, \quad x > 1,$$

which proves that D is the domain of positivity of the function a.

Finally, we find the radius of convergence of the power series of $f(z)$ about $z = \bar{S}^{-2}$. It is equal to the distance d of $\bar{S}^{-2}$ from the boundary of D (see Fig. 3).

To compute it, we minimize d^2, i.e., we minimize the function:

$$h(x) = d^2(x) = (x - \bar{S}^{-2})^2 + \left(\frac{x^2 - 1}{2} - 0\right)^2 = (x - \bar{S}^{-2})^2 + \frac{(x^2 - 1)^2}{4}.$$

We have $h'(x) = 2(x - \bar{S}^{-2}) + x(x^2 - 1) = x^3 + x - 2\bar{S}^{-2}$. Solving $x^3 + x - 2\bar{S}^{-2} = 0$ for x, we find

$$x_0 = \frac{-3^{1/3}\bar{S}^{4/3} + \left(9 + \sqrt{3}\sqrt{27 + \bar{S}^4}\right)^{2/3}}{3^{2/3}\bar{S}^{2/3}\left(9 + \sqrt{3}\sqrt{27 + \bar{S}^4}\right)^{1/3}}.$$

Then computing the distance gives that the radius of convergence in the z-variable is

$$R_z = \sqrt{(x_0 - \bar{S}^{-2})^2 + \frac{(x_0^2 - 1)^2}{4}}.$$

Since $z = (1 - 2w)/\bar{S}^2$, the power series of $f(w)$ around $w = 0$ converges for all w such that

$$\left|\frac{1 - 2w}{\bar{S}^2} - \frac{1}{\bar{S}^2}\right| < R_z \quad \text{or} \quad |w| < \frac{\bar{S}^2 R_z}{2}.$$

Therefore, the function $f(x)$ is analytic for all $x < x^*$, and its power series around $x = 0$ has radius of convergence $r_c = \bar{S}^2 R_z/2$.

Appendix G. Derivation of the Taylor series for $Q(x)$

Recall that $Q(x)$ can be expressed in the form:

$$Q(x) = \begin{cases} \sum_{n=0}^{\infty} a_n x^n & \text{if } -r \leqslant x \leqslant r, \\ \sum_{n=0}^{\infty} a_n r^n & \text{if } x > r, \\ \sum_{n=0}^{\infty} (-1)^n a_n r^n & \text{if } x < -r. \end{cases} \tag{58}$$

Define

$$L(x) \equiv \frac{-r - \psi(x)}{\tilde{\lambda}(x)}, \quad U(x) \equiv \frac{r - \psi(x)}{\tilde{\lambda}(x)},$$

and

$$f_1(x) \equiv \frac{1}{\tilde{\lambda}(x)} \sum_{n=0}^{\infty} a_n \int_{-r}^{r} y^n e^{-(y-\psi(x))^2/(2\tilde{\lambda}(x)^2)}\,dy = \sum_{n=0}^{\infty} a_n \int_{L(x)}^{U(x)} (\tilde{\lambda}(x)y + \psi(x))^n e^{-y^2/2}\,dy,$$

$$f_2(x) \equiv \frac{1}{\tilde{\lambda}(x)} \sum_{n=0}^{\infty} a_n r^n \int_{r}^{\infty} e^{-(y-\psi(x))^2/(2\tilde{\lambda}(x)^2)}\,dy = \sum_{n=0}^{\infty} a_n r^n \int_{U(x)}^{\infty} e^{-y^2/2}\,dy,$$

$$f_3(x) \equiv \frac{1}{\tilde{\lambda}(x)} \sum_{n=0}^{\infty} (-1)^n a_n r^n \int_{-\infty}^{-r} e^{-(y-\psi(x))^2/(2\tilde{\lambda}(x)^2)}\,dy = \sum_{n=0}^{\infty} (-1)^n a_n r^n \int_{-\infty}^{L(x)} e^{-y^2/2}\,dy.$$

For $-r \leqslant x \leqslant r$, (26) is equivalent to

$$\sum_{n=0}^{\infty} a_n x^n = e^{-\gamma x} + \frac{K_3}{\sqrt{2\pi}}(f_1(x) + f_2(x) + f_3(x)). \tag{59}$$

G.1. *The Taylor series of $f_1(x)$ at zero*

Calculate

$$\begin{aligned}
g_1(x) &\equiv \int_{L(x)}^{U(x)} (\tilde{\lambda}(x)y + \psi(x))^n e^{-y^2/2}\,dy = \int_{L(x)}^{U(x)} (\tilde{\lambda}(x)y + \psi(x))^n \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{y^2}{2}\right)^k dy \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \int_{L(x)}^{U(x)} (\tilde{\lambda}(x)y + \psi(x))^n y^{2k}\,dy = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \sum_{j=0}^{n} \binom{n}{j} \psi(x)^{n-j} \tilde{\lambda}(x)^j \int_{L(x)}^{U(x)} y^{2k+j}\,dy \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^k}{2^k(2k+j+1)k!} \psi(x)^{n-j} \tilde{\lambda}(x)^j (U(x)^{2k+j+1} - L(x)^{2k+j+1}) \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^k}{2^k(2k+j+1)k!} \cdot \frac{\psi(x)^{n-j}}{\tilde{\lambda}(x)^{2k+1}} \{(r-\psi(x))^{2k+j+1} - (-r-\psi(x))^{2k+j+1}\}.
\end{aligned}$$

Simplify

$$\begin{aligned}
&(r-\psi(x))^{2k+j+1} - (-r-\psi(x))^{2k+j+1} \\
&\quad = (-1)^{2k+j} \sum_{i=0}^{2k+j+1} (1 + (-1)^{i+1}) \binom{2k+j+1}{i} \psi(x)^{2k+j+1-i} r^i.
\end{aligned}$$

Thus,

$$g_1(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{n} \sum_{i=0}^{2k+j+1} \binom{n}{j} \binom{2k+j+1}{i} \frac{((-1)^{j+k} + (-1)^{i+j+k+1}) r^i}{2^k(2k+j+1)k!} \cdot \frac{\psi(x)^{2k+1+n-i}}{\tilde{\lambda}(x)^{2k+1}}.$$

G.2. *The Taylor series of $f_2(x) + f_3(x)$ at zero*

$$f_2(x) + f_3(x) = \sum_{n=0}^{\infty} a_n r^n \left(\int_{U(x)}^{\infty} e^{-y^2/2}\,dy + (-1)^n \int_{-\infty}^{L(x)} e^{-y^2/2}\,dy \right).$$

Calculate

$$\begin{aligned}
g_2(x) &\equiv \int_0^{\infty} e^{-y^2/2}\,dy - \int_0^{U(x)} e^{-y^2/2}\,dy = \frac{\sqrt{2\pi}}{2} - \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k(2k+1)k!} U(x)^{2k+1} \\
&= \frac{\sqrt{2\pi}}{2} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^k(2k+1)k!} U(x)^{2k+1}
\end{aligned}$$

and

$$g_3(x) \equiv \int_{-\infty}^{0} e^{-y^2/2}\,dy + \int_{0}^{L(x)} e^{-y^2/2}\,dy = \frac{\sqrt{2\pi}}{2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k(2k+1)k!} L(x)^{2k+1}.$$

Then

$$\begin{aligned} &g_2(x) + (-1)^n g_3(x) \\ &= \frac{\sqrt{2\pi}}{2}(1+(-1)^n) + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^k(2k+1)k!} (U(x)^{2k+1} + (-1)^{n+1}L(x)^{2k+1}) \\ &= \frac{\sqrt{2\pi}}{2}(1+(-1)^n) + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^k(2k+1)k!} \frac{(r-\psi(x))^{2k+1} + (-1)^{n+1}(-r-\psi(x))^{2k+1}}{\tilde{\lambda}(x)^{2k+1}}. \end{aligned}$$

Simplify

$$\begin{aligned} &(r-\psi(x))^{2k+1} + (-1)^{n+1}(-r-\psi(x))^{2k+1} \\ &= (-1)^{2k+1} \sum_{j=0}^{2k+1} ((-1)^j + (-1)^{n+1}) \binom{2k+1}{j} \psi(x)^{2k+1-j} r^j. \end{aligned}$$

Thus,

$$g_2(x) + (-1)^n g_3(x) = \frac{\sqrt{2\pi}}{2}(1+(-1)^n) + \sum_{k=0}^{\infty} \sum_{j=0}^{2k+1} \binom{2k+1}{j} \cdot \frac{\psi(x)^{2k+1-j}}{\tilde{\lambda}(x)^{2k+1}}.$$

G.3. Recurrence relations for differentiation

(a) $\tilde{\lambda}(x) = c\sqrt{1-2x} - \sigma$:

$$\tilde{\lambda}(0) = c - \sigma, \quad \tilde{\lambda}'(0) = -c, \quad \tilde{\lambda}^{(n)}(0) = (2n-3)\tilde{\lambda}^{(n-1)}(0) \quad \text{for } n = 2,3,4,\ldots$$

(b) $\Lambda_k(x) = (\tilde{\lambda}(x))^k$ for $k>0$:

$$\Lambda_{k+1}^{(0)}(0) = \tilde{\lambda}(0)^{k+1}, \quad \Lambda_{k+1}^{(n)}(0) = \sum_{i=0}^{n} \binom{n}{i} \Lambda_k^{(i)}(0)\tilde{\lambda}^{(n-i)}(0) \quad \text{for } n = 1,2,3,\ldots.$$

(c) $\Lambda_{-k}(x) = (\tilde{\lambda}(x))^{-k}$ for $k>0$:

$$\Lambda_{-k}^{(0)}(0) = (\Lambda_k^{(0)}(0))^{-1},$$

$$\Lambda_{-k}^{(n)}(0) = -(\Lambda_k^{(0)}(0))^{-1} \sum_{i=0}^{n-1} \binom{n}{i} \Lambda_{-k}^{(i)}(0) \cdot \Lambda_k^{(n-i)}(0) \quad \text{for } n = 1,2,3,\ldots.$$

(d) $\psi(x) = \phi x + \sigma(K_2 - \gamma)\tilde{\lambda}(x)$:

$$\psi(0) = \sigma(K_2-\gamma)(c-\sigma), \quad \psi'(0) = \phi - \sigma(K_2-\gamma)\cdot c,$$
$$\psi''(0) = -\sigma(K_2-\gamma)\cdot c,$$

$$\psi^{(n)}(0) = (2n-3)\psi^{(n-1)}(0) \quad \text{for } n = 3,4,5,\ldots.$$

(e) $\Psi_k(x) = (\psi(x))^k$ for $k>0$:

$$\Psi_{k+1}^{(0)}(0) = [\psi(0)]^{k+1}, \quad \Psi_{k+1}^{(n)}(0) = \sum_{i=0}^{n} \binom{n}{i} \Psi_k^{(i)}(0)\psi^{(n-i)}(0) \quad \text{for } n = 1,2,\ldots$$

The recurrence relations (c) and (e) are applied to compute the Taylor series of the functions $\psi(x)^s/\lambda(x)^t$ around zero, where s and t are non-negative integers.

$$\frac{\psi(x)^s}{\tilde{\lambda}(x)^t} = \sum_{l=0}^{\infty} b_{s,t,l} x^l,$$

where

$$b_{s,t,l} = \frac{1}{l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}\left(\frac{\psi(x)^s}{\tilde{\lambda}(x)^t}\right)\bigg|_{x=0} = \frac{1}{l!}\sum_{i=0}^{l}\binom{l}{i}\Psi_s^{(i)}(0)\Lambda_{-t}^{(l-i)}(0).$$

G.4. The linear system for undetermined coefficients

Now (59) is equivalent to the system of linear equations:

$$\begin{aligned} a_l = {} & \frac{(-1)^l\gamma^l}{l!} + \sum_{n=0}^{\infty} a_n\Bigg\{\delta_{l,0}\frac{K_3(1+(-1)^n)r^n}{2} + \sum_{k=0}^{\infty}\frac{K_3}{\sqrt{2\pi}\sigma^{2k+1}2^k k!} \\ & \times\left(\sum_{j=0}^{n}\sum_{i=0}^{2k+j+1}\binom{n}{j}\binom{2k+j+1}{i}\frac{[(-1)^{j+k}+(-1)^{i+j+k+1}]r^i}{2k+j+1}\cdot b_{2k+1+n-i,2k+1,l}\right. \\ & \left.+\sum_{j=0}^{2k+1}\binom{2k+1}{j}\frac{((-1)^{j+k}+(-1)^{n+k+1})r^{n+j}}{2k+1}\cdot b_{2k+1-j,2k+1,l}\right)\Bigg\}, \end{aligned} \tag{60}$$

where

$$\delta_{l,0} = \begin{cases} 1 & \text{if } l=0, \\ 0 & \text{if } l\neq 0. \end{cases}$$

Appendix H. Proof of Lemma 3

It is equivalent to show that

$$\begin{aligned} 1-2r+\bar{S}^2 > 2\bar{S}\sqrt{1+2r} & \iff 4r^2-4(\bar{S}^2+1)r+\bar{S}^4+2\bar{S}^2+1 > 4\bar{S}^2+8\bar{S}^2 r \\ & \iff 4r^2-4(3\bar{S}^2+1)r+\bar{S}^4-2\bar{S}^2+1>0 \\ & \iff r^2-(3\bar{S}^2+1)r+\tfrac{1}{4}(\bar{S}^4-2\bar{S}^2+1)>0. \end{aligned}$$

The equation $r^2-(3\bar{S}^2+1)r+\frac{1}{4}(\bar{S}^4-2\bar{S}^2+1)=0$ has real roots

$$\tfrac{1}{2}+\bar{S}\left(\tfrac{3}{2}\bar{S}\pm\sqrt{2}\sqrt{\bar{S}^2+1}\right).$$

Lemma 3 is proved.

Appendix I. Proof of Theorem 3

Let C_r be the circle of radius r centered at the origin in the complex plane $\mathbb{C}$. The Cauchy integral formula yields

$$Q^{(k)}(0) = \frac{k!}{2\pi i}\oint_{C_r}\frac{Q(z)}{z^{k+1}}\,\mathrm{d}z \quad \text{for } k=0,1,2,\ldots. \tag{61}$$

Let $z=x+yi$ be a point on C_r, which satisfies $-r\leqslant x\leqslant r$, $-r\leqslant y\leqslant r$, and $x^2+y^2=r^2$. By the hypothesis, we have $r<(1-\bar{S}^2)/2$. Write

$$u+vi=\lambda(z)=\frac{1}{\bar{S}}\sqrt{1-2z}-1=\frac{1}{\bar{S}}\sqrt{1-2x-2yi}-1 \quad \text{with } u+1\geqslant 0,$$

which is equivalent to the equations:

$$(u+1)^2-v^2=(1-2x)/\bar{S}^2 \quad \text{and} \quad (u+1)v=-y/\bar{S}^2.$$

These equations imply

$$(u+1)^4 - \frac{1-2x}{\bar{S}^2}(u+1)^2 - \frac{y^2}{\bar{S}^4} = 0 \quad \text{and} \quad v^4 + \frac{1-2x}{\bar{S}^2} v^2 - \frac{y^2}{\bar{S}^4} = 0.$$

If $y \neq 0$, then the quadratic formula yields

$$(u+1)^2 = \frac{1-2x+\sqrt{1-4x+4r^2}}{2\bar{S}^2} \quad \text{and} \quad v^2 = \frac{-1+2x+\sqrt{1-4x+4r^2}}{2\bar{S}^2}.$$

Applying $(1-2r)^2 = 1-4r+4r^2 \leqslant 1-4x+4r^2 \leqslant 1+4r+4r^2 = (1+2r)^2$, we get

$$(1-2r)/\bar{S}^2 \leqslant (u+1)^2 \leqslant (1+2r)/\bar{S}^2 \quad \text{and} \quad 0 \leqslant v^2 \leqslant 2r/\bar{S}^2,$$

where the first inequality implies further that

$$0 < \lambda(r) = \sqrt{1-2r}/\bar{S} - 1 \leqslant u \leqslant \sqrt{1+2r}/\bar{S} - 1 = \lambda(-r).$$

So

$$0 < \lambda(r)^2 \leqslant u^2 + v^2 \leqslant (\sqrt{1+2r}/\bar{S} - 1)^2 + 2r/\bar{S}^2.$$

We can also estimate $u^2 - v^2$:

$$\begin{aligned} u^2 - v^2 &= (u+1)^2 - v^2 - 2u - 1 = \frac{1-2x}{\bar{S}^2} + 1 - 2(u+1) \\ &= \frac{1-2x+\bar{S}^2}{\bar{S}^2} - 2(u+1) \\ &\geqslant \frac{1-2r+\bar{S}^2}{\bar{S}^2} - \frac{2\sqrt{1+2r}}{\bar{S}} = \frac{1-2r+\bar{S}^2-2\bar{S}\sqrt{1+2r}}{\bar{S}^2}. \end{aligned}$$

Write $1/\lambda(z)^2 = A(z) + B(z)i$ and $\psi(z) = C(z) + D(z)i$, where

$$\begin{aligned} &A(z) = \frac{u^2-v^2}{(u^2+v^2)^2}, \quad B(z) = -\frac{2uv}{(u^2+v^2)^2}, \\ &C(z) = \phi x + \sigma^2(K_2-\gamma)u, \quad D(z) = \phi y + \sigma^2(K_2-\gamma)v. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{Re}\left\{\frac{[t-\psi(z)]^2}{\lambda(z)^2}\right\} &= A(z)[t-C(z)]^2 + 2B(z)D(z)[t-C(z)] - A(z)D(z)^2 \\ &= A(z)\left(t - \frac{A(z)C(z)-B(z)D(z)}{A(z)}\right)^2 - \frac{D(z)^2(A(z)^2+B(z)^2)}{A(z)} \\ &= A(z)\left(t - \frac{A(z)C(z)-B(z)D(z)}{A(z)}\right)^2 - \frac{D(z)^2}{u^2-v^2}. \end{aligned}$$

By (26), for any $z \in C_r$ we have

$$
\begin{aligned}
|Q(z)| &\leqslant |e^{-\gamma z}| + \frac{K_3}{\sqrt{2\pi}\sigma|\lambda(z)|}\int_{-\infty}^{\infty} \|Q\| |e^{-[t-\psi(z)]^2/(2\sigma^2\lambda(z)^2)}| \, dt \\
&\leqslant e^{\gamma r} + \frac{K_3\|Q\| e^{-D(z)^2/(u^2-v^2)}}{\sqrt{2\pi}\sigma\sqrt{u^2+v^2}}\int_{-\infty}^{\infty} e^{-(A(z)/(2\sigma^2)[t-(A(z)C(z)-B(z)D(z))/A(z)]^2}\, dt \\
&= e^{\gamma r} + \frac{K_3\|Q\| e^{-D(z)^2/(u^2-v^2)}}{\sqrt{u^2+v^2}\sqrt{A(z)}} \leqslant e^{\gamma r} + K_3\|Q\|\sqrt{\frac{u^2+v^2}{u^2-v^2}} \\
&\leqslant e^{\gamma r} + K_3\|Q\|\sqrt{\frac{(\sqrt{1+2r}-\bar{S})^2+2r}{1-2r+\bar{S}^2-2\bar{S}\sqrt{1+2r}}} \\
&\leqslant e^{\gamma r} + K_3\|Q\|\sqrt{\frac{1+4r+\bar{S}^2-2\bar{S}\sqrt{1+2r}}{1-2r+\bar{S}^2-2\bar{S}\sqrt{1+2r}}} \leqslant B_r.
\end{aligned}
$$

By Cauchy's integral formula (61), we get

$$
|Q^{(k)}(0)| \leqslant \frac{k!}{2\pi}\oint_{C_r}\frac{|Q(z)|}{r^{k+1}}\, dz = \frac{B_r k!}{2\pi}\cdot\frac{2\pi r}{r^{k+1}} = \frac{B_r k!}{r^k}.
$$

If $0<\mu<1$ and $|x|<\mu r$, then we get

$$
|R_n(x)| \leqslant \sum_{k=n+1}^{\infty}\frac{|Q^{(k)}(0)|}{k!}(\mu r)^k = B_r\sum_{k=n+1}^{\infty}\mu^k = \frac{B_r\mu^{n+1}}{1-\mu}.
$$

Theorem 3 is proved.

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