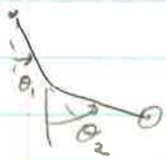
# Resonant Wave Interactions

Model problem - double pendulum

eqs. of motion:




$$\frac{d^2\theta_1}{dt^2} + \omega_1^2\theta_1 = a\left\{(\theta_1-\theta_2)\frac{d^2\theta_2}{dt^2} - \left(\frac{d\theta_2}{dt}\right)^2\right\}$$

$$\frac{d^2\theta_2}{dt^2} + \omega_2^2\theta_2 = b\left\{(\theta_1-\theta_2)\frac{d^2\theta_1}{dt^2} + \left(\frac{d\theta_1}{dt}\right)^2\right\}$$

- for small but finite oscillations.
- anticipate normal mode solutions to LHS of the form

$$\theta_1 = \phi_1 \cos(\omega_1 t + \psi_1)$$

$$\theta_2 = \phi_2 \cos(\omega_2 t + \psi_2)$$

- RHS represents weak interactions of normal modes
  - The quadratic terms involve products of normal modes

$$\phi_1 \cos(\omega_1 t + \psi_1)\,\phi_2 \cos(\omega_2 t + \psi_2)$$

$$= \tfrac{1}{2}\phi_1\phi_2\left[\cos\{(\omega_1-\omega_2)t + (\psi_1-\psi_2)\}\right.$$

$$\left. + \cos\{(\omega_1+\omega_2)t + (\psi_1+\psi_2)\}\right]$$

- therefore we have trigonometric fns of $(\omega_1 \pm \omega_2)$

- Therefore small terms of the order $\phi_1 \phi_2$ with periods $2\pi / |\omega_1 \pm \omega_2|$ arise on RHS.
- Similarly, terms like

$$\phi_1^2 \text{ and } \phi_2^2 \quad \text{with periods } \frac{\pi}{\omega_1} \text{ and } \frac{\pi}{\omega_2}$$

also arise from RHS.

- Consider modes on RHS as forcing to linear system defined by LHS.
- Generally, the response is the same order as the forcing. EXCEPT when,

$$\omega_2 = 2\omega_1 \quad \text{or} \quad \omega_1 = 2\omega_2$$

→ conditions for INTERNAL RESONANCE.

(also when $\omega_1 = 3\omega_2,\ 4\omega_2$, etc)

Suppose $\omega_2 = 2\omega_1$, then normal modes,

$$\begin{cases} \theta_1 = A_1 e^{i\omega_1 t} + A_1^* e^{-i\omega_1 t} \\ \theta_2 = A_2 e^{2i\omega_1 t} + A_2^* e^{-2i\omega_1 t} \end{cases}$$

now, for $A_1$ + $A_2$ small,

- $$\frac{d^2\theta_1}{dt^2} + \omega_1^2 \theta_1 \simeq a\, \theta_1 \frac{d^2\theta_2}{dt^2}$$

and

$$\frac{d^2\theta_2}{dt^2} + 4\omega_1^2\theta_2 \cong b\left\{\theta_1\frac{d^2\theta_1}{dt^2} + \left(\frac{d\theta_1}{dt}\right)^2\right\}$$

and now $|\ddot{A}_1| \ll \omega_1|\dot{A}_1| \ll \omega_1^2|A_1|$

$|\ddot{A}_2| \ll \omega_1|\dot{A}_2| \ll \omega_1^2|A_2|$

ie, slowly varying Amplitudes.

subst. for $\theta_1$, $\theta_2$ and equating like coefs of $e^{\pm i\omega_1 t}$ and $e^{\pm i2\omega_1 t}$,

$$\left\{\begin{array}{ll} 2i\omega_1 \dfrac{dA_1}{dt} = -4a\omega_1^2 A_1^* A_2 & \text{(a)} \\ \& & \\ 4i\omega_1 \dfrac{dA_2}{dt} = -2b\omega_1^2 A_1^2 & \end{array}\right.$$

• check on slow amplitude variation: $\left.\begin{array}{l}\dot{A}_1 = \mathcal{O}(A_1A_2) \\ \dot{A}_2 = \mathcal{O}(A_1^2)\end{array}\right\}$

which is small in limit $A_1, A_2 \to 0$.

eliminating $A_2$,

$$\frac{d}{dt}\left(\frac{1}{A_1^*}\frac{dA_1}{dt}\right) = -ab\omega_1^2 A_1^2$$

let $A_1 = \phi e^{i\psi}$, $A_1^* = \phi e^{-i\psi}$, then

$$\frac{1}{\phi}\frac{d^2\phi}{dt^2} - \frac{1}{\phi^2}\left(\frac{d\phi}{dt}\right)^2 - 2\left(\frac{d\psi}{dt}\right)^2 = -ab\omega_1^2\phi^2$$

and,

$$\frac{d^2\psi}{dt^2} + \frac{2}{\phi}\frac{d\phi}{dt}\frac{d\psi}{dt} = 0$$

sol.

$$\boxed{\psi = k_1\int_0^t \phi^{-2}\,dt + k_2}$$

$\therefore$ for $A_1 = \phi e^{i\psi}$,

$$\therefore \boxed{A_1 = \phi\exp\left\{ik\int_0^t \phi^{-2}\,dt + k_2\right\}}$$

and subs into (a),

$$\boxed{A_2 = -\frac{1}{2a\omega_1}\left(\frac{1}{\phi}\frac{d\phi}{dt} + \frac{ik_1}{\phi^2}\right)\exp\left\{2ik_1\int_0^t \phi^{-2}\,dt + 2k_2\right\}}$$

Shows: a slow periodic interchange of energy between two fundamentals.

## Resonant wave interactions

in traveling waves, the wave character is given by

$u = Ae^{i\underset{\sim}{k}\underset{\sim}{x} - \omega t}$, where $\omega$ is real frequency and $\underset{\sim}{k}$ is the wave-number vector.

Ⓑ ∴ two waves may interact at second order to excite a third wave not only if their frequencies add, but also if their wave numbers add, namely

'wave triad' → $\omega_3 = \omega_1 + \omega_2$ and $\underset{\sim}{k}_3 = \underset{\sim}{k}_1 + \underset{\sim}{k}_2$

where $\omega_n = F(\underset{\sim}{k}_n) \overset{\text{generally}}{=} \underset{\sim}{k}_n c$ , $n = 1, 2, 3$

this says resonance occurs if $F(\underset{\sim}{k}_1 + \underset{\sim}{k}_2) = F(\underset{\sim}{k}_1) + F(\underset{\sim}{k}_2)$

for example if $\boxed{\omega_2 = 2\omega_1 \quad \text{and} \quad \underset{\sim}{k}_2 = 2\underset{\sim}{k}_1}$

## Model Problem

Consider a simple model eq (one space variable)

(1) $$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u = \underbrace{u^2}_{\text{nonlinear (neglect) in beginning}}$$

normal mode,

$$u = Ae^{i(kx - \omega t)} + A^* e^{-i(kx - \omega t)}$$

linear solution gives $\omega^2 = 1 - k^2 + k^4$

- for weakly nonlinear problem (keep $u^2$ term)

(2) $$u = \sum_k A(k, t)\, e^{i\{kx - \omega(k)t\}}$$

$A$ ← slowly varying fn of $t$

- $\omega$ satisfies linear sol, $\omega^2 = 1 - k^2 + k^4$

$$A(-k, t) = A^*(k, t)$$

subst. into (2) into (1)

$$\sum_{k_1} \frac{d^2A_1}{dt^2}\, e^{i(\;)} - 2i\omega_1 \frac{dA_1}{dt}\, e^{i(\;)} + \overbrace{\{-\omega_1^2 + (1-k_1^2+k_1^4)\}}^{0 \text{ from linear sol.}} A_1\, e^{i(k_1x-\omega_1 t)}$$

$$= \sum_{\substack{\omega_2+\omega_3=\omega_1 \\ k_2+k_3=k_1}} 2A_2A_3\, e^{i(k_1x-\omega_1 t)}$$

$$\therefore \sum_{k_1}\left(\frac{d^2A_1}{dt^2} - 2i\omega_1\frac{dA_1}{dt}\right) e^{i(k_1x-\omega_1t)} = \sum_{\substack{\omega_1+\omega_2+\omega_3=0 \\ k_1+k_2+k_3=0}} 2A_2^*A_3^*\, e^{i(k_1x-\omega_1t)}$$

or,

$$\boxed{\frac{d^2A_1}{dt^2} - 2i\omega_1\frac{dA_1}{dt} = 2A_2^*A_3^*}$$

again, $\dot{A}_1 = \mathcal{O}(A_2A_3)$ so that $\ddot{A}_1$ can be neglected as $A_2, A_3 \to 0$

$$\therefore \quad \frac{dA_1}{dt} = i\omega_1^{-1} A_2^* A_3^*$$

similarly for $A_2$ + $A_3$,

$$\boxed{\begin{aligned} \frac{dA_2}{dt} &= i\omega_2^{-1} A_3^* A_1^* \\ \frac{dA_3}{dt} &= i\omega_3^{-1} A_1^* A_2^* \end{aligned}}$$

for example, combine last two eqs.

$$A_2^*\omega_2 \frac{dA_2}{dt} = A_3^* \,\omega_3 \frac{dA_3}{dt}$$

adding the complex conjugate of these, deduce that

$$\frac{d}{dt}\,\omega_2 |A_2|^2 = \frac{d}{dt}\,\omega_3 |A_3|^2$$

with similar combinations for $|A_3|$ and $|A_1|$

$$\frac{d}{dt}\,\omega_1 |A_1|^2 = \frac{d}{dt}\,\omega_2 |A_3|^2$$

integrating, $\int_{t=0}^{1} dt$

$$\omega_1 \left(|A_1|^2 - A_{10}^2\right) = \omega_2 \left(|A_2|^2 - A_{20}^2\right) = \omega_3 \left(|A_3^2| - A_{30}^2\right)$$

where $A_{\substack{10\\2\\3}} = |A_{\substack{1\\2\\3}}|$ at $t=0$

note: this is consistant with conservation of 'energy', that

$$\frac{1}{2}\left(\omega_1^2 |A_1|^2 + \omega_2^2 |A_2|^2 + \omega_3^2 |A_3|^2\right) = \text{const.}$$

with $-\omega_1 + \omega_2 + \omega_3 = 0$

- consider stability of single wave of freq. $\omega_1$ and wave number $k_1$
  - with $A_2$ + $A_3$ small

$$\rightarrow A_1 = \text{const}, \quad A_2 = A_3 = 0$$

then

$$\frac{dA_1}{dt} = 0$$

$$\frac{d^2 A_2}{dt^2} - \frac{|A_1|^2}{\omega_2 \omega_3} A_2 = 0$$

$$\frac{d^2 A_3}{dt^2} - \frac{|A_1|^2}{\omega_2 \omega_3} A_3 = 0$$

$A_1 = \text{const}$

$\omega_2 \omega_3 > 0 \implies A_2 \sim e^{\frac{|A_1|}{\sqrt{\omega_1 \omega_2}} t}$ Unstable

$\omega_2 \omega_3 < 0 \implies A_2 \sim e^{i \frac{|A_1|}{\sqrt{\omega_1 \omega_2}} t}$ Stable (periodic)

similarly for $A_3$

For $\omega_2 \omega_3 > 0$, wave $A_1$ is unstable subject to parasitic growth of $A_2$ + $A_3$