1. Linear Stability of Lorenz eqs

$$\begin{cases} A_1' = -A_1 - B_1 - A_0 B_1 \\ B_1' = P_r \{ -\lambda A_1 - B_1 \} \\ A_0' = -b A_0 + A_1 B_1 \end{cases}$$

sol: $A_0' = -b A_0 \Rightarrow A_0 = e^{-bt}$

for the other two eqs, put in $\begin{cases} A_1 = e^{-\tau t} \\ B_1 = e^{-\tau t} \end{cases}$

then

$$-\tau \cancel{e^{\tau t}} = -\cancel{e^{\tau t}} - \cancel{e^{\tau t}}$$

$$-\tau \cancel{e^{\tau t}} = -P_r \lambda \cancel{e^{\tau t}} - P_r \cancel{e^{\tau t}}$$

eigenvalues are found from

$$\begin{vmatrix} 1-\tau & -1 \\ \lambda P_r & \tau - P_r \end{vmatrix} = 0 = (1-\tau)(\tau - P_r) + \lambda P_r$$
$$= \tau - \tau^2 - P_r - \tau P_r + \lambda P_r = 0$$

then

stable if $\tau = 0$, $\lambda = 1 = \dfrac{R}{R_L}$ $\therefore \boxed{R = R_L}$

2. Steady Bifurcation Sol. $\left(\frac{d}{dt} = 0\right)$

$$\begin{cases} 0 = -A_1 - B_1 - A_0 B_1 \\ 0 = \lambda A_1 + B_1 \\ 0 = -b A_0 + A_1 B_1 \end{cases}$$

from eq. 3, $A_0 = \dfrac{A_1 B_1}{b}$

subs. into 2, $-A_1 - B_1 - \dfrac{A_1 B_1^2}{b} = 0$

from 2, $B_1 = -\lambda A_1$

subs, $-A_1 + \lambda A_1 - \dfrac{\lambda^2 A_1^3}{b} = 0$

or, $(\lambda - 1)A_1 = \dfrac{\lambda^2 A_1^3}{b}$

To get Landau eq.,

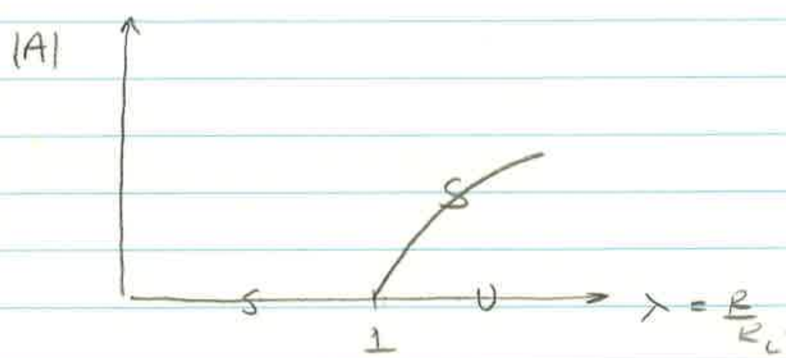
start with, $A_1' = -A_1 - \overset{-\lambda A_1}{\cancel{B_1}} - \overset{\frac{A_1 \overset{\lambda^2 A_1^2}{B_1^2}}{b}}{\cancel{A_0 B_1}}$

$$\boxed{A_1' = (\lambda - 1)A_1 - \frac{\lambda^2}{b} A_1^3}$$ ← Landau eq. (steady limit cycle)

For this,

$A_1 = 0$ is basic state

$A_1^2 = \dfrac{b(\lambda - 1)}{\lambda^2}$ is bifurcation solution for $\lambda > 1$



- Landau eq. says that the new solution in the region of the branch remains stable (for ever).

- <u>Stability of Bifurcation Sol.</u>

let' $A_1 = \bar{A}_1 + a_1$

$B_1 = \bar{B}_1 + b_1$

$A_0 = \bar{A}_0 + a_0$

subst. into Lorenz eqs. and linearize.

$$a_1' = -(\bar{A}_1 + a_1) - (\bar{B}_1 + b_1) - (\bar{A}_0 + a_0)(\bar{B}_1 + b_1)$$

$$a_1' = -a_1 - b_1 - \bar{A}_0 b_1 - \bar{B}_1 a_0 + \cancelto{0}{\text{nonlinear terms}}$$

$$\therefore \begin{cases} a_1' = -a_1 - b_1 - \bar{A}_0 b_1 - \bar{B}_1 a_0 \\ b_1' = P_r(-r a_1 - b_1) \\ a_0' = -b a_0 + \bar{A}_1 b_1 + \bar{B}_1 a_1 \end{cases}$$

<u>linear</u>

subs. $(a_1, b_1, a_0) = (a_1, b_1, a_0)e^{\sigma t}$

then

$$\begin{cases} (\sigma+1) a_1 + (1+\bar{A}_0) b_1 + \bar{B}_1 a_0 = 0 \\ P_r r a_1 + (\sigma + P_r) b_1 = 0 \\ \bar{B}_1 a_1 - \bar{A}_1 b_1 + (\sigma + b) a_0 = 0 \end{cases}$$

or,

$$\begin{vmatrix} \sigma+1 & 1+\bar{A}_0 & \bar{B}_1 \\ P_r r & \sigma+P_r & 0 \\ -\bar{B}_1 & -A_1 & \sigma+b \end{vmatrix} = 0$$

obtain a cubic eq. for σ, namely.

$$\sigma^3 + \sigma^2(1+P_r+b) + \sigma b(P_r+\lambda) + 2P_r b(\lambda-1) = 0$$

this applies for $\lambda > 1$. $\boxed{\text{i.e. } (R > R_L)}$
$b = b(k^2, \pi^2, \ldots)$ $\longrightarrow$ $b = \frac{4\pi^2}{\pi^2+k^2}$, $R_L = \frac{(k^2+\pi^2)^3}{k^2}$

- at critical k^2, $\boxed{b = \frac{8}{3}}$ @ $P_r = 10$. @ critical pt.

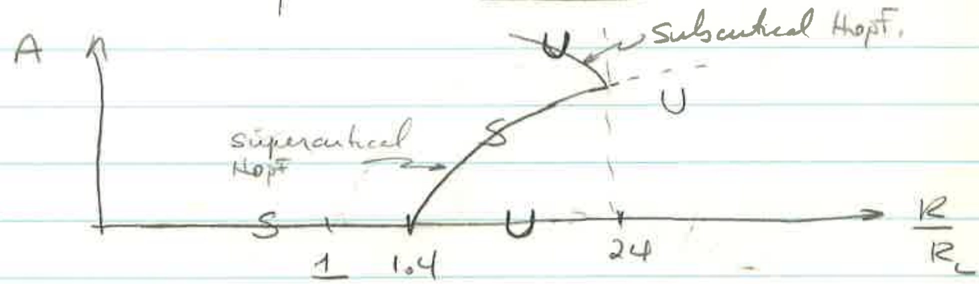
When $1 < \lambda < \lambda_1$, cubic has 3 real, neg. roots.

$\lambda_1 < \lambda < \lambda_2$ 1 real neg. root, 1 complex conjg. pair with neg. real part. Still stable.

Values are $\lambda_1 = 1.4 = R/R_L$

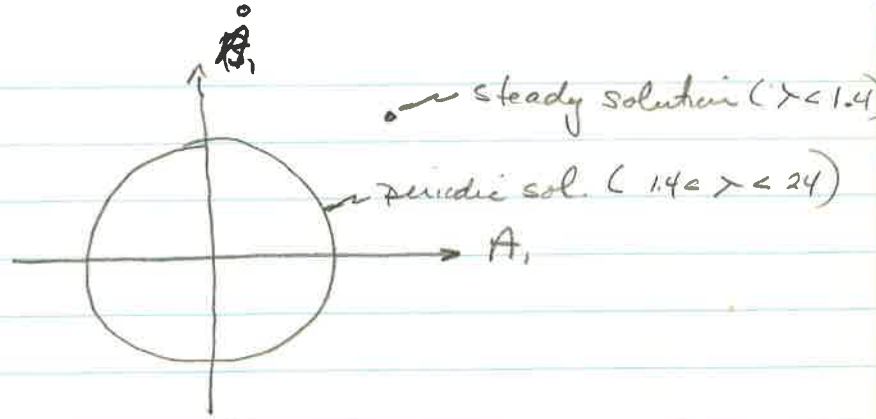
$\lambda_2 = 24$

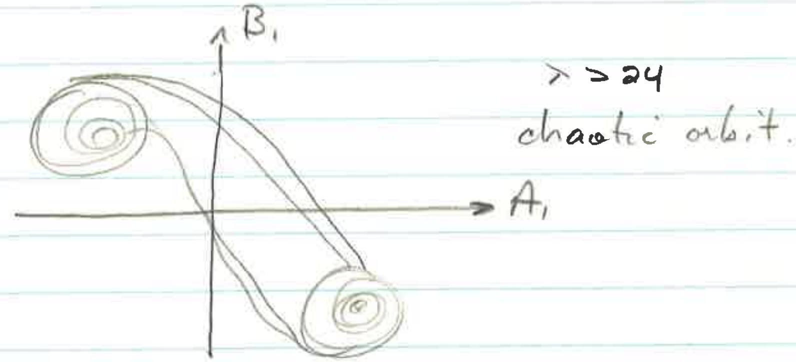
$\lambda_2 < \lambda$ 1 neg. real, 1 complex conjg pair with pos. real part. unstable



- For $\lambda = 24$, there will be a chaotic orbit, or so-called aperiodic sol which goes back and forth between two or several periodic-like orbits. -- called a strange attractor.

Phase space





As opposed to Landau approach, which says that you obtain a chaotic behavior (turbulence?) through a (large) successive no. of supercritical bifurcations, implying large number of modes (high dimension system),

we have for the Lorenz system, chaotic behavior after two bifurcations, one being subcritical. This is a low dimensional chaos.