$$\epsilon = \sqrt{\frac{R-R_L}{R_2}}$$ ← gives eq. of curve $u = \epsilon A$

- This is what happens in certain circumstances, even though these results are not from the N-S eqs.

- Problem — we can conclude nothing about the stability of the bifurcation solution, since ϕ_t was dropped from eq. It is that term which yields the necessary stability information.

Landau - Stuart - Watson Procedure

$$-\phi_t + \nabla^6\phi - R\phi_{xx} = 0$$

if $\phi = A(t)\cos kx \sin z$, the eq. becomes,

time derivative

$$-\dot{A}\cos kx \sin z - (k^2+1)^3 A\cos kx \sin z + ARk^2\cos kx \sin z = 0$$

then,

$$\dot{A} = \{ Rk^2 - (k^2+1)^3 \} A$$

using $(k^2+1)^3 \equiv R_L k^2$

then $\dot{A} = k^2(R-R_L)A$

$$\therefore \boxed{\dot{A} = \sigma A} \quad ; \quad \sigma \equiv k^2(R-R_L)$$

- So growth rate (σ) depends on whether $R \gtrless R_L$ and the same result is reached using a general $A(t)$ time dependence viz. $e^{\sigma t}$ time dependence.

- In 1944 Landau proposed the amplitude eq. of the form,

$$\dot{A} = F(A)$$

where $F(A) \cong \sigma A$ when A is small.

- The first specific proposal for $\dot{A}$ was

~like a Landau const.

$$\dot{A} = \sigma A - a_{11} A^3 + \cdots \leftarrow \text{Ricatti Eq.}$$

Take a look at $\dot{A} = \sigma A - a_{11} A^3$, use change of variables. $y = \frac{1}{A^2}$

then, $\dot{y} = -\frac{2}{A^3}\dot{A}$

and $-\frac{A^3 \dot{y}}{2} = \sigma A - a_{11} A^3$

$$-\frac{\dot{y}}{2} = \frac{\sigma}{A^2} - a_{11} = \sigma y - a_{11}$$

$$\therefore \boxed{\dot{y} + 2\sigma y = 2a_{11}} \leftarrow \text{so now it's linear.}$$

sol: $y = \frac{a_{11}}{\sigma} + Ce^{-2\sigma t}$ C = const.

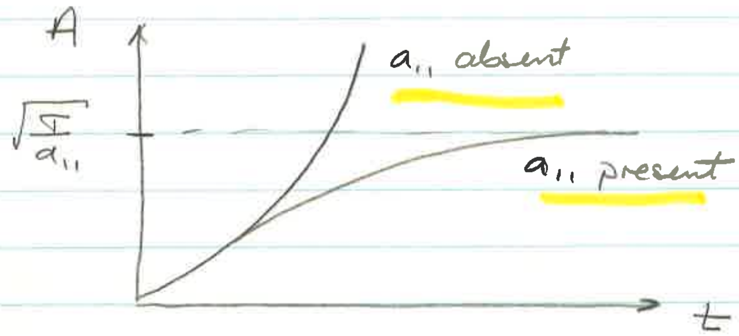
$$\therefore \boxed{A = \frac{1}{\sqrt{\frac{a_{11}}{\sigma} + Ce^{-2\sigma t}}}}$$

Cases:

For $\dot{A} = F(A)$, $a_{11} = 0 \rightarrow A = \frac{1}{\sqrt{c}} e^{\sigma t}$

$\rightarrow \infty$ as $t \to \infty$
$\sigma > 0$

, $a_{11} > 0 \rightarrow A \rightarrow \sqrt{\frac{\sigma}{a_{11}}}$ as $t \to \infty$
$\sigma > 0$



Now, take the model eq: $\boxed{\mathcal{L}\phi = \phi\phi_z}$; $\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \dots$

$\mathcal{L}\phi \equiv \nabla^6\phi - R_L\phi_{xx}$

$\mathcal{L}\phi_1 = 0$ and $\phi_1 = A\cos kx \sin z$ → provided $\dot{A} = \sigma A$

Also,

$$\mathcal{L}\phi_2 = \overbrace{\phi_1\,\phi_{1z}}^{RHS} = \frac{A^2}{4}(1+\cos 2kx)\sin 2z$$

and $\phi_2 = A^2\{(\ )\sin 2z + (\ )\cos 2kx \sin 2z\}$

with $\dot{A} = \sigma A$ still.

Next eq:

$$\mathcal{L}\phi_3 \equiv -\frac{\partial\phi_3}{\partial t} + \nabla^6\phi_3 - R\phi_{3xx} = \overbrace{(\phi_1\phi_2)_z}^{RHS.}$$

$$= A^3(\ )\sin z\cos kx + \dots$$

$$\phi_3 = A \sin z \cos kx + \cdots$$

so that $$\frac{dA}{dt} - k^2(R-R_L)A = \underbrace{(\quad)}_{-a_{11}} A^3$$

$$\boxed{\frac{dA}{dt} = \underbrace{-k^2(R-R_L)}_{\sigma} A - a_{11} A^3}$$

Time-independent soln : $A = 0$

$$A^2 = \frac{k^2}{a_{11}}(R-R_L)$$

- The A^2 sol is exactly the solution we get through bifurcation theory. So the bifurcation soln is the soln. of the Landau eq. So the Landau approach includes the bifurcation approach.

Stability of solution, $A = \sqrt{\frac{k^2}{a_{11}}(R-R_L)}$

- To do stability analysis we consider perturbations.

- What is stability of solution $A = \sqrt{\frac{k^2(R-R_L)}{a_{11}}}$

(we guessed from bifurcation theory that it was stable).

- Perturbation eq of A: $A = \bar{A} + a$

as before, $\frac{dA}{dt} = k^2(R-R_L)A - a_{11}A^3$

subst,

$$\frac{da}{dt} = k^2(R-R_L)\bar{A} + k^2(R-R_L)a$$
$$- a_{11}(\bar{A}+a)^3$$

steady sol $= \bar{A} = \sqrt{\frac{k^2(R-R_L)}{a_{11}}}$

$$= k^2(R-R_L)a - a_{11}3\bar{A}^2 a$$

$$= k^2(R-R_L)a - 3k^2(R-R_L)a$$

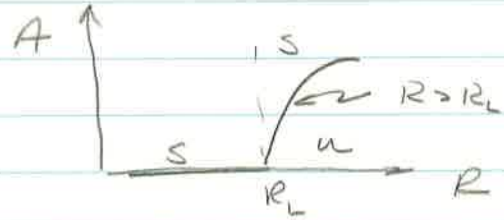
$$\frac{da}{dt} = -2k^2(R-R_L)a$$

$\therefore$ Stable $\left(\frac{da}{dt} < 0\right)$ if $R > R_L$

Recall that this solution only existed if $R > R_L$

$\therefore$ guess from bifurcation theory was correct.

- $\therefore$ For the conditions of existance, the solution is stable



Now suppose $\boxed{a_{11} < 0}$ eg. $a_{11} = -b_1$

Then the solution eq becomes,

$$k^2(R-R_L)A + b_{11}A^3 = 0$$

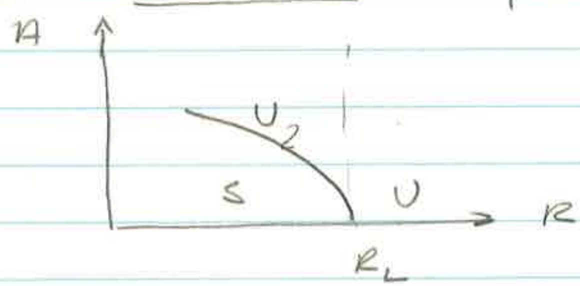
solutions, $A = 0$

$$A^2 = \frac{-k^2(R-R_L)}{b_{11}}$$

A must be real, which is only true if

$$R < R_L \qquad (b_{11} > 0)$$

This gives subcritical bifurcation,



and, $\frac{dA}{dt} = k^2(R-R_L) + b_{11}A^3 = -2k^2(R-R_L)a$

- → unstable since $R < R_L$

- Thus, the sign of a_{11} decides whether you have a supercritical or subcritical bifurcation.

a_{11} is the Landau constant.

For $\frac{dA}{dt} = \sigma A \ominus a_{11} A^3$

$a_{11} > 0$ Taylor - Couette (super)

$a_{11} \gtrless 0$ Benard (depending on structure of the problem)

$a_{11} < 0$ Channel flow. (sub)

Bifurcation Theorem

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(R)\, x_j + F_i(x_1, \ldots x_n) \quad ; i=1,\ldots n$$

in vector form:

(1) $$\frac{dx}{dt} = Ax + F$$, where $A = A(R)$ and

A is a real parameter.

F is a polynomial function of $x_1, x_2, \ldots x_n$ of degree 2 at least.

$F(0) = 0 \Rightarrow x = 0$ is a solution of (1)

Consider linearized problem,

$$\frac{dx}{dt} = Ax$$

sol.,

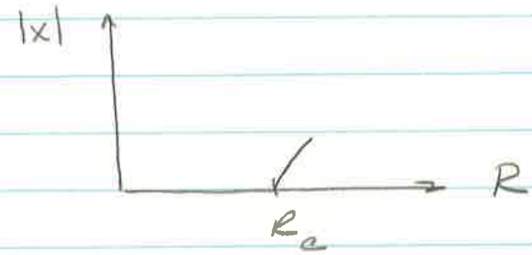
$$x = x_0\, e^{\sigma t} \quad ; \quad \sigma = s = A$$

Stability is decided by eigenvalues of A.

1. If all eigenvalues of A have negative real parts, then $x \equiv 0$ is a stable (asymptotically) solution of ① for sufficiently small disturbances.

2. If A has an eigenvalue with a positive real part, then $x \equiv 0$ is an unstable solution of ①.

3. If as R increases through some R_c, a single eigenvalue of A changes from real negative to real positive, then a solution bifurcates from $x=0$ at $R=R_c$.

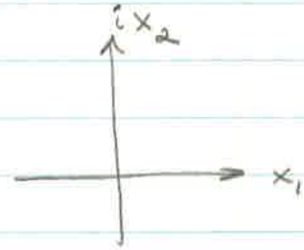


4. Under conditions of ③, and if all eigenvalues have real parts when $R<R_c$, the bifurcation solution is stable if supercritical and unstable if subcritical

How about if σ is complex?

$$\frac{dx}{dt} = Ax \quad , \quad x = e^{\sigma t} \quad ; \quad \sigma = s + i\omega$$

Canonical forms:



$$\frac{dx_1}{dt} = s x_1 + \omega x_2$$

$$\frac{dx_2}{dt} = -\omega x_1 + s x_2$$

in complex notation, $\quad \xi = x_1 + i x_2$

$$\frac{d\xi}{dt} = \sigma \xi$$

at critical pt., when $s = 0$,

subst.

$$\frac{dx_1}{dt} = \omega x_2 \quad , \quad \frac{dx_2}{dt} = -\omega x_1$$

$$\frac{d^2 x_1}{dt^2} + \omega^2 x_1 = 0$$

Sol: $\quad x_1 = \begin{cases} \cos \omega t \\ \sin \omega t \end{cases}$

Thus, eigenfunction of linear problem at criticality is periodic, with period $\frac{2\pi}{\omega}$.

eg. in plane Poiseuill flow, linear exponential growth given by $e^{i\alpha(x-ct)}$ $\rightarrow$ temporal growth