- by this scenario, we have a repeated bifurcation sequence → turbulence.

- note: $l > 0$ Bernard convection
  $l > 0$ Taylor cells.
  $l \not< 0$ Unbounded jets + shear layers.
  $l < 0$ Boundary layers, channel flows, pipe flows.

- Landau's theory comes from implied assumption that that there is ~~an~~ a unique and discrete most unstable mode.
  → applicable to a bounded region where we expect linear eigenvalue problem with normal modes which are both discrete and complete.

  → not applicable in cases where normal modes depend continuously on wave number.

Bifurcations

Example: $\frac{1}{R} u'' + \sin u = 0 \quad ; \quad 0 \le y \le \pi, \quad R > 0$

- one solution is $u=0$.

- linear theory for small displacements from $u=0$.

$$\frac{1}{R} u'' + u = 0$$

sol: $u = A \sin\sqrt{R}\, y + B \cos\sqrt{R}\, y$

apply B.C. : @ $y=0,\ u=0 \rightarrow B=0$

@ $y=\pi,\ u=0 \rightarrow \sqrt{R} = n$ = integer 1, 2, ...

or $R = n^2$

small amplitude expansion:

$$u = \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots$$

$$R = n^2 + \epsilon R_1 + \epsilon^2 R_2 + \cdots$$

ϵ is a small parameter which is a measure of amplitude

then $u'' + R \sin u = 0$

gives $u'' + R\left\{u - \frac{u^3}{3!} + \cdots\right\} = 0$

or,

$$\epsilon u_1'' + \epsilon^2 u_2'' + \epsilon^3 u_3'' + \cdots + \left\{n^2 + \epsilon R_1 + \epsilon^2 R_2 + \cdots\right\} \cdot \left\{\epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 - \frac{\epsilon^3 u_1^3}{3!} \cdots\right\}.$$

equate like powers of ϵ,

$$\begin{cases} u_1'' + u_1 = 0 \\ u_2'' + u_2 = -R_1 u_1 \\ u_3'' + u_3 = R_1 u_2 - R_2 u_1 + \dfrac{u_1^3}{3!} \ \ldots \end{cases}$$

BC: $u_i(0) = 0$
$u_i(\pi) = 0$
$i = 1, 2, 3$

solutions:

① $u_1 = A \sin y$

② then $u_2'' + u_2 = -R_1 A \sin y$

$$u_2 = \alpha \sin y + \beta \cos y + \cancel{\gamma}^{-2R_1 A} y \cos y$$

BC: $u_2(0) = u_2(\pi) = 0$

$\Downarrow$ $\beta = 0$

$\Downarrow$ $0 = -2R_1 A \pi \cos \pi$

$\therefore R_1 = 0$

③ next eq. becomes.

$$u_3'' + u_3 = -R_2 u_1 + \frac{u_1^3}{3!} = R_2 A \sin y + \frac{1}{3!}(A^3 \sin^3 y)$$

$$= R_2 A \sin y + \frac{A^3}{3!} \sin y + \frac{A^3}{3!} \sin 3y$$

BC: $u_3(0) = u_3(\pi) = 0$

solvability cond: $-R_2 A + \dfrac{A^3}{3!} = 0$

if $A \neq 0$ then $R_2 = \dfrac{A^2}{3!}$

checking what we have:

$$u = \epsilon A \sin y + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots$$

$$R-1 = \epsilon \cancel{R_1}^{\,0} + \epsilon^2 R_2 + \cdots = k\epsilon^2 A^2$$

$$|u| = \epsilon A + \cdots \text{ H.O.T}$$

$$R-1 = k(\epsilon A)^2 + \overset{\text{H.O.T}}{\cdots} = k|u|^2$$