Model Problem

$$-\phi_t + \nabla^6\phi - R\phi_{xx} = \phi\phi_z$$

$$\phi = \phi_{zz} = \phi_{zzzz} = 0 \quad \text{on } z = 0, \pi$$

$$|\phi| \text{ bounded as } x \to \pm\infty$$

- According to linear theory:

$$\phi = \sum \beta_{np} e^{\sigma_{np} t} \sin pz \cos nkx$$

sol:

$$\phi = e^{\sigma t} \sin z \cos kx \qquad (n=1, p=1)$$

$$\sigma = k^2(R - R_L)$$

$$R_L = \frac{(1+k^2)^3}{k^2}, \qquad \underset{\substack{\downarrow \\ n=p=1}}{R_{L,min}} = \frac{27}{4}$$

now,

$$\mathcal{L}\phi = \phi\phi_z = -\phi_t + \nabla^6\phi - R\phi_{xx}$$

→ Try straight out expansion:

First iterate: $\mathcal{L}\phi_1 = 0$

2nd iterate: $\mathcal{L}\phi_2 = \phi_1\phi_{1z} = e^{2\sigma t} \sin z \cos z \cos^2 kx$

$$= \frac{1}{4} e^{2\sigma t} \sin 2z (1 + \cos 2kx)$$

3rd iterate: $\mathcal{L}\phi_3 = \phi_1\phi_{2z} + \phi_2\phi_{1z}$

$$\phi_2 = e^{2\sigma t}\{k_0 + k_2 \cos 2kx\} \sin 2z$$

- it has double growth rate.

$$\mathcal{L}\phi_3 = \frac{\partial}{\partial z}(\phi_1\phi_2) = e^{3\sigma t}\frac{\partial}{\partial z}\left\{\sin z \cos kx \,(k_0 + k_2\cos 2kx)\cdot \sin 2z\right\}$$

$$= e^{3\sigma t}\{\underbrace{\sin z \cos kx}_{} + \cdots\}$$

→ replication of 1st order problem ⟹ unbounded sol.

→ this is bad ± increased growth rate!

Bifurcation Approach:

- Since eigenfunction at criticality is steady (ie, time dependence, $\sigma_i = 0$), try to find a solution to nonlinear problem near criticality.

- Object is to solve: $\nabla^6\phi - R\phi_{xx} = \phi\phi_z$
(same prob. w/out ϕ_t).

- Let ϵ be a small parameter,

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \cdots$$

$$R = R_L + \epsilon R_1 + \epsilon^2 R_2 + \cdots$$

add + subtract $R_L\phi_{xx}$

$$\mathcal{L}\phi \equiv \nabla^6\phi - \overset{\text{a const.}}{R_L}\phi_{xx} = (R - R_L)\phi_{xx} + \phi\phi_z$$

Then, $\mathcal{L}\{\epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3\} = (\epsilon R_1 + \epsilon^2 R_2)(\epsilon\phi_{1xx} + \epsilon^2\phi_{2xx} + \epsilon^3\phi_{3xx}) + (\epsilon\phi_1 + \epsilon^2\phi_2 + \cdots)(\epsilon\phi_{1z} + \epsilon^2\phi_{2z} + \cdots)$

Equate coeffs. of like powers of ϵ:

$$\mathcal{L}\phi_1 = 0 \qquad : O(\epsilon^1)$$

$$\mathcal{L}\phi_2 = R_1\phi_{1xx} + \phi_1\phi_{1z} \qquad : O(\epsilon^2)$$

$$\mathcal{L}\phi_3 = R_1\phi_{2xx} + R_2\phi_{1xx} + \underline{\phi_2\phi_{1z} + \phi_1\phi_{2z}} \qquad : O(\epsilon^3)$$

Here

1st eq. has eigenvalue of (this has been done many times).

$$\phi_1 = \cos kx \sin z$$

$$\nabla^2\phi_1 = -(k^2+1)\cos kx \sin z$$

$$\nabla^6\phi_1 = -(k^2+1)^3\cos kx \sin z = -R_L k^2 \cos kx \sin z$$

where $R_L = \dfrac{(k^2+1)^3}{k^2}$

2nd eq.,

$$\mathcal{L}\phi_2 = -k^2 R_1 \cos kx \sin z + \cos^2 kx \sin z \cos z$$

$$= \text{- - - -} + \frac{1}{4}(1+\cos 2kx)\sin 2z$$

→ from here on, $R_1 = 0$, otherwise we cannot have sol. (i.e., will not remain bounded as variable grows)

$$\therefore \nabla^6\phi_2 - R_L\phi_{2xx} = \frac{1}{4}\sin 2z + \frac{1}{4}\cos 2kx \sin 2z$$

Sol: $\phi_2 = A\sin 2z + B\cos 2kx \sin 2z$

then: (subs. & equating like terms):

$$-64A\sin 2z = \frac{1}{4}\sin 2z \implies A = -\frac{1}{256}$$

$$B\{-(4k^2+4)^3 + 4k^2R_L\} = \frac{1}{4}$$

or,

$$B\{-64(k^2+1)^3 + 4k^2R_L\} = \frac{1}{4}$$

$$\text{or,}\quad B\Big\{-\underbrace{64k^2R_L}_{(k^2+1)^3} + 4k^2R_L\Big\} = \frac{1}{4}$$

$$\therefore\quad B = -1\big/(240k^2R_L)$$

$$\therefore\quad \phi_2 = -\frac{1}{256}\sin 2z - \frac{1}{240k^2R_L}\sin 2z\cos 2kx$$

3rd Eq:

- For the eq. for ϕ_3, note $\phi_2\phi_{1z} + \phi_1\phi_{2z} = \frac{\partial}{\partial z}(\phi_1\phi_2)$.
- Now, $\phi_1 = \cos kx \sin z$, then,

$$\phi_1\phi_2 = -\frac{1}{256}\sin 2z\sin z\cos kx - \frac{1}{240k^2R_L}\sin 2z\sin z\,\underbrace{\cos^2 kx}_{\cos kx}$$

$$= -\frac{1}{512}\cos kx\,(\cos z - \cos 3z) - \frac{1}{960k^2R_L}(\cos z - \cos 3z)\,(\cos x + \cos 3x)$$

$$\mathcal{L}\phi_3 = \underbrace{-k^2R_2\sin z\cos kx}_{} \overset{=}{\underset{\text{for solution}}{\;}} + \underbrace{\left(\frac{1}{512} + \frac{1}{960k^2R_L}\right)\sin z\cos kx}_{}$$

$$+\ (\text{other terms that don't influence our answer for } \phi_3)$$

solvability condition

$$\boxed{R_2 = \frac{1}{k^2}\left(\frac{1}{512} + \frac{1}{960k^2R_L}\right)}\quad \text{for nonsecular } \phi_3.$$

This procedure could be carried out further, but we have what we need.

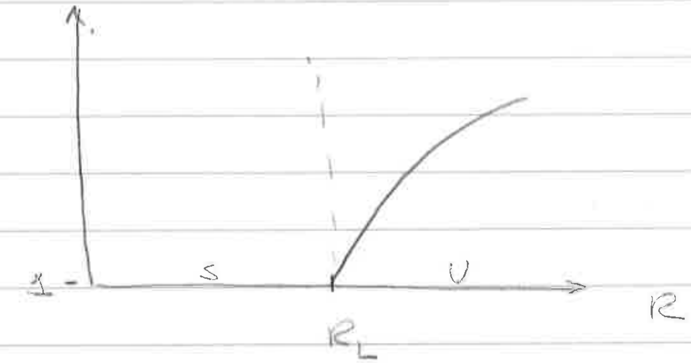
There exists a solution, time independent, of the form

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots$$
$$R = R_L + \epsilon^2 R_2 + \cdots$$

- Note that $R_2 > 0$. So that

$$R - R_L = \epsilon^2 R_2 > 0$$

- Therefore the solution exists only for $R > R_L$.



The earlier solution of this disturbance eq. was $u \equiv 1$. So we have two solutions,

(a) $u = 1$
(b) $u = 1 + \epsilon\phi_1 + \cdots$

(a) Time dependent w/ no spatial dependence; exists for all R.

(b) Exists for $R > R_L$, spatial dependence in x + z, time-independent.