Benard Problem

governing eqs.

$$\begin{cases} Pr^{-1}\{\underset{\sim}{V}_t + \underset{\sim}{U}\cdot\nabla\underset{\sim}{V}\} = -\nabla P + \nabla^2\underset{\sim}{V} + R\theta\underset{\sim}{k} \\ \theta_t + \underset{\sim}{V}\cdot\nabla\theta = \nabla^2\theta + w\underset{\sim}{k} \\ \nabla\cdot\underset{\sim}{V} = 0 \end{cases}$$

linear stability ::

$$\theta = \left\{A\cos\frac{\sqrt{3}}{2}kx\cos\frac{1}{2}ky + B\cos ky\right\}\sin\pi z$$

$\Rightarrow$ note: fn of x, y, z.

- similarly for $\underset{\sim}{V}$

$k^2 = k_x^2 + k_y^2$ — any geometry was possible with linear theory.. These were dependent on k^2 (rolls, rectangles, hexagons).

$A = 0,\ B \neq 0 \quad \Rightarrow$ roll

$A \neq 0,\ B = 0 \quad \Rightarrow$ rectangle

$A = \pm 2B \quad \Rightarrow$ hexagon

- How do we get the preferences on structure?
- Get the Landau eq. for A' and B'

$$A' = \sigma A - \alpha A^3 - \gamma AB^2$$

$$B' = \sigma B - \beta B^3 - \frac{1}{2}\gamma A^2 B$$

interaction terms.
from sin & cos terms.

α, β, γ are positive constants

$$\gamma + \beta = 4\alpha$$
$$\tau = R - R_L$$

- one solution is $A = B = 0$ $\longrightarrow$ stable $\boxed{\tau < 0}$
  (basic state)
  unstable $\boxed{\tau > 0}$

Other steady solutions

$\frac{d}{dt} = 0$

$$\begin{cases} \boxed{0} = \tau A - \alpha A^3 - \gamma A B^2 \\ \boxed{0} = \tau B - \beta B^3 - \frac{1}{2}\gamma A^2 B \end{cases}$$

Solutions:

1. $A = B = 0$

2. $B = 0$, $A^2 = \dfrac{\tau}{\alpha}$ $(\tau > 0)$

3. $A = 0$, $B^2 = \dfrac{\tau}{\beta}$ $(\tau > 0)$

4. $A \neq 0$, $B \neq 0$ $(\tau > 0)$

4.

$$\begin{cases} \tau = \alpha A^2 + \gamma B^2 \\ \tau = \frac{1}{2}\gamma A^2 + \beta B^2 \end{cases}$$

$\leftarrow$ divide thru by A

combining;

$$(\alpha - \tfrac{1}{2}\gamma) A^2 = (\beta - \gamma) B^2$$

or,

$$(\tfrac{1}{4}\gamma + \tfrac{1}{4}\beta - \tfrac{1}{2}\gamma) A^2 = (\beta - \gamma) B^2$$

or, $\frac{1}{4}(\cancel{\beta-\gamma})A^2 = (\cancel{\beta-\gamma})B^2$

or, $\boxed{A = \pm 2B}$ ← hexagons.

- To do stability analysis use:

$$A = \bar{A} + a$$
$$B = \bar{B} + b$$

substitute and linearize.

$$\begin{cases} a' = \sigma a - 3\alpha \bar{A}^2 a - \gamma \bar{B}^2 a - 2\gamma \bar{A}\bar{B} b \\ b' = \sigma b - 3\beta \bar{B}^2 b - \gamma \bar{A}\bar{B} a - \frac{1}{2}\gamma A^2 b \end{cases}$$

now substitute in A & B for cases ② → ④
and calculate eigenvalues (as done before).

This exercise shows that,

Hexagon → unstable
rolls → stable
rectangle → unstable

- Why do we see hexagons then? Because we modeled problem with free surface on top.

free
rigid

- In reality, free surface influenced by surface tension. This stabilizes hexagon.

given by $\quad e^{-i\alpha c_r t} \quad ; \quad \omega = \alpha c_r$

Hopf Bifurcation Theorem

3a. If as R increases through R_c, the real part of one pair of complex conjugate eigenvalues of A changes sign, then a solution bifurcates from $x \equiv 0$, $R = R_c$. This solution is periodic with freq. $\omega = \omega(R)$ which goes to ω_0 as $R \to R_c$, where ω_0 is a freq. of linear eigenfunction at criticality.

4a. If in addition to 3a all the eigenvalues of A have neg. real parts when $R < R_c$, then the bifurcation sol. is stable if supercritical and unstable if subcritical.

Thermal Convection Problem

$$\begin{cases} P_r^{-1} \{ \underset{\sim}{V}_t + \underset{\sim}{V} \cdot \nabla \underset{\sim}{V} \} = -\nabla p + \nabla^2 \underset{\sim}{V} + R\theta \hat{k} \\ \theta_t + \underset{\sim}{V} \cdot \nabla \theta = \nabla^2 \theta + w \\ \nabla \cdot \underset{\sim}{V} = 0 \end{cases}$$

Free - Free boundary conditions $\rightarrow$ $w = \dfrac{\partial u}{\partial z} = \dfrac{\partial w}{\partial z} = 0$

and, $\theta = 0$ $\longrightarrow$ on $z = 0, 1$

2-D problem: $v = 0$, $\frac{\partial}{\partial y} = 0$

by continuity , $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$

stream Fn: $u = \psi_z$, $w = -\psi_x$

Two eqs. are:

after taking curl →

$$\begin{cases} P_r^{-1}\left\{ \nabla^2\psi_t + \psi_z \nabla^2\psi_x - \psi_x \nabla^2\psi_z \right\} = \nabla^4\psi - R\theta_{(x)} \\ \theta_t + \psi_z\theta_x - \psi_x\theta_z = \nabla^2\theta - \psi_x \end{cases}$$

— nonlinear terms

Linear eigenfunction:

$\theta \sim \cos kx \sin \pi z$ $\qquad\qquad$ $\psi \approx \sin kx \sin \pi z$

$$R_L = \frac{(k^2+\pi^2)^3}{k^2}$$

Nonlinear Method:

1. - Ignore $\frac{\partial}{\partial t}$ term (criticallity), look for a small, finite amplitude steady solution.

2. - Seek a Landau eq. by writing

$$\theta = A(t) \cos kx \sin \pi z$$

generally: $\theta = \sum_{n=0}^{\infty} A_n e^{\sigma_n t} \cos kx \sin \pi z$

$$\psi = B(t) \sin kx \sin \pi z$$

Compute nonlinear contributions due to linear terms.

$$\psi_z \nabla^2\psi_x - \psi_x \nabla^2\psi_z = \pi B \sin kx \cos \pi z \, (-k^2-\pi^2)\, kB \cos kx \cdot$$

$$\cdot \sin\pi z - \left\{ Bk \cos kx \sin\pi z \left((-k^2-\pi^2) \cdot \right.\right.$$

$$\left.\left. \cdot\, \pi B \sin kx \cos \pi z \right) \right\} \quad = \underline{\underline{0}}$$

∴ these terms drop out.

$$\psi_z \theta_x - \psi_x \theta_z = B\pi \sin kx \cos\pi z \, (-kA \sin kx \sin\pi z) -$$

$$\left\{ Bk \cos kx \sin\pi z \cdot \pi A \cos kx \cos\pi z \right\}$$

$$= -AB\pi k \left(\sin^2 kx + \cos^2 kx \right) \sin\pi z \cos\pi z$$

$$= -\frac{AB\pi k}{2} \sin 2\pi z$$

(note: no x dependence only z dependence)

- subst. into eqs. of motion, solve for θ and ψ.
- Solution will be $\psi = 0$ and $\theta \sim A_0 \sin 2\pi z$

$$\Downarrow$$

$$B_0 = 0$$

③ Now leave in $\frac{\partial}{\partial t}$ term, put in

$$\psi = B_1 \sin kx \sin\pi z + \cancel{B_0}^{\,0} + \cancel{\text{harmonics}}^{\text{truncate}}$$

$$\theta = A_1 \cos kx \sin\pi z + A_0 \sin 2\pi z + \cancel{\text{harmonics}}^{\text{truncate}}$$

substitute into disturbance eqs.

$$\frac{d}{dt}$$

$$Pr^{-1}\left\{-(k^2+\pi^2)\,B_1'\,\sin kx\,\sin\pi z\right\} = (k^2+\pi^2)^2 B_1 \sin kx \cdot$$

$$\cdot \sin \pi z + R k_0 A_1 \sin kx \sin\pi z$$

or,

$$\boxed{-Pr^{-1}(k^2+\pi^2)\,B_1' = (k^2+\pi^2)^2 B_1 + R k_0 A_1} \qquad (a)$$

Other eq:

$$A_1' \cos kx \sin \pi z + A_0' \sin 2\pi z - \tfrac{1}{2} A_1 B_1 \pi k \sin 2\pi z$$

$$- B_1 k \cos kx \sin\pi z \cdot A_0\, 2\pi \cos 2\pi z = -(k^2+\pi^2)\cdot$$

$$\cdot A_1 \cos kx \sin\pi z - A_0\, 4\pi^2 \sin 2\pi z - k B_1 \cos kx \sin \pi z$$

note: $\sin\pi z \cos 2\pi z = \frac{1}{2}\left\{-\sin\pi z + \cancel{\sin 3\pi z}\right\}$ neglect harmonics

equating like terms,

$$\boxed{\begin{aligned} A_1' + A_0 B_1 k\pi &= -(k^2+\pi^2)A_1 - kB_1 \qquad (b)\\ A_0' - \tfrac{1}{2}\pi k A_1 B_1 &= -4\pi^2 A_0 \qquad (c)\end{aligned}}$$

put $\tau = (k^2+\pi^2)t$, $A_1 = k a_1$, $B_1 = (k^2+\pi^2) b_1$

$$A_0 = \frac{a_0}{\pi}$$

we get,

$$-Pr^{-1}(k^2+\pi^2)^3 b_1' = (k^2+\pi^2)^3 b_1 + R k^2 a_1$$

which becomes,

$$\boxed{-P_r^{-1} b_1' = b_1 + \frac{R}{R_L} a_1} \qquad (a')$$

and, $(k^2+\pi^2)ka_1' + k(\pi^2+k^2)a_0 b_1 = -(k^2+\pi^2)ka_1 - k(k^2+\pi^2)b_1$

or,

$$\boxed{a_1' + a_0 b_1 = -a_1 - b_1} \qquad (b')$$

and, $\frac{k^2+\pi^2}{\pi} a_0' - \frac{1}{2}\pi k^2 (\pi^2+k^2) a_1 b_1 = -\frac{4\pi^2}{\pi} a_0$

or,

$$\boxed{a_0' = -\frac{4\pi^2}{k^2+\pi^2} a_0 + \frac{1}{2}\pi^2 k^2 a_1 b_1} \qquad (c')$$

$$\begin{cases} a_1' = -a_1 - b_1 - a_0 b_1 & \\ b_1' = P_r\{-\gamma a_1 - b_1\} & ; \quad \gamma = \frac{R}{R_L} \\ a_0' = -b a_0 + \frac{1}{2}\pi^2 k^2 a_1 b_1 & ; \quad b = \frac{4\pi^2}{\pi^2+k^2} \end{cases}$$

if $a_1 = \frac{A_1}{\sqrt{\frac{1}{2}\pi^2 k^2}}$, $b_1 = \frac{B_1}{\sqrt{\frac{1}{2}\pi^2 k^2}}$ etc.

then,

$$\begin{cases} A_1' = -A_1 - B_1 - A_0 B_1 \\ B_1' = P_r\{-\gamma A_1 - B_1\} \\ A_0' = -bA_0 + A_1 B_1 \end{cases}$$

$\Longrightarrow$ Lorenz eqs.