Even manifolds

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Definitions

A $2k$-dimensional manifold M^{2k} is called *mod-2-even* if the intersection pairing $H_k(M;\mathbb{F}_2) \otimes H_k(M;\mathbb{F}_2) \to \mathbb{F}_2$ has trivial squares.

If M^{2k} is oriented there is an integral intersection pairing $H_k(M;\mathbb{Z}) \otimes H_k(M;\mathbb{Z}) \to \mathbb{Z}$ and M is called *even* provided this pairing has even squares.

History

Wu proved in 1950 that M^{2k} is mod-2-even if and only if the Wu class v_k vanishes. In fact he proved the evaluation map $H_k(M;\mathbb{F}_2) \xrightarrow{v_k} \mathbb{F}_2$ is the mod-2 squaring map.

Donaldson's work in the early 80's raised the question of which forms could be forms for even 4-manifolds. This gave rise to the $\frac{11}{8}$ths conjecture for Spin 4-manifolds and the $\frac{10}{8}$ths conjecture for even 4-manifolds.

One question was whether an even manifold could have index 8. (By van der Blij the signature of any even manifold is divisible by 8.) Habegger constructed such an example and then the algebraic geometers pointed out that Enriques surfaces have this property as well.

Furuta proved a bit more than the $\frac{10}{8}$ths conjecture for Spin 4-manifolds and using this Bohr and Lee &Li proved the $\frac{10}{8}$ths conjecture for even 4-manifolds with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}/2^k\mathbb{Z}$.

The parity of the integral squaring function is given by the composition $H_k(M;\mathbb{Z}) \longrightarrow H_k(M;\mathbb{F}_2) \xrightarrow{v_k} \mathbb{F}_2$ so

M is even if and only if $v_k \in H^k(M;\mathbb{F}_2)$ vanishes under the map in the Universal Coefficients Theorem

$$H^k(M;\mathbb{F}_2) \to Hom(H_k(M;\mathbb{Z}), \mathbb{F}_2)$$

This was also observed by Bohr and Lee & Li in their studies of even 4-manifolds.

Let P_∞ denote the Prüfer group $\mathbb{Z}/2^\infty\mathbb{Z}$, the direct limit of the finite cyclic 2-groups.

Theorem

Let $\iota\colon \mathbb{F}_2 \to P_\infty$ be the injection. Then M^{4k} is even if and only if $\iota_*(v_{2k}) = 0 \in H^{2k}(M; P_\infty)$.

Proof.

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Ext}\big(H_{2k-1}(M;\mathbb{Z});\mathbb{F}_2\big) & \longrightarrow & H^{2k}(M;\mathbb{F}_2) & \longrightarrow & \mathrm{Hom}(H_{2k}(M;\mathbb{Z});\mathbb{F}_2) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathrm{Ext}\big(H_{2k-1}(M;\mathbb{Z});P_\infty\big) & \longrightarrow & H^{2k}(M;P_\infty) & \longrightarrow & \mathrm{Hom}(H_{2k}(M;\mathbb{Z});P_\infty) & \to 0
\end{array}$$

The right-hand vertical map is injective since ι is. The lower right horizontal map is injective since the injectivity of P_∞ implies that the $\mathrm{Ext}\big(H_{2k-1}(M;\mathbb{Z});P_\infty\big)$ in the lower left vanishes. □

Mod 2 classes which vanish with Prüfer coefficients?

Given any mod 2 cohomology class $x \in H^k(X; \mathbb{F}_2)$ let $\hat{x} = \iota_*(x)$.
The following are equivalent:

- $\hat{x} = 0$.
- The composition $H_k(X;\mathbb{Z}) \to H_k(X;\mathbb{F}_2) \xrightarrow{x} \mathbb{F}_2$ is trivial.
- There exists a Prüfer character ψ making
$$\begin{array}{ccc} H_k(X;\mathbb{F}_2) & \xrightarrow{x} & \mathbb{F}_2 \\ \downarrow \beta & & \downarrow \iota \\ H_{k-1}(X;\mathbb{Z}) & \xrightarrow{\psi} & P_\infty \end{array}$$
commute.

More on hat-classes

Notice that as soon as the Bockstein $H_k(X;\mathbb{F}_2) \to H_{k-1}(X;\mathbb{Z})$ is non-zero, there exist non-trivial classes x with $\hat{x} = 0$.

The condition $\hat{x} = 0$ implies $\delta x = 0 \in H^{k+1}(X;\mathbb{Z})$.

A Prüfer character determines x.

Say two Prüfer characters are *commensurate* if the induce the same map ${}_2H_{k-1}(X;\mathbb{Z}) \to \mathbb{F}_2 \subset P_\infty$.

The class x determines the Prüfer character up to commensurability.

$\hat{v}_k$ structures.

Define $BSO\langle \hat{v}_k \rangle$ as the homotopy fibre of $BSO \to K(P_\infty, k)$ made into a fibration $BSO\langle \hat{v}_k \rangle \to BSO$.

Given a bundle $\xi\colon X \to BSO$ a $\hat{v}_k$ *structure* on ξ is a lift of ξ to $BSO\langle \hat{v}_k \rangle$.

The set of lifts is an $H^{k-1}(X; P_\infty)$ torsor.

The set of lifts with a fixed Prüfer character is an ${}_2H^{k-1}(X; P_\infty)$ torsor.

Given a Prüfer character ψ_1 for a $\hat{v}_k$ structure on a bundle and given a second commensurate Prüfer character ψ_2, there is a lift of the underlying bundle whose Prüfer character is ψ_2.

Manifold structures and bundle structures

An orientable $4k$-dimensional manifold is even if and only if its tangent bundle has a $\hat{v}_{2k}$-structure.

An orientable $4k+1$-dimensional manifold with a $\hat{v}_{2k}$-structure has trivial de Rham invariant.

Covers of even manifolds are even.

The case of $\hat{v}_2 = 0$.

For oriented bundles, $w_2 = v_2$. Several things happen in this case which fail in general.

- The Whitney sum of oriented bundles with $\hat{v}_2$ structures has a $\hat{v}_2$ structure.
- $v_2 = 0$ if and only if the bundle has a Spin structure. A Spin structure induces a unique $\hat{v}_2$ structure which will have trivial Prüfer character.
- $\hat{v}_2 = 0$ implies the bundle has a Spin^c structure. A $\hat{v}_2$ structure induces a unique Spin^c structure and the Prüfer character determines the first Chen class.
- $\hat{v}_2$ structures with trivial Prüfer character correspond bijectively to Spin structures.

Additional results for $\hat{v}_2$

A bundle over a finite CW complex which has a $\hat{v}_2$ structure has a finite 2^k cyclic cover for which the induced structure is Spin (Bohr and Lee & Li).

An orientable bundle with a $\hat{v}_2$ structure has a $\hat{v}_{4k+2}$ structure for all integers $k > 0$.

This is an analogue of the result that $v_{4k+2} = 0$ for a Spin bundle.

If $w_2(\xi) = x \cup x$, ξ has a $\hat{v}_2$ structure.

Proof.

Let λ be a line bundle with $w_1(\lambda) = x$. Then $\xi \oplus \lambda \oplus \lambda$ is Spin so ξ is $\hat{v}_2$ if and only if $\lambda \oplus \lambda$ is. But $\lambda \oplus \lambda$ pulls back from $K(\mathbb{Z}/2\mathbb{Z}, 1)$ and $H^2\big(K(\mathbb{Z}/2\mathbb{Z}, 1); P_\infty\big) = 0$. □

Corollary

The complexification of a real vector bundle has a $\hat{v}_2$ structure.

Groups of Schur multiplicator 0.

A finite group G has Schur multiplicator 0 if and only if $H_2(G;\mathbb{Z}) = 0$. Generalize to any discrete group.

Theorem

If G is a discrete group of Schur multiplicator 0 and if M is a simply-connected Spin manifold on which G acts freely, M/G is $\hat{v}_2$.

Examples

Any fundamental group of a rational homology 3-sphere has Schur multiplicator 0. There are two families of finite 2-group examples, cyclic 2-groups and quaternionic 2-groups.

Group actions on manifolds

Theorem

Let G be a finite 2 group of Schur multiplicator 0 and let M be a simply-connected $8k+4$ dimensional, Spin manifold on which G acts freely. Suppose the signature of M, $\sigma(M)$ satisfies $\sigma(M) \equiv 16 \bmod 32$. Then

$$G \text{ is } \mathbb{Z}/2\mathbb{Z} \text{ or } \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Lemma

If G is cyclic or quaternion of order 2^k and M is a simply-connected $8k+4$ dimensional, Spin manifold on which G acts freely then 2^{k+3} divides $\sigma(M)$. If 2^{k+4} does not divide $\sigma(M)$, M/G can not be Spin by Ochanine.

Bordism

There is a bordism theory of $\hat{v}_2$ manifolds which is closely related to Spin^c bordism. Stong proves that Spin^c bordism has 2-torsion subgroup $\mathbf{T}_k$ in degree k which is finite of exponent 2 and has a free-abelian part isomorphic to the polynomial algebra $\mathbb{Z}[c_1, p_1, \cdots]$ where c_1 has degree 2 and p_i has degree $4i$ for all positive integers i. Write $\mathbf{P}_*$ for the subalgebra generated by the p_i and $\mathbf{C}_*$ for the rest. Then $MSpin^c_k = \mathbf{P}_k \oplus \mathbf{C}_k \oplus \mathbf{T}_k$.

$$MSO\langle\hat{v}_2\rangle_k = \mathbf{P}_k \oplus \mathbf{T}_k \oplus \Sigma^{-1}(\mathbf{C}_k \otimes P_\infty)$$