Fibrations, cofibrations and related results, II

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Configuration spaces, braids and applications

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More results in the $M \times \mathbb{R}$ case

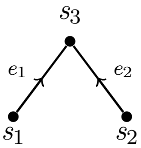
To describe the various Thom spaces which go into the decomposition of $\Sigma \mathit{Conf}\,(M \times \mathbb{R}, S)$, begin by discussing 1-dimensional CW complexes. Given a finite set S, an *ordered 1-complex* Γ is a CW complex with vertex set S and a set of edges $\mathcal{E}(\Gamma)$. Each edge is oriented and the set of edges is ordered.

Given an edge $e \in \mathcal{E}(\Gamma)$ define $A_e = A_{s_2,s_1}$ where e starts at vertex s_1 and ends at vertex s_2. Define $A_\Gamma = A_{e_1} \cdots A_{e_k}$ where $e_1, \ldots, e_k$ are the edges of Γ in order. These conventions set up a bijection between products of the A's and ordered 1-complexes. It can be shown that

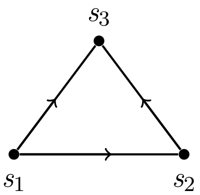
$$A_\Gamma \neq 0 \text{ if and only if } H_1(\Gamma) = 0.$$

Hence $A_\Gamma \neq 0$ if and only if each path component of Γ is a tree or a single vertex. If we continue the arboreal theme by calling components with single vertices *seeds*, then $A_\Gamma \neq 0$ if and only if Γ is a *forest*.

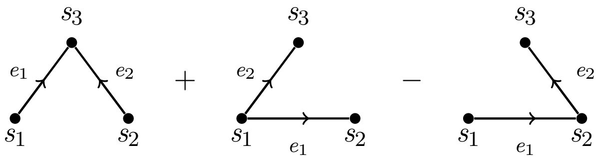
The key to the proof of the previous result is the graphical version of the three-term relation which can be described using ordered 1-complexes. Say that a vertex s_3 *supports an incoming three-term relation* provided there are at least two edges which have s_3 as an incoming end. There may well be additional vertices and edges which are not drawn in the picture.



Draw a new edge from s_1 to s_2, provided $e_1 < e_2$, to get the triangle on the next page.



The three-term relation says that a combination of three ordered 1-complexes is 0. They are obtained by combining the three ways of deleting an edge from the triangle, and reordering an edge or two.



$$A_{\Gamma_3} \quad + \quad A_{\Gamma_1} \quad - \quad A_{\Gamma_2} \quad = \quad 0$$

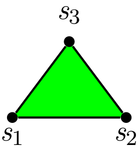
Theorem

Given a vertex which supports a three-term relation then for the three graphs described above

$$H_*(\Gamma_3) \cong H_*(\Gamma_2) \cong H_*(\Gamma_1)$$

A graph partitions its set of vertices by saying two are equivalent if and only if they lie in the same path component. All three graphs yield the same partition.

Proof.



$\square$

Certain collections of ordered 1-complexes give a basis for $H^*(Conf(\mathbb{R}^n, S); \mathbb{Z})$. Clearly the ordered 1-complexes in a basis must be a forest, but there are more forests than basis elements whenever $|S| > 2$.

One basis is given by the *admissible forests*. To define when an forest is admissible, it is first necessary to order S. Then we can orient an edge by starting at the smaller vertex and going to the larger. We can order the edges using lexicographical order. A forest is *admissible* provided no vertex supports an incoming three-term relation using the above orientations and ordering.

Theorem

If $\mathcal{A}(S)$ is the set of admissible forests on the ordered vertex set S then the elements A_Γ for all $\Gamma \in \mathcal{A}(S)$ are an additive basis for $H^*(Conf(\mathbb{R}^n, S); \mathbb{Z})$, $n \geqslant 2$.

For any forest Γ there is a diagonal

$$\Delta_\Gamma \colon X^{\pi_0(\Gamma)} \to X^S$$

defined by $(\Delta_\Gamma(\iota))(s) = \iota([s])$ where $[s] \in \pi_0(\Gamma)$ is the path component of Γ containing s. If X is a manifold, let ν_γ be the normal bundle of $X^{\pi_0(\Gamma)}$ in X^S. Note it is a sum of various tangent bundles of X pulled back to $X^{\pi_0(\Gamma)}$.
Let $\mathfrak{A}(S)$ be a set of forests such that the collection A_Γ, $\Gamma \in \mathfrak{A}(S)$ is a basis for $H^*\big(Conf\,(\mathbb{R}^n, S)\,;\mathbb{Z}\big)$.
Then

$$\Sigma\, Conf\left(M \times \mathbb{R}^1, S\right) \cong \bigvee_{\Gamma \in \mathfrak{A}(S)} \Sigma\, T(\nu_\Gamma)$$

Remark: The admissible basis has an additional property that there is an algorithm for writing any forest as a linear combination of admissible forests.

The top representation

The sub-group of $H^*\big(Conf\,(\mathbb{R}^n, S)\,;\mathbb{Z}\big)$ generated by all A_Γ with the associated partition fixed form a subgroup of $H^{(n-1)(|S|-r)}\big(Conf\,(\mathbb{R}^n, S)); \mathbb{Z}\big)$ where r is number of path components of Γ, which is also the number of elements in the partition.
Hence the highest non-trivial cohomology group of $Conf\,(\mathbb{R}^n, S)$ is in dimension $(n-1)(|S|-1)$. Classes A_Γ in this dimension come from forests which are connected and vice versa. From this one sees that $H^*\big(Conf\,(\mathbb{R}^n, S)\,;\mathbb{Z}\big)$ is built up out of tensor products of top dimensional groups for various subsets of S.

Example

Let $S = \{1, 2, 3, 4, 5\}$ and let $\{\{1, 2, 3\}, \{4, 5\}\}$ be a partition. Then a summand of $H^{3(n-1)}\big(Conf\,(\mathbb{R}^n, S)\,; Z\big)$ is a tensor product of the top group for 3 points tensor the top group for 2 points.

The top representation (continued)

Recall every forest partitions the set S and by taking the cardinality of each set in the partition, we get a partition of the integer $|S|$. Given any two forests with the same integer partition, there are permutations of S which take one to the other.

Hence under the action of the symmetric group, the cohomology decomposes into equivariant summands corresponding to integer partitions of $|S|$. The cohomological degree of the corresponding A_Γ can be determined from the integer partition.

Leher & Solomon wrote down the Poincaré character for the rational representation on $H^*(\mathit{Conf}\,(\mathbb{R}^n, S)\,;\mathbb{Q})$.

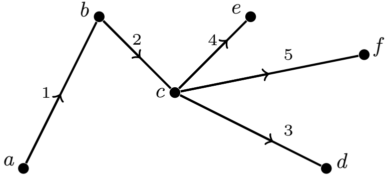
Fred & I worked out the representation over $\mathbb{Z}$ as a sum of tensor products of representations induced from Young subgroups: i.e. subgroups of the form

$$\Sigma_{S_1} \times \cdots \times \Sigma_{S_k} \subset \Sigma_{S_1 \amalg \cdots \amalg S_k}$$

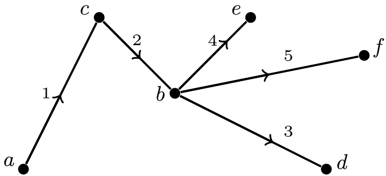
The top representation (continued)

The top representation comes from the partition with one subset (or one integer). For example, one basis for this group consists of the admissible trees. Notice however that a permutation applied to an admissible tree is often not admissible.

Here is an admissible tree on $\{a, b, c, d, e, f\}$ ordered alphabetically.



The permutation $\begin{pmatrix} abcdef \\ acbdef \end{pmatrix}$ applied to the above admissible tree gives the tree

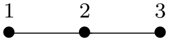


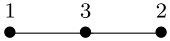
which is no longer admissible: the orientation on the edge between b and c has the "wrong" orientation.

If we reorient this edge "correctly", then c supports an incoming three-term relation.

There are other bases for this top rep which are useful. A *rooted tree* is a tree with a distinguished vertex. A tree is called *linear* provided every vertex has valence 1 or 2. There must be exactly two vertices of valence 1. Fix one of the vertices of valence one, say $\mathbf{v}$.

A *linear rooted tree with root* $\mathbf{v}$ is a linear tree with vertex set S with one vertex of valence 1 being $\mathbf{v}$. Direct the edges so that you start at $\mathbf{v}$ and just keep going. Number the edges in the order in which they appear along the tree starting at the root. Here are the two rooted trees on $\{1, 2, 3\}$:





Theorem

The set of linear trees with root $\mathbf{v}$ *is a basis for* $H^{(n-1)(|S|-1)}(Conf\,(\mathbb{R}^n, S)\,;\mathbb{Z})$

This shows that the top rep as an integral representation of the of the symmetric subgroup of Σ_S fixing $\mathbf{v}$ is free.

$C(M, X)$

Fix a countably infinite set, say $\mathbf{N}$. Given a space M and a based space $(X, *)$ consider the space of maps $f\colon \mathbf{N} \to M \times X$. Define the *support* of f to be the subset of $\mathbf{N}$ such that $f(s) \neq *$. Let $E(M, X)$ be the subspace of functions whose support is a finite subset, say S, and such that the composition $S \xrightarrow{f} M \times X \to M$ is injective.

Define

$$C(M, X) = E(M, X) / \approx$$

where $\approx$ is the equivalence relation generated by the following two types of relations:

1. $f_1 \approx f_2$ if f_1 and f_2 have the same support and are equal when restricted to that support
2. $f_1 \approx f_2$ if there exists a bijection $\phi\colon \mathbf{N} \to \mathbf{N}$ such that $f_1 \circ \phi = f_2$

$C(M, X)$ (canonical identifications)

Since any two countably infinite sets are bijectively equivalent, the choice of set $\mathbf{N}$ is not usually important: if it is we will write $C_{\mathbf{N}}(M, X)$.
Any bijection $\phi\colon \mathbf{N}_1 \to \mathbf{N}_2$ induces a homeomorphism $C_{\mathbf{N}_2}(M, X) \to C_{\mathbf{N}_1}(M, X)$. Thanks to relation (2), any two ϕ induce identical maps. In particular, any two versions of this construction can be canonically identified.

In a few pages we will also need the following related remark. Define the braid space $B(M, S)$ to be $\mathit{Conf}(M, S)/\Sigma_S$. Given another finite set T of the same cardinality, any choice of bijection $\phi\colon T \to S$ induces a homeomorphism $\phi\colon \mathit{Conf}(M, S) \to \mathit{Conf}(M, T)$ which descends to a homeomorphism $B(M, S) \to B(M, T)$.
The remark is that two different ϕ's induced the same map on the braid spaces so they may be canonically identified. In the sequel we will write $B_k(M)$ whenever the index set has cardinality k.

$C(M, X)$ (continued)

Filter $C(M, X)$ by letting $F_k(M, X) \subset C(M, X)$ be the image of all functions in $E(M, X)$ whose support has at most k elements. Notice both relations (1) and (2) preserve the cardinality of the support of the functions.

Define $D_k(M, X)$ to be the cofibre of the inclusion $F_{k-1}(M, X) \subset F_k(M, X)$. If $(X, *)$ is an NDR pair then so is $(F_k(M, X), F_{k-1}(M, X))$ and we can identify the cofibre. Fix a finite set of cardinality k, $S \subset \mathbf{N}$. The composition $\mathit{Conf}(M, S) \times X^S \to F_k(M, X)$ is onto and factors through the orbit space $\mathit{Conf}(M, S) \times_{\Sigma_S} X^S$.

Therefore map $\mathit{Conf}(M, S) \times_{\Sigma_S} X^S \to D_k(M, X)$ is onto and if $F\Delta \subset X^S$ is the set of points with at least one coordinate the base point, then

$$\left(\mathit{Conf}(M, S) \times_{\Sigma_S} X^S\right) \Big/ \left(\mathit{Conf}(M, S) \times_{\Sigma_S} F\Delta\right) \to D_k(M, X)$$

is a homeomorphism.

$C(M, X)$ (continued)

Any other choice of finite set of cardinality k gives a similar identification and any choice of bijection induces the same map. With a bit of fiddling, one can rewrite $D_k(M, X)$ as

$$D_k(M, X) = Conf(M, S) \ltimes_{\Sigma_S} X^{[S]}$$

where $X^{[S]}$ denotes the S-fold smash product.

We would like to extend the natural map $\mathfrak{f}_k \colon F_k(M, X) \to D_k(M, X)$ to a map $C(M, X) \to D_k(M, X)$ but this is not usually possible.
It is however possible to do so stably.

Stable splitting of $C(M, X)$

To describe the extension, first try the most naive thing you (I?) can think of: given f and T any finite set of cardinality k, define $f|_T \colon \mathbf{N} \to M \times X$ by $f|_T(s) = \begin{cases} f(s) & s \in T \\ (m, *) & s \notin T \end{cases}$ where $m \in M$ is any point you like. Note $f|_T$ is a point in $F_k(M, X)$ and hence also a point in $D_k(M, X)$. The obvious problem is which subset to take. The solution in situations like this where there is no natural choice is to take all choices.

Let $\mathbf{N}' = \binom{\mathbf{N}}{k}$ denote the set of all subsets of $\mathbf{N}$ of cardinality k. Note $\mathbf{N}'$ is also countably infinite.

Define a map

$$h_k\colon C_{\mathbf{N}}(M,X) \to C_{\mathbf{N}'}(B_k(M), D_k(M,X))$$

as follows. Recall that a point in $E(M,X)$ is a map $f\colon \mathbf{N} \to M \times X$ satisfying some additional conditions. We need to define a map $h_k(f)\colon \mathbf{N}' \to B_k(M) \times D_k(M,X)$. An element of $\mathbf{N}'$ is a set $S \subset \mathbf{N}$ of cardinality k. Therefore $f|_S$ is a point in $F_k(M,X)$ and hence a point in $D_k(M,X)$, denote this point by $[f]_S$.

If S is in the support of f, let $\langle f\rangle_S \in B_k(M)$ denote the image in $B_k(M)$ of the point in $Conf(M,S)$ given by the composition

$$S \subset \mathbf{N} \xrightarrow{\ f\ } M \times X \to M$$

If S is not in the support of f, define $\langle f\rangle_S$ to be any point you like in $B_k(M)$. Define

$$h_k(f)(S) = \langle f\rangle_S \times [f]_S$$

Check that the support of $h_k(f)$ is the set of all subsets of cardinality k contained in the support of f. The additional requirements to be a point in $C_{\mathbf{N}'}(B_k(M), D_k(M,X))$ can be checked.
Since $B_k(M)$ is a manifold, it embeds in $\mathbb{R}^K$ for some K and so there is a map

$$h_k\colon C_{\mathbf{N}}(M,X) \to C_{\mathbf{N}'}\left(\mathbb{R}^K, D_k(M,X)\right)$$

Moreover, $D_k(M,X)$ is path-connected and so a theorem of Peter May supplies a map
$C_{\mathbf{N}'}\left(\mathbb{R}^K, D_k(M,X)\right) \to \Omega^K\Sigma^K D_k(M,X)$ and so we get a map

$$h_k\colon C_{\mathbf{N}}(M,X) \to \Omega^K\Sigma^K D_k(M,X)$$

and a commutative diagram

$$\begin{array}{ccc} F_k(M,X) & \xrightarrow{\quad \mathfrak{f}_k \quad} & D_k(M,X) \\ \cap & & \downarrow \\ C(M,X) & \xrightarrow{\quad h_k \quad} & \Omega^K \Sigma^K D_k(M,X) \end{array}$$

To belabor the point, we could adjoint h_k and note

$$\Sigma^K \mathfrak{f}_k \colon \Sigma^K F_k(M,X) \subset \Sigma^K C(M,X) \xrightarrow{\mathrm{ad} h_k} \Sigma^K D_k(M,X)$$

The K certainly increases as k increases, but if we pass to the stable world we get an equivalence

$$QC(M,X) \cong Q \bigvee_{k=1}^{\infty} D_k(M,X)$$

There are many results concerning this construction and its pieces.

Remark: $QC(M, S^0) = Q \bigvee_{k=1}^{\infty} B_k(M)$

Remark: For an appropriate K there is a map

$$\Sigma^{k \cdot K} D_k(M, X) \to D_k\left(M, \Sigma^K X\right)$$

which is a homotopy equivalence.

Remark: A theorem of Jeff Caruso says

$$\Omega C(M, \Sigma X) \cong C(M \times \mathbb{R}, X)$$

This generalizes the case in which $M = \mathbb{R}^n$ due to Peter May.

Remark: If X is path connected, then a theorem of Bödigheimer says that $C(M, X)$ is weakly-homotopy equivalent to a space of sections of a certain bundle $E \to M$. The bundle is formed from the tangent bundle $T \to M$ as follows. Take the fibre-wise one point compactification of the tangent bundle: take $T \amalg M \to M$ and topologize so that each fibre is the one-point compactification of the fibre of T. This bundle is denoted $\hat{T}$ and it has a section at infinity. Then E is the reduced fibre-wise smash of $\hat{T}$ with X. This a bundle with fibre the reduced suspension $\Sigma^n X$. The reduced suspension has a base point and so E has a section at infinity, σ_∞. The *support* of a section $\sigma\colon M \to E$ is the set of all points $m \in M$ such that $\sigma(m) \neq \sigma_\infty(m)$. A section has compact support provided the closure of the support is compact.
Bödigheimer says that $C(M, X)$ is weakly-homotopy equivalent to the space of sections with compact support of $E \to M$.

Corollary

$C(M, X)$ *is a proper homotopy invariant of* M.