Local Surgery: Foundations and Applications

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In sections 1 through 7 of this paper we collect the basic results of local surgery theory. Sections 1 through 6 merely collect results found in Quinn [16]. We incorporate a twist motivated by Barge's work [3], and rearrange the material to suit our needs in sections 7, 8, and 9. The theory parallels the integral theory until one goes to calculate the normal map set. Here Quinn found an extra obstruction (see section 6).

Section 7 is a general section in which we try to handle Quinn's extra obstruction and the surgery obstruction simultaneously. We give two applications of the general theory to embedding theory in sections 8 and 9. Hopefully more applications will be forthcoming.

We must apologize to the many people who have worked in this area but are not mentioned here. A combination of ignorance and lack of space prevents a detailed look at the historical foundations of local surgery. Our thanks go to Frank Quinn for helpful conversations on the material in [16].

§1. Basics.

We begin by fixing some notation. We let P denote an arbitrary subset of primes in Z, and we let P' denote the complementary set. We let R denote the subring of Q consisting of all rationals with

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denominators relatively prime to the primes in P, and we use R'
to denote the complementary subring.

We use a localization process which preserves the geometry coming from π_1. If X is a CW complex, consider the map

$$1.1) \quad u: X \to K(\pi_1, 1) = B\pi$$

which classifies the universal cover. We convert u to a fibration and apply the fibrewise localization functor of Bousfield - Kan [4] p. 40. We get a commutative diagram

$$\begin{array}{ccccc} \tilde{X} & \to & X & \to & B\pi \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{X}_{(P)} & \to & X_{(P)} & \to & B\pi \end{array}$$

where $\tilde{X}_{(P)}$ is the usual localization of the simply connected space $\tilde{X}$.

A map $f: X \to Y$ is a P-equivalence if the induced map $f_P: X_{(P)} \to Y_{(P)}$ is a homotopy equivalence. A space is P-local if the map $X \to X_{(P)}$ is a homotopy equivalence.

§2. Local Poincaré spaces.

We say that a P-local space, denoted X, is a simple P-local P. D. space if there exists a finite complex, K, and a P-equivalence $\rho: K \to X$, together with

i) a homomorphism $w_1: \pi_1 X \to Z/2$ and

ii) a class $[X] \in H_m(X; R^t)$ such that

$$\xi \cap : \mathrm{Hom}_\Lambda(C_{m-*}(X); R\pi) \to C_*(X) \otimes_\Lambda R\pi$$

is a simple equivalence, where $\Lambda = Z\pi$ and ξ is a chain representative for [X]. For more details, see Anderson [1] p. 39 and Wall [24] p. 21. In particular, the notion of a simple P-local Poincaré n-ad should be clear.

Remark: The choice of K and ρ determines the P-local simple homotopy type of X.

Definition 2.1: An oriented P-local Poincaré space consists of a simple P-local P. D. space X; a specific choice of [X]; and a fixed P-local simple homotopy type for X. We denote such a gadget by (X;[X]), suppressing the simple type.

§3. Normal maps.

We agree to let C stand for O, PL, or TOP: then $BSC_{(P)}$ denotes the localization of the classifying space BSC. Given a C-manifold, M, we have the map $u: M \to B\pi$ (1.1). The homomorphism $w_1: \pi_1 M = \pi \to Z/2$ gives rise to a line bundle λ over $B\pi$. If ν_M denotes the normal bundle of M, $\nu_M \oplus u^*(-\lambda)$ is orientable. Hence we get a map

$$3.1) \quad \eta_M: M \to BSC \times B\pi$$

from which we can recover both u and ν_M. In fact, w_1 can be used to get a map $\mu: BSC \times B\pi \to BC$ such that $\mu \circ \eta_M = \nu_M$ and $(2^{nd}\text{ projection}) \circ \eta_M = u$.

Definition 3.2: An "oriented" manifold is a manifold M together with a choice of class $[M] \varepsilon H_m(M, \partial M; Z^t)$.

Remark: The bundle $\nu_M \oplus u^*(-\lambda)$ is now oriented.

Definition 3.3: A degree 1, P-normal map is a map $f\colon M \to X$ and a map $\zeta_P\colon X \to BSC_{(P)}$ such that

i)
$$\begin{array}{ccc} M & \xrightarrow{f} & X \\ \downarrow \eta_M & & \downarrow \zeta_P \times u \\ BSC \times B\pi_1 M & \to & BSC_{(P)} \times B\pi_1 X \end{array}$$
commutes

ii) $f^* w_1$ is the first Stiefel-Whitney class of M, and

iii) $f_*[M] = [X]$.

There is an obvious generalization to n-ads. This permits us to define the set of bordism classes of degree 1, P-normal maps over the oriented Poincaré complex $(X;[X])$.

We denote this set by $N(X;[X])$.

§4. Surgery.

Our goal is to define and interpret a surgery obstruction map

4.1) $\sigma_* : N(X;[X]) \to L_m^s(R\pi_1 X; w_1)$.

To begin, we form the pullback
$$\begin{array}{ccc} E(\zeta_P) & \to & BSC \\ | & & | \\ X & \xrightarrow{\zeta_P} & BSC_{(P)} \end{array}$$

Given a degree 1, P-normal map $f\colon M \to X$ and ζ_P, we get a map $\hat{f}\colon M \to E(\zeta_P)$. We will need

Lemma 4.2: Let K be a finite complex and let $g\colon K \to F$ be a map. Suppose there exists a finite complex, L, and a P-equivalence $L \to F_{(P)}$. Then there exists a finite complex L_∞ such that

i) g factors as $K \xrightarrow{g_\infty} L_\infty \xrightarrow{r_\infty} F$

ii) r_∞ is a P-equivalence.

<u>Proof</u>: We shall define a series of spaces L_i and maps g_i, r_i such that $g = r_i \circ g_i$. Let $L_0 = K$; $g_0 = 1_K$; $r_0 = g$.

Since $\pi_1 F$ is finitely presented, we can attach a finite number of cells to L_0 to get a complex L_1 and a map $r_1 \colon L_1 \to F$ which is an isomorphism on π_1. The map g_1 is the obvious inclusion $K = L_0 \subset L_1$.

Suppose we have constructed L_i, g_i, and r_i so that $(r_i)_P \colon (L_i)_{(P)} \to F_{(P)}$ is an i-equivalence. Then $\pi_{i+1}(F_{(P)}, (L_i)_{(P)})$ is a finitely generated $R\pi$-module (e.g. [24] Lemma 2.3 (b)). We can choose a finite set of elements in $\pi_{i+1}(F, L_i)$, attach cells to get L_{i+1}, and extend the maps. As usual, $(r_{i+1})_P$ is now an (i+1)-equivalence.

Construct L_i, g_i, r_i for $i = \max(\dim L, 2)$. Then Lemma 2.3 of Wall [24] shows that $\pi_{i+1}(F_{(P)}, (L_i)_{(P)})$ is s-free over $R\pi$. By adding more (i+1)-cells to L_i, we can assume it free and choose elements in $\pi_{i+1}(F, L_i)$ to give a basis for $\pi_{i+1}(F_{(P)}, (L_i)_{(P)})$. Then $L_\infty = L_{i+1}$; $g_\infty = g_{i+1}$; $r_\infty = r_{i+1}$ satisfy all the requirements. //

Once upon a time we had a map $\hat{f} \colon M \to E(\zeta_P)$. Use Lemma 4.2 to find a finite complex K and a factorization of $\hat{f} \colon M \xrightarrow{g} K \xrightarrow{r} E(\zeta_P)$. Over $E(\zeta_P)$ we have a C-bundle, $\zeta \oplus \lambda$, where $\zeta \colon E(\zeta_P) \to BSC$ and λ is the line bundle given by

$\lambda: E(\zeta_P) \to X \xrightarrow{u} B\pi \xrightarrow{Bw_1} RP^{\infty}$. The bundle $\zeta \oplus \lambda$ restricts to a bundle $r^*(\zeta \oplus \lambda)$. With this bundle over K, the map $g: M \to K$ becomes a normal map in the sense of Anderson [1] and so has a well-defined surgery obstruction. Using an n-ad version of Lemma 4.2, we see that the obstruction in $L^s_m(R\pi; w_1)$ depends only on the degree 1, P-normal map. We get

<u>Theorem 4.3</u>: The map σ_* (4.1) has the property that $\sigma_*(f, \zeta_P) = 0$ iff $f: M \to X$ is normally bordant to a simple P-equivalence (provided, as usual, dimension $M \geq 5$).

Even more is true. Let

$$\begin{array}{ccc} M & \xrightarrow{f} & F \\ & \searrow^{\nu_M} \quad \swarrow_{\zeta} & \\ & BC & \end{array}$$

commute, and suppose there is a P-equivalence $\beta: F \to X$ such that $\beta \circ f$, and $\zeta_P \circ \beta_P^{-1}$ give a degree 1, P-normal map $M \to X$. Then, if $\sigma_*(\beta \circ f, \zeta_P \circ \beta_P^{-1}) = 0$, $f: M \to F$ is normally bordant over F to a map $f_1: M_1 \to F$ which is a simple P-equivalence. Furthermore, if F is a finite complex, then f_1 can be chosen to be $\left[\frac{m-2}{2}\right]$-connected.

<u>Proof</u>: One uses Lemma 4.2 and the material in Anderson [1] to prove all but the last sentence. This follows as in Cappell - Shaneson [6] Addendum to 1.7, p. 293. //

Remark: Theorem 4.3 has a straightforward n-ad version. The experts can amuse themselves by considering non-simple, P-local,

P. D. spaces; doing surgery to get P-equivalences with exotic torsions; introducing Γ-groups [6]; etc.

Remark: If we define $\mathcal{S}_{\mathcal{C}}(X;[X])$ to be the set of degree 1, simple P-equivalences $f: M \to X$ (M a $\mathcal{C}$-manifold) modulo the relation of P-local s-cobordism, then the usual long exact sequence (e.g. Wall [24] 10.3 and 10.8) is valid.

§5. The local Spivak normal fibration and local lifts.

As usual X is a P-local Poincaré space. Let $\rho: K \to X$ be a P-equivalence from a finite complex K. We can embed K in some large sphere and take a regular neighborhood $(N^{m+k}, \partial N)$. Make the inclusion map $\partial N \to N$ into a fibration, and let F denote the fibre.

We can localize the entire fibration and it is easy to redo Spivak [21] to prove that $F_{(P)}$ is a local sphere and that the associated stable spherical fibration

$$\nu_X: X \to N_{(P)} \to BSG_{(P)} \times K(R^*,1)$$

is unique (R^*= units of R). (Recall that $BSG_{(P)} \times K(R^*,1)$ is the classifying space for P-local spherical fibrations, Sullivan [22] p. 4.14 and May [13].)

More is available from our geometry. Instead of considering $F_{(P)}$ we can use Serre class theory and compute $H_*(F;Z)$ modulo the class of P'-torsion groups. One easily discovers that $H_*(F;Z)$ is P'-torsion, $* \neq k-1$, and $H_{k-1}(F;Z)$/Torsion is a rank 1 abelian group. The cohomology groups have a similar description. The universal coefficients theorem and Fuks [8], p.111, Prop. 85.4, then show that $H_{k-1}(F;Z)/P'\text{-torsion} = Z$. Hence the map ν_X factors through $BSG_{(P)} \times K(Z^*,1)$, and the map $X \to K(Z^*,1) = RP^\infty$ is just w_1. Hence, just as for manifolds, we can define a map

$$5.1) \quad \eta_X: X \to BSG_{(P)} \times B\pi$$

Over $BSG_{(P)} \times B\pi$ we have the universal fibration $\mu_P \times \lambda$. If we pull this fibration back over X, we get ν_X and we can form the Thom spectrum $\mathfrak{J}(\nu_X)$.

Note: All Thom spectra are indexed so that the Thom class has dimension 0.

In $\pi_m(\mathfrak{J}(\nu_X))$ there are elements c_X, which, once we orient X, map to [X] under the Hurewicz and Thom maps. We choose one of these once and for all and refer to it as the local reduction of the Thom spectrum for the Spivak normal fibration of X.

Definition 5.2: We define $Lift(\eta_X)$ to be the set of lifts of η_X to $BSC_{(P)} \times B\pi$. We suppress which C as it is either clear from context or irrelevant.

We have the usual map

$$5.3) \quad \ell: N(X;[X]) \to Lift(\eta_X)$$

The map ℓ is defined as follows. The map $X \to BSC_{(P)} \times B\pi$ is given by $\zeta_P \times u$ and the specific equivalence of the underlying local spherical fibration with η_X is specified by choosing the equivalence which takes the reduction of $\mathfrak{J}(\nu_M)$ to c_X using the map $\mathfrak{J}(\nu_M) \to \mathfrak{J}(\zeta_P \times \lambda)$ induced by our normal map. Kahn [11] and May [13] may be profitably consulted here.

Remark: If $Lift(\eta_X) \neq \phi$ it is in one to one correspondence with $[X, (G/C)_{(P)}]$.

Remark: If $P \neq \phi$, there is no reason to suppose that ℓ is an isomorphism. Anderson [1] considers a less natural definition of degree 1, P-normal map and gets a map similar to ℓ but taking values in the set of lifts of ν_X to BC. He claims, Thm. 3 p. 51, that his map is an isomorphism, but we are unable to follow his proof (in particular, the first two lines).

§6. Normal maps again.

We need to calculate $N(X;[X])$ since the map ℓ (5.3) is no longer an isomorphism. This was done by Quinn [16] and we display the result following Barge [3]. Rather than interrupt the presentation later, we pause to prove

Lemma 6.1: Consider the square of connected CW complexes

$$6.2) \qquad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

Suppose that g induces an isomorphism on π_1. Further suppose that f is a P-equivalence and that C and D are P-local spaces.

Then, if 6.2 is a fibre square, it is a cofibre square. If $\pi_1 A = 0$, then the converse holds.

Proof: Define F to be the fibre of f. Show that $H_*(F;Z)$ is P'-torsion. As in [16], the spectral sequence $H_*(D,C;H_*(F;Z))$ $H_*(B,A)$ shows $H_*(D,C) = H_*(B,A)$. The converse is easy. //

To fix notation, let $\mathfrak{T}\pi$ denote the Thom spectrum of the line bundle λ over $B\pi$. (We should probably call it $\mathfrak{T}(\pi_1,w_1)$, but we won't.) Given a lift $X \to BSC_{(P)} \times B\pi$ the composite

$$S^m \xrightarrow{c_X} \mathfrak{T}(\nu_X) \to MSC_{(P)} \wedge \mathfrak{T}\pi$$

defines a homomorphism

$$6.3) \quad \beta_P\colon \mathrm{Lift}(\eta_X) \to MSC_m(\mathfrak{T}\pi;R) .$$

We also have a map

$$6.4) \quad \beta\colon N(X;[X]) \to MSC_m(\mathfrak{T}\pi)$$

defined by sending
$$\begin{array}{ccc} M & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \eta_M} & & \\ BSC \times B\pi_1 M & & \end{array}$$
to the composite

$$S^m \xrightarrow{c_M} \mathfrak{T}(\nu_M) \to MSC \wedge \mathfrak{T}\pi_1 M \to MSC \wedge \mathfrak{T}\pi .$$

Clearly
$$\begin{array}{ccc} N(X;[X]) & \xrightarrow{\ell} & \mathrm{Lift}(\nu_X) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta_P} \\ MSC_m(\mathfrak{T}\pi) & \to & MSC_m(\mathfrak{T}\pi;R) \end{array}$$
commutes. Hence a necessary condition for a lift to be in the image of ℓ is that β_P of it must correspond to an honest manifold. This is also sufficient as 6.5 below shows.

To fix notation, let $\beta'\colon N(X;[X]) \to MSC_m(\mathfrak{T}\pi;R')$ denote β followed by the obvious coefficient homomorphism. If $\alpha \in N(X;[X])$ is given, $\ell(\alpha)$ determines a map $X \to BSC_{(P)} \times B\pi$. If λ denotes the line bundle over X induced from the fixed one on $B\pi$, we get a homomorphism

$$\theta_\alpha\colon \quad \pi_{m+1}(\mathfrak{T}\lambda;R) \oplus MSC_{m+1}(\mathfrak{T}\pi;R') \to MSC_{m+1}(\mathfrak{T}\pi;Q) .$$

Quinn's Theorem 2.3, as reformulated by Barge, now reads

Theorem 6.5: There is an exact sequence of sets

$$N(X;[X]) \xrightarrow{\ell \times \beta'} \mathrm{Lift}(\nu_X) \times MSC_m(\mathfrak{J}\pi;R') \rightarrow MSC_m(\mathfrak{J}\pi;Q) .$$

The group $MSC_{m+1}(\mathfrak{J}\pi;Q)$ acts on $N(X;[X])$ so that two elements α_1 and $\alpha_2 \in N(X;[X])$ satisfy $(\ell \times \beta')(\alpha_1) = (\ell \times \beta')(\alpha_2)$ iff α_1 and α_2 lie in the same orbit under this action.

The isotropy subgroup of an element α is just the image of θ_α.

Proof: The proof is clear from studying Quinn [16] and Barge [3]. Lemma 6.1 is used extensively. //

Remark: Quinn [16] has also proved an n-ad version of 6.5.

§7. Surgery again.

Ranicki [17] has defined a symmetrization map

$$1+T : L_m^s(R\pi;w_1) \rightarrow L^m(R\pi;w_1) .$$

The goal of this section is to understand $1+T$ composed with the surgery obstruction map 4.1. We shall do this in terms of a homomorphism $\sigma^*: MSC_m(\mathfrak{J}\pi) \rightarrow L^m(R\pi;w_1)$ and an element $\sigma^*(X;[X]) \in L^m(R\pi;w_1)$, both defined by Ranicki [17] (or Mischenko [14] if $2 \in P'$). The formula is

$$7.1) \quad (1+T)\ \sigma_*(\ \ ) = \sigma^*\ \beta(\ \ ) - \sigma^*(X;[X])$$

This gives a solution to our problem, but we wish more. We want to define maps

$$\sigma'_*: MSC_m(\mathfrak{J}\pi;R') \to L^m(R\pi;w_1) \otimes R'$$

$$\tau^*: Lift(\nu_X) \to L^m(R\pi;w_1) \otimes R$$

such that

Theorem 7.2: The diagram

$$\begin{array}{ccc} N(X;[X]) & \longrightarrow & Lift(\nu_X) \times MSC_m(\mathfrak{J}\pi;R') \\ \downarrow (1+T)\ \sigma_* & & \downarrow \tau^* \times \sigma'_* \\ 0 \to L^m(R\pi;w_1) & \longrightarrow & L^m(R\pi;w_1)\otimes R \oplus L^m(R\pi;w_1)\otimes R' \end{array}$$

commutes.

Remark: If we think of $Lift(\nu_X)$ as the P-part of the set of normal maps, and of $MSC_m(\mathfrak{J}\pi;R')$ as the P'-part of the set of normal maps, then Theorem 7.2 says that the P-local part of the symmetrized surgery obstruction is determined by the P-local part of the normal map set, with a similar statement for P'.

The map σ'_* is easily defined: one just takes the map $\sigma^*(\ \ ) - \sigma^*(X;[X]): MSC_m(\mathfrak{J}\pi) \to L^m(R\pi;w_1)$ and localizes it with respect to P'. The map τ^* is almost as easy. Take the map $\Psi: Lift(\nu_X) \xrightarrow{\beta_P} MSC_m(\mathfrak{J}\pi;R) \xrightarrow{(\sigma^*)_P} L^m(R\pi;w_1) \otimes R$ and let $\tau^*(\ \ ) = \Psi(\ \ ) - \sigma^*(X;[X])_{(P)}$. The proof of Theorem 7.2 is easy.

Remark 7.3: The map $L^s_m(R\pi;w_1) \otimes R' \to L^m(R\pi;w_1) \otimes R'$ is an isomorphism by Ranicki [17], so we have determined the P'-local part of the surgery obstruction from the P'-local part of the normal map set.

Remark 7.4: If $2 \in P'$, the map $L^s_m(R\pi;w_1) \to L^m(R\pi;w_1)$ is an isomorphism. Hence we can determine each part of the surgery obstruction from the corresponding part of the normal map set.

Remark: If $2 \in P$, there is a very involved construction of a map τ_*: Lift$(\nu_X) \to L^s_m(R\pi;w_1)$ so that we can compute σ_* from σ'_* and τ_*. We neither need nor pursue this refinement here.

§8. A metastable embedding theorem.

Dax [7], Laramore [12], Salomonsen [20], Rigdon [18], Rigdon-Williams [19], etc. have shown that the best metastable embedding codimension is a 2-local phenomenon. This suggests the following "converse"

Theorem 8.1: Given a smooth manifold, M^m, whose Novikov higher signature (defined below) vanishes, there exists a smooth manifold, N^m, and a map $f: N \to M$ such that

i) N embeds in S^{m+k} if $m+3 \leq 2k$

ii) f is a $(\frac{1}{2})$-local equivalence

iii) f is $\left[\frac{m-2}{2}\right]$-connected

iv) $f^*\nu_M = \nu_N$.

Definition 8.2: The Novikov higher signature of a manifold M is defined to be

$$\mathcal{L} \setminus (\eta_M)_*([M]) \in H_*(B\pi;Z^t_{(2)})$$

where $\eta_M: M \to BSC \times B\pi$ is the map 3.1 and $\mathcal{L}$ is the Morgan-Sullivan L-class in $H^{4*}(BSC; Z_{(2)})$ [15].

Remark: In the proof of 8.1 we assume only that $\sigma^*(M;[M])$ is an odd torsion element in $L^m(R\pi; w_1)$, where R denotes $Z[\frac{1}{2}]$ for the rest of sections 8 and 9. The Novikov higher signature is more easily calculated than $\sigma^*(M;[M])$. The relation between them is supplied by

<u>Lemma 8.3</u>: There is a homomorphism

$$A: H_*(B\pi; Z^t_{(2)}) \to L_*(R\pi; w_1) \otimes Z_{(2)}$$

such that $A(\mathcal{L} \setminus (\eta_M)_*([M])) = \sigma^*(M;[M]) \otimes 1$.

<u>Proof</u>: Ranicki's methods define an assembly map $\mathbb{L}^\circ(R) \wedge \mathfrak{J}\pi \to \mathbb{L}^\circ(R\pi; w_1)$ and a map $MSTOP \wedge \mathfrak{J}\pi \to \mathbb{L}^\circ(R) \wedge \mathfrak{J}\pi$ so that the composite $S^m \xrightarrow{c_M} \mathfrak{J}(\nu_M) \to MSTOP \wedge \mathfrak{J}\pi \to \mathbb{L}^\circ(R) \wedge \mathfrak{J}\pi \to \mathbb{L}^\circ(R\pi; w_1)$ is just $\sigma^*(M;[M])$. See [17] for more details.

In [23] we showed that $\mathbb{L}^\circ(R)$ is a product of Eilenberg-MacLane spectra. Anderson [2] has shown that

$$\pi_*(\mathbb{L}^\circ(R)) = \begin{cases} Z \oplus Z/2 & * \equiv 0 \pmod 4 \\ 0 & * \not\equiv 0 \pmod 4 \end{cases} \qquad (R = Z[\tfrac{1}{2}])$$

Classical quadratic form theory and the methods of [23] provide classes $L_i \in H^{4i}(\mathbb{L}^\circ(R); Z_{(2)})$ and $h_i \in H^{4i}(\mathbb{L}^\circ(R); Z/2)$ which give the decomposition. The map $MSTOP \to \mathbb{L}^\circ(R)$ is described at 2 by the fact that the h_i restrict to 0 and the L_i restrict to the Morgan-Sullivan L-class. This proves 8.3. //

Remark: This proof was our original motivation for [23].

We need one more lemma.

Lemma 8.4: If $(X;[X])$ is an oriented P-local Poincaré space ($2 \in P'$) then

$$\sigma^*(X;[X]) = \sigma^*(X;4^i[X]) .$$

Proof: Miscenko's version of symmetric L-theory with 2 invertible, [14], shows that $\sigma^*(X;[X])$ is determined by $C_*(X)$ and the map $\xi\cap : C^*(X) \to C_{m-*}(X)$. Multiplication by 2^i gives a chain map $C_*(X) \to C_*(X)$ which induces an equivalence from $C_*(X)$ and $4^i[X]$ to $C_*(X)$ and $[X]$. Hence they have the same σ^*. //

We can now prove 8.1. Our first goal is to produce a finite complex having the homotopy properties N is to enjoy.

Let V denote the pullback

$$\begin{array}{ccc} V & \to & BO(r) \\ \downarrow & & \downarrow \\ M & \to & BO \end{array}$$

r fixed below.

We wish to find a finite complex X and a map $g: X \to V$ such that the composite $X \to V \to M$ is an r-connected, $\frac{1}{2}$- equivalence. If k is odd, set $k = r$. If k is even, set $k-1 = r$. Note r is odd.

To begin, let X_r be an r-skeleton for M. It is easy to map $X_r \to V$ so that $X_r \to V \to M$ is the inclusion, hence r-connected. Define X_i and $g_i: X_i \to V$ inductively by adding i-cells to X_{i-1} so that g_i is $\frac{1}{2}$-locally, i-connected. Since the map $V \to M$ is $\frac{1}{2}$-locally, $(2r+1)$-connected, this is easy to do up to X_m since $m \leq 2k-2$.

Now $\pi_{m+1}(V,X_m) \to \pi_{m+1}(V_{(\frac{1}{2})},(X_m)_{(\frac{1}{2})}) \to \pi_{m+1}(M_{(\frac{1}{2})},(X_m)_{(\frac{1}{2})}) \to 0$ and, clearly, $\pi_{m+1}(M_{(\frac{1}{2})},(X_m)_{(\frac{1}{2})})$ is s-free over $R\pi$. As usual, we may assume that it is free. One can then choose elements in $\pi_{m+1}(V,X_m)$ to give a basis in $\pi_{m+1}(M_{(\frac{1}{2})},(X_m)_{(\frac{1}{2})})$ and attach cells to get X and g: X → V as required.

Over X there is a k-plane bundle, ν^k, and a stable bundle equivalence $h^*\nu_M = \nu^k$, where h is X → V → M. Hence we get a stable map $\mathfrak{J}(\nu^k) \to \mathfrak{J}(\nu_M)$ which is easily seen to be a $\frac{1}{2}$-equivalence. Hence there exists an element $c \in \pi_m(\mathfrak{J}(\nu^k))$ such that c goes to $4^e c_M$ for some positive integer e. We also have the stabilization map $\pi_{m+k}(T(\nu^k)) \to \pi_m(\mathfrak{J}(\nu^k))$, where $T(\nu^k)$ is the Thom space.

Since $m \leq 2k-2$, Theorem 0.2 of [25] assures us that we can find an integer, d, such that, for all $i \geq d$ we have an element $\gamma_i \in \pi_{m+k}(T(\nu^k))$ which goes to $4^i c_M$ under stabilization and the map $\mathfrak{J}(\nu^k) \to \mathfrak{J}(\nu_M)$.

Associated to each γ_i we get a normal map α_i: $N_i \to M$ which is degree 1 if we consider the $\frac{1}{2}$-local oriented Poincaré space $(M;4^i[M])$. If $\sigma_*(\alpha_i) = 0$, and if $m \geq 5$, then, since $m \leq 2k-3$, Levine's work (see [19], Embedded Surgery Lemma) shows that we can do the surgery inside S^{m+k}, proving 8.1.

But we can calculate $(1+T)\ \sigma_*(\alpha_i)$. It is

$\sigma^*(N_i;[N_i]) - \sigma^*(M;4^i[M])$. Lemmas 8.3 and 8.4 show that this is $4^i\sigma^*(M;[M]) - \sigma^*(M;[M])$. Since the order of $\sigma^*(M;[M])$ is odd, we can choose $i \geq d$ so that $\sigma_*(\alpha_i) = 0$ (see 7.4).

If $m < 5$, then M itself embeds in S^{m+k}, $m \leq 2k-3$. //

§9. The False - Hirsch conjecture is half true.

We begin with a local analogue of a theorem of Browder [5].

Theorem 9.1: Let M^m be a manifold such that

i) ν_M desuspends to a bundle η^k with $k \geq 2$;

ii) the image of c_M in $\pi_m(\mathcal{T}(\nu_M)_{(P)})$ comes from an element in $\pi_{m+k}(T(\eta^k)_{(P)})$.

Then M^m embeds in Σ^{m+k+1} with normal bundle $\eta \oplus \epsilon^1$, where Σ^{m+k+1} is a framed P-local homotopy sphere which bounds a framed P-local homotopy disc.

We postpone the proof to discuss a corollary. Hirsch [9] has conjectured that every framed manifold M^m embeds in S^{m+k} with $m \leq 2k$. The statement that they embed with trivial normal bundle is known to be false (e.g. [10]) and is usually labeled the False-Hirsch conjecture (even though it was never conjectured by Hirsch.) We can prove

Theorem 9.2: Every framed manifold of dimension m embeds in a framed $\frac{1}{2}$-homotopy sphere Σ^{m+k} with trivial normal bundle if $m \leq 2k-1$. We may choose Σ to bound a framed $\frac{1}{2}$-disc.

Proof of 9.2: Clearly ν_M desuspends to ϵ^{k-1} for any k we wish, so the problem is to desuspend the normal invariant when $P' = \{2\}$.

The space $T(\epsilon^{k-1})$ is $S^{k-1} \vee X$ for some space X. Theorem 0.2 of [25] shows that, if $m \leq 2(k-1)+1$, the image of c_M in $\pi^S_{m+k-1}(X_{(P)})$ desuspends to $\pi_{m+k-1}(X_{(P)})$. The image of c_M in $\pi^S_{m+k-1}(S^{k-1}_{(P)})$ desuspends to $\pi_{m+k-1}(S^{k-1}_{(P)})$ by classical EHP sequence arguments.

Hence 9.1 can be applied to yield the result. //

To prove 9.1 we will need

<u>Theorem 9.3</u>: Assume (X,A) is a CW pair with X and A simply connected. Let $\beta \in \pi^S_r(X,A)$ and $\alpha \in \pi_r(X_{(P)},A_{(P)})$ be elements whose images agree in $\pi^S_r(X_{(P)},A_{(P)})$. Then, if $r \geq 6$, there is a framed manifold $(W^r,\partial W)$ and maps $f\colon (W,\partial W) \to (X,A)$ and $g\colon (W,\partial W) \to (D^r_{(P)},S^{r-1}_{(P)})$ such that

i) the framed bordism class represented by W and f is just β;

ii) g is a P-local equivalence such that $f_P \circ g_P^{-1}$ is α.

<u>Proof of 9.3</u>: Lemma 6.1 implies that the following fibre square is also a cofibre square.

$$\begin{array}{ccc} (E,E_1) & \xrightarrow{j} & (X,A) \\ \downarrow{\pi} & & \downarrow \\ (D^r_{(P)},S^{r-1}_{(P)}) & \xrightarrow{\alpha_P} & (X_{(P)},A_{(P)}) \end{array}$$

Hence there is an element $\mathfrak{b} \in \pi_r^S(E,E_1)$ such that $j(\mathfrak{b}) = \beta$ and $\pi(\mathfrak{b}) = \iota_P$ where ι_P denotes the stable homotopy class of the localization map $(D^r,S^{r-1}) \to (D^r_{(P)},S^{r-1}_{(P)})$.

The element $\mathfrak{b}$ corresponds to a framed bordism class. Let it be represented by a manifold $(W_1^r,\partial W_1)$ and a map $f_1\colon (W_1,\partial W_1) \to (E,E_1)$. Since π is a P-equivalence, f_1 corresponds to a degree 1 normal map. The 2-ad version of Theorem 4.2 gives a local π-π theorem, so we can do the surgery (provided $r \geq 6$) to get a P-equivalence $f\colon (W,\partial W) \to (E,E_1)$ still representing $\mathfrak{b}$. //

Proof of 9.1: If $m+k+1 < 6$, it is easy to prove 9.1 case by case, so assume that $m+k+1 \geq 6$. The proof which follows is essentially Browder's [5] (also see [24] Thm. 11.3).

Let $c \in \pi_{m+k}(T(\eta^k)_{(P)})$ be the element promised in 9.1 ii). Let $\alpha \in \pi_{m+k+1}(\mathrm{Cone}(T(\eta^k)_{(P)}),T(\eta^k)_{(P)})$ be the unique element whose boundary is c. Define $\beta \in \pi_{m+k+1}(\mathrm{Cone}(T(\nu_M)),T(\nu_M))$ to be the element whose boundary is c_M. Theorem 9.3 applies so we can find $(W,\partial W)$ and a map $g\colon \partial W \to T(\eta^k)$.

$$\begin{array}{ccccc} S(\eta^k \oplus \epsilon^1) & \to & T(\eta^k) & \to & T(\eta^k) \cup_g W \\ \downarrow & & & & \downarrow \\ D(\eta^k) & & \longrightarrow & & A \end{array}$$

is defined to be a pushout. Browder's proof shows that A is a P-local S^{m+k+1} and that there is an element $\gamma \in \pi^S_{m+k+1}(A)$ going to c_M under the map $A \to T(\eta^k \oplus \epsilon^1)$. Let $\beta \in \pi^S_{m+k+2}(\mathrm{Cone}\ A,A)$

be the element whose boundary is γ. Let $\alpha \in \pi^S_{m+k+2}(\text{Cone } A_{(P)}, A_{(P)})$ be the stabilization of $(D^{m+k+2}_{(P)}, S^{m+k+1}_{(P)}) \rightarrow (\text{Cone } A_{(P)}, A_{(P)})$.

Use 9.3 again to get Σ_1^{m+k+1} and a P-equivalence $g\colon \Sigma_1 \rightarrow A$ so that g represents γ. Following Browder, make the composite

$$\Sigma_1 \rightarrow A \rightarrow T(\eta \oplus \epsilon^1)$$

transverse to the zero section. The result is normally bordant to M, so we finish just as Browder does using the local π-π theorem in place of the integral π-π theorem. //

§10: References.

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