SMOOTHING THEORY AND FREEDMAN'S WORK ON FOUR MANIFOLDS

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Introduction: The Freedman-Casson handle theorem [F] used an unusual combination of smooth and topological techniques that resulted in the topological classification of almost smooth 1-connected closed four manifolds. (A compact connected manifold M is almost smooth, if M_0 = M - interior point, is smooth.) In this paper we combine smoothing theory with Freedman's results to further study the structure of topological and almost smooth manifolds.

In section 1 we give a preliminary discussion of almost smooth manifolds and show, using Freedman's completion of Scharlemann's transversality theorem, that if M is a compact four manifold, then $M \# k(S^2 \times S^2)$, some k, is s-cobordant to an almost smooth manifold. (Theorem A)

In sections 2-5 we give consequences of our main result that $\pi_i(Top_4/O_4) = \pi_i(Top/O)$, $i = 2, 3$. Some of these consequences are:

1. A smoothing of $M_0 \times R$, M a four manifold, is isotopic to a product smoothing provided M_0 admits some smoothing. (Theorem B)

2. If V is a cobordism between almost smooth four manifolds, then V has a topological handle decomposition on $\partial_- V$. (Theorem C)

3. An s-cobordism between almost smooth four manifolds becomes a topological product by adding $S^2 \times S^2$'s along the cobordism. (Theorem D)

4. Let M be a closed 5-manifold. Then the tangent microbundle of M splits off a line bundle. (Corollary of Theorem E)

Finally, in section 5 we prove our main result.

Remark: Quinn [Q] has proved that $\pi_i(Top_4/O_4) = 0$, $i = 0, 1, 2$. This implies that every four manifold is almost smoothable. Our proof that $\pi_2(Top_4/O_4) = 0$ is independent of Quinn's.

1. Remarks on Almost Smooth 4-Manifolds.

If M is a topological manifold, a smoothing of M is a pair (U,α) where U is a smooth manifold and $\alpha : M \to U$ is a homeomorphism. Two such (U_1,α_1) and (U_2,α_2) are isotopic rel ∂M if there is an isotopy $G : M \times I \to M$ such that $G_0 = 1_M$, $G_t|\partial M = 1_{\partial M}$ and $\alpha_2 G_1 \alpha_1^{-1} : U_1 \to U_2$ is a diffeomorphism (where $G_t(x) = G(x,t)$).

An almost smoothing of M is a smoothing (U,α) of M minus one interior point from each compact component. If M is compact and connected, denote an almost smoothing by (U,α,p), where p is the interior point. A homotopy class $w(p,q)$ of paths from p to q in M determines a bijective correspondence between isotopy classes of smoothings rel ∂M of $m - p$ and $M - q$. In fact there is an ambient isotopy $G : M \times I \to I$ such that $G_0 = 1_M$, $G_t|\partial M = 1_{\partial M}$, $G_1(p) = q$ and $G_t(p)$, $0 \le t \le 1$, is a path in $w(p,q)$. If (U,α) is a smoothing of $M - p$, $(U,\alpha G_1^{-1})$ is a smoothing of $M - q$. If G' is another such isotopy, then G' is isotopic rel endpoints to G" with $G''_t(p) = G_t(p)$, and it is easy to see that this implies the two smoothings of $M - q$ are isotopic. This gives an action of the fundamental group on the smoothings of $M - p$. Just as for homotopy groups we will often suppress the base point and simply write M_0 for M minus any interior point p.

If (U,α) is a smoothing of M, we will sometimes identify M with U via α, and write M_α for M with this smoothing.

Note that a four manifold is smoothable if and only if it is a handlebody. Freedman has shown there are four manifolds which are not smoothable and hence not handlebodies. This suggests we investigate the following notion:

Call a compact 4-manifold M an <u>almost handlebody</u> if one can find a compact contractible 4-manifold W in the interior of M so that $\overline{M - W}$ is a smooth manifold with boundary. Thus M is a handlebody except for one exotic 4 handle W. Clearly, every almost handlebody is almost smooth. The converse is unknown in general, but we will say more about it later. For the present we note the following:

1.1. Scharlemann's transversality theory [S] as completed by Freedman [F] allows one to deform a map $f : V^{k+4} \to T(\xi^k)$ of a topological manifold V into the Thom space of a k-plane bundle ξ to a map g topologically transverse to the zero section. This process yields an almost handlebody M^4 as preimage of the zero section.

1.2. The arguments of Freedman and Quinn [FQ] in the smooth case show that if the Wall obstruction vanishes, one may do surgery mod $\# S^2 \times S^2$'s on a normal degree one map $f : M \to X$, M an almost handlebody, so as to end up with a simple homotopy equivalence of an almost handlebody M' with $X \# k(S^2 \times S^2)$, some k. In fact their method requires surgery only on 0 and 1 spheres.

<u>Theorem A</u>: If M is a compact 4 manifold, there is a k such that $M \# k(S^2 \times S^2)$ is s-cobordant rel ∂ to an almost handlebody.

<u>Proof</u>: By 1.1 with ξ the normal bundle of M and $V^{k+4} = S^{k+4}$, we obtain an almost handlebody N and a degree one normal map $f : N \to M$, normally cobordant to 1_M. By 1.2 we may assume f is a simple homotopy equivalence, when we replace M by $M \# k(S^2 \times S^2)$. Now we wish to do surgery on the normal cobordism to make it an s-cobordism, but in general there is a surgery obstruction. On the other hand, every surgery obstruction can be realized mod $\# S^2 \times S^2$ by a normal cobordism of N to N', at least if N is smooth [CS]. By first removing the interior of the exotic 4-handle in N, and realizing the surgery obstruction on the resultant smooth manifold, we end up with a normal cobordism mod $S^2 \times S^2$'s of N to another almost handlebody N', such

that the surgery obstruction for the normal cobordism from N' to $M \# k(S^2 \times S^2)$, some k, vanishes, enabling us to construct an s-cobordism.

Remark: Alternately, starting with $M \times [0,\infty)$ and making the projection onto $[0,\infty)$ transverse to say $M \times 1$, we could construct an almost handlebody N in $M \times (0,1)$, and modify it so that mod $S^2 \times S^2$'s the cobordism from M to N is an s-cobordism. Compare [CSL], where the argument is done in the smooth case.

2. Bundle Reductions and the Product Structure Theorem.

Let $j : BTop_4 \to BTop$ and $\hat{j} : BO_4 \to BO$ be the maps induced by the inclusion of Top_4 in Top. Note that j may be considered a map of fibrations:

$$\begin{array}{ccc} BO_4 & \xrightarrow{\hat{j}} & BO \\ \downarrow & & \downarrow \\ BTop_4 & \xrightarrow[j]{} & BTop \end{array},$$

with fibres Top_4/O_4 and Top/O, respectively.

Notation: If (X,A) is a relative CW complex, we let $(X,A)^i = A \cup$ cells of dimension $\leq i$.

Proposition 2.1: Let (X,A) be a relative CW complex of dimension at most four. Let $\xi : X \to BTop_4$ and suppose $\xi_0 = \xi|(X,A)^3$ lifts to $\hat{\xi}_0 : (X,A)^3 \to BO_4$. Then the correspondence $\hat{\xi}$ to $\hat{j}\hat{\xi}$ induces a surjection of the homotopy classes of lifts of ξ extending $\xi_0|A$ onto the homotopy classes of lifts of $j\xi$ extending $\hat{j}\hat{\xi}_0|A$.

Addenda: By replacing A by $(X,A)^2$ and using the fact that $\pi_i(Top/O) = 0$ for $i < 3$ we have by 2.1:

2.1a: ξ lifts to BO_4 extending $\xi_0|(X,A)^2$ if and only if $j\xi$ lifts to BO extending $j\xi_0|A$.

By replacing A by $(X,A)^3$ and using the fact that $\pi_4(Top/O) = 0$ we have:

2.1b: ξ lifts to BO_4 extending $\hat{\xi}_0$ if and only if $j\xi$ lifts to BO extending $\hat{j}\hat{\xi}_0$. Any two such lifts of $j\xi$ are homotopic $rel(X,A)^3$; and in particular if $\hat{\xi}$ is such a lift of ξ and $\hat{\eta}$ such a lift of $j\xi$, $\hat{j}\hat{\xi}$ is homotopic to $\hat{\eta}$ $rel(X,A)^3$.

Proof of 2.1 (using the main theorem): Since $\pi_i(Top/O) = 0$ for $i < 3$ we may assume up to homotopy that any lift $\hat{\eta}$ of $j\xi$ to BO extending $j\xi_0|A$ actually extends $\hat{j}\hat{\xi}_0|(X,A)^2$. Since $\pi_3(Top_4/O_4) \to \pi_3(Top/O)$ is surjective, we may change $\hat{\xi}_0$ over the 3 cells of (X,A) so that $\hat{j}\hat{\xi}_0$ is homotopic $rel(X,A)^2$ to $\hat{\eta}|(X,A)^3$, and hence we can assume $\hat{\eta}$ agrees with $\hat{j}\hat{\xi}_0$ over $(X,A)^3$. Since $\pi_3(Top_4/O_4) \to \pi_3(Top/O)$ is injective $\hat{\xi}_0$ extends to a lift $\hat{\xi}$ of ξ. Since $\pi_4(Top/O) = 0$, $\hat{j}\hat{\xi}$ is homotopic to $\hat{\eta}$ $rel(X,A)^3$.

If M is a 4-manifold, the Kirby-Siebenmann obstruction $\kappa \in H^4(BTop;Z_2)$ yields a class $\kappa(M) \in H^4(M,\partial M;Z_2)$ which can be viewed as the obstruction to smoothing $M \times R$ rel $\partial M \times R$.

The following is an immediate consequence of 2.1.

Proposition 2.2: Let M be a 1-connected almost smoothed closed 4-manifold with $\kappa(M) = 0$. Then the corresponding lift of τ_{M_0} to BO_4 extends to a lift of τM.

Remark: The proposition says there is no bundle theoretic obstruction to extending the smoothing to M. Nevertheless, a recent result of Donaldson on Spin manifolds, shows that the smoothing does not always extend.

Smoothing theory and 2.1 will allow us to prove the following weak product structure theorem:

Theorem B: Let M be an almost smoothed 4-manifold, and suppose we are given a smoothing of $M_0 \times R$ which is the product smoothing on $\partial M \times R$. Then there exists (a possibly different) smoothing of M_0, unchanged on the boundary, so that the product smoothing of $M_0 \times R$ is isotopic rel $\partial M \times R$ to the given smoothing.

Remark: The reason this theorem is called weak is that the new smoothing of M_0 is unique only up to concordance — not isotopy or even sliced concordance.

Addendum B1: Let $C \subset M$ be a proper closed subset. Under the hypothesis of Theorem B and supposing $C \subset M_0$ (which can always be arranged — see section 1) and that the smoothing of $M_0 \times R$ restricts to the product smoothing on $U \times R$, U a neighborhood of C in M_0, then we can conclude that the new smoothing of M_0 agrees with the original smoothing on a neighborhood of C.

If M is smoothed and we are given a smoothing of $M \times R$ one cannot guarantee that this smoothing is isotopic to a product smoothing on all of $M \times R$, even though all bundle obstructions vanish. However we can show:

Addendum B2: There is an integer $k \geq 0$ with the following property. With the hypothesis of Theorem B, and assuming $M = X \# k(S^2 \times S^2)$ for some smooth compact connected 4-manifold X; if we are given a smoothing of all of $M \times R$, which is the product smoothing on $\partial M \times R$, then there is a smoothing of all of M such that the product smoothing on $M \times R$ is isotopic rel $\partial M \times R$ to the given smoothing.

Remark B3: The relative version of Addendum B2 holds provided $C \subset X_0 \subset M$.

<u>Addendum B4</u>: Let N^4 be the twisted S^3-bundle over S'. There exists a smooth 4-manifold, M^4, and a homotopy equivalence $f : M^4 \to N^4$ which is not homotopic to a diffeomorphism iff $k = 0$.

<u>Proof of Theorem B</u>: The classifying map $\tau : M \to BTop_4$ of the tangent microbundle of M satisfies: a) $\tau_0 = \tau|M_0$ lifts to BO_4 – using the almost smoothing of M, b) $j\tau_0$ lifts to BO – using the smoothing of $M_0 \times R$, so that if $\hat{\tau}_0$ is the lift of τ_0 and $\hat{\eta}_0$ is the lift of $j\tau_0$, then $\hat{j}\hat{\tau}_0|\partial M = \hat{\eta}_0|\partial M$. By 2.1 there is a lift $\hat{\tau}_0'$ of τ_0 so that $\hat{\tau}_0'|\partial M = \tau_0|\partial M$ and $\hat{j}\hat{\tau}_0'$ is homotopic to $\hat{\eta}_0$ rel ∂M. It follows from smoothing theory [L], that there is a smoothing of M_0 satisfying the conclusion.

<u>Proof of B1</u>: Take a smooth compact submanifold $A^4 \subset U$, with $C \subset \text{Int } A$. Then the same argument as above with $\partial M \cup A$ replacing ∂M, proves B1.

<u>Proof of B2</u>: The classifying map for the tangent bundle of M factors up to homotopy as follows:
$M = X \# k(S^2 \times S^2) \xrightarrow{q} X \vee k(S^2 \times S^2) \xrightarrow{\tau \vee \tau'} BTop_4$, where q is the quotient map and τ (resp. τ') classifies the tangent bundle of X (resp. $k(S^2 \times S^2)$). Since $k(S^2 \times S^2)$ has a trivial stable tangent bundle, $j\tau'$ is homotopically trivial by a standard based homotopy. Since $\pi_2(Top/0) = 0$, any lift of $j\tau_M$ defines a lift of $j\tau$. The lift is unique up to homotopy since $\pi_4(Top/0) = 0$. Thus the smoothing of $M \times R$ defines a lift $\hat{\eta}$ of $j\tau$ so that if $\hat{\tau}$ is the lift of τ given by the smoothing of X, $\hat{j}\hat{\tau}|\partial X = \eta|\partial X$.

Since Top/0 is a $K(Z_2,3)$, the difference between $\hat{\eta}$ and $\hat{j}\hat{\tau}$ defines a class $\alpha \subset H^3(X,\partial X;Z_2)$. We assume $\alpha \neq 0$, since otherwise the result is trivial. The dual of α is represented by a smoothly embedded S^1 in X. The normal tube of S^1 is either $E_+ = D^3 \times S^1$ or

E_- = the unoriented D^3 bundle over S^1. Let E denote whichever one we have. Then α is the image of the generator γ of $H^3(E,\partial E;Z_2) = H^3(D^3,\partial D;Z_2)$. In particular, we can assume $\hat{\eta} = \hat{j}\hat{\tau}$ on X - Int E. Then $\hat{\eta}|E$ is in the unique non-standard homotopy class δ of lifts of $j\tau_E$ rel ∂E. To realize the lift $\hat{\eta}$ of $j\tau$, it would be sufficient to change the smoothing on E rel ∂E from the standard smoothing represented by $\tau|E$ to one defining a lift $\hat{\sigma}$, where $\hat{j}\hat{\sigma}$ is in δ. By 2.1, a lift $\hat{\sigma}$ always exists such that $\hat{j}\hat{\sigma}$ is in δ. However, in general, $\hat{\sigma}$ only defines a smoothing of $E \# k(S^2 \times S^2)$ for some k [LS], but nevertheless with the induced product smoothing of $(E \# k(S^2 \times S^2)) \times R$ representing δ. Thus if for E_+ we chose σ_+ with $\hat{j}\hat{\sigma}_+ \in \delta_+$ and similarly for E_-, then letting $k = \max(k_+,k_-)$, we can always get a smoothing of $X \# k(S^2 \times S^2)$ satisfying B2.

<u>Proof of B4</u>: If k = 0 we can use the homeomorphism h promised by B2 for f.

Given f it is an easy surgery theoretic calculation to show that M^4 is topologically h-cobordant to N^4 but M^4 is not smoothly h-cobordant to N^4.

By a theorem of Quinn [Q], any sufficiently large cover of this h-cobordism is a product. Here we can find a smooth manifold $\tilde{M}^4$ and a homeomorphism $h : \tilde{M}^4 \to N^4$ by taking a large odd cover. This shows $k_- = 0$. The double cover of this picture shows $k_+ = 0$.

3. Handlebody Theory.

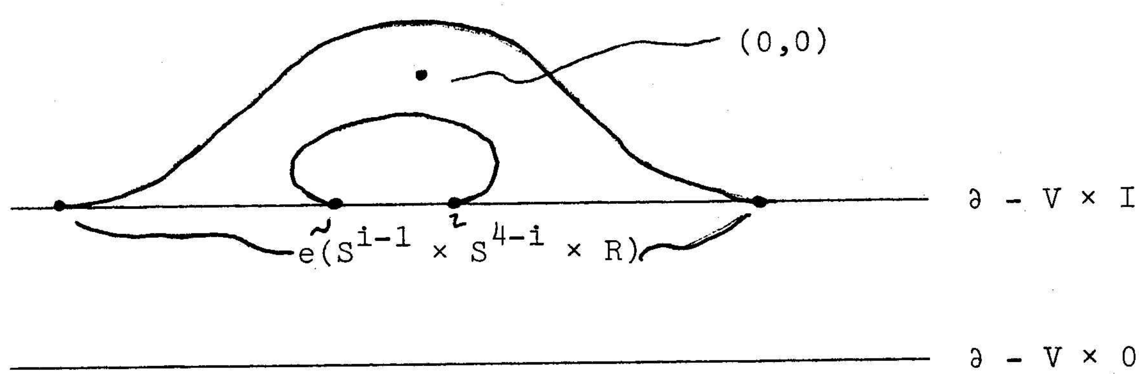
<u>Remark</u>: These results are largely superseded by Quinn's results [Q].

<u>Proposition 3.1</u>: Let V be a five dimensional compact cobordism between the four manifolds ∂_-V and ∂_+V which is a product between their boundaries. If V is a topological handlebody on ∂_-V, there

is a 4-plane bundle η over V_0 such that $\eta \oplus 1 = \tau(V_0)$ and $\eta|\partial_\pm V = \tau(\partial_\pm V)$.

Proof: It suffices to prove 3.1 when $V = \partial_- V \times I \cup_f D^i \times D^{5-i}$. In fact, by induction up the handles this will construct a bundle η over $V - F$, F a finite collection of interior points, which we can assume contains at least one point from each component of V. By a standard argument V_0 may be engulfed rel ∂V into $V - F$.

Now we can always define a smooth structure on a neighborhood of $f(S^{i-1} \times D^{5-i}) \subset \partial_- V$ so the attachment is smooth with rounded corners; and in particular we have an embedding $e : S^{i-1} \times S^{4-i} \times R \to \partial_- V \times 1$.



Define η over $W = \partial_- V \times I \cup_e R^i \times S^{4-i}$ by gluing $T(R^i \times S^{4-i})$ to $T(\partial_- V) \times I$ by $T(e)$. Since $V_0 = V - (0,0)$, $(0,0) \in D^i \times D^{5-i}$, has W as a deformation retract, the result follows.

Corollary 3.2: Under the hypothesis of 3.1 and assuming $\partial_+ V \neq \emptyset$, there is a 4-plane bundle ξ over V such that $\xi \oplus 1 = \tau(V)$, $\xi|\partial_- V = \tau(\partial_- V)$ and $\xi|(\partial_+ V)_0 = \tau(\partial_+ V)_0$.

Proof: Engulf V in V_0 by pushing in on an interval from a base point in $\partial_+ V$ to the base point in V. Let ξ be the pull back of η by the engulfing.

Proposition 3.3: Let X be a 4-complex and ξ_1, ξ_2 topological 4-plane bundles over X. If a) ξ_1 and ξ_2 are stably equivalent, and

and b) ξ_1 and ξ_2 have lifts to BO_4 over the 3-skeleton X^3; then ξ_1 and ξ_2 are equivalent over X^3 and ξ_1 lifts to BO_4 over X if and only if ξ_2 does.

<u>Proof</u>: $\pi_i BO_4 \to \pi_i BO$ is an isomorphism for $i \leq 3$ and $\pi_i BO \to \pi_i BTop$ is an isomorphism for $i \leq 3$. Thus the lifts $\hat{\xi}_1$ and $\hat{\xi}_2$ of $\xi_1 | X^3$ and $\xi_2 | X^3$ to BO_4 are homotopic over X^2. Since $\pi_3 BO_4 = 0$, $\hat{\xi}_1$ and $\hat{\xi}_2$ are homotopic. Hence ξ_1 and ξ_2 are equivalent over X^3. The last statement of the proposition follows from 2.1 and hypothesis a).

<u>Remark</u>: In order for ξ_1 and ξ_2 to be equivalent it is necessary and sufficient that they have the same Euler class.

<u>Proposition 3.4</u>: Let V be a compact h-cobordism between 4-manifolds which is a product along the boundary, and suppose V is a handlebody on $\partial_- V$. Then

a) $\partial_- V$ is almost smoothable if and only if $\partial_+ V$ is almost smoothable.

b) $\tau(\partial_- V)$ reduces to a vector bundle if and only if $\tau(\partial_+ V)$ reduces to a vector bundle.

<u>Proof</u>: By 3.2, there is a 4-plane bundle ξ over V such that $\xi | \partial_\pm V = \tau(\partial_\pm V)$. Since V is an h-cobordism $\xi = r^* \tau(\partial_- V)$, where $r : V \to \partial_- V$ is the retraction. In particular, $\tau(\partial_+ V) = r_+^* \tau(\partial_- V)$, where $r_+ = r | \partial_+ V$. Since $\partial_+ V$ has the homotopy type of a 4-complex X with $(\partial_+ V)_0$ homotopy equivalent to X^3 [W], the result follows from 3.3 and the fact that if V is a handlebody on $\partial_- V$ then it is a handlebody on $\partial_+ V$.

<u>Proposition 3.5</u>: Suppose there is a compact 4-manifold which is not almost smoothable. Then

a) there is a compact s-cobordism V^5 which does not have a handle decomposition on $\partial_- V$, and

b) there is a compact manifold V^5 with boundary such that V is not a handlebody on ∂V.

Proof:

a) By Theorem A, if M is the compact 4-manifold of the hypothesis, then $M \# k(S^2 \times S^2)$ is s-cobordant to an almost smoothable compact manifold. But if M is not almost smoothable, neither is $M \# k(S^2 \times S^2)$. Hence by 3.4, the s-cobordism cannot have a handle decomposition.

b) Let V be the s-cobordism in a). Suppose V is a handlebody on ∂V. By 3.2, there is a 4-plane bundle ξ on V which restricts to $\tau(\partial V)_0$ on $(\partial V)_0$. Since $\xi = r^*\tau(\partial_- V)$, $\xi|(\partial V)_0 = \tau(\partial V)_0$ has a vector bundle reduction. But this implies $\partial_+ V$ is almost smoothable, giving a contradiction.

Theorem C: Let V be a compact cobordism between almost smoothable 4-manifolds which is a product along the boundary. Then V has a topological handle decomposition on $\partial_- V$.

Proof:

1. We may assume $\partial_- V$ and $\partial_+ V$ are non-empty:

Just remove one or two open discs from V as necessary to make $\partial_- V$ and $\partial_+ V$ non-empty. Obviously if the new V has a handle decomposition on $\partial_- V$ so does the original cobordism.

2. We may assume $\tau(V)$ reduces to a vector bundle rel L, $L = \partial(\partial_- V) \times I$ the (possibly empty) "lateral" surface of V:

Suppose the obstruction $\kappa(V) \in H^4(V,L;Z_2)$ to extending the reduction of $\tau(V)|L$ (induced by the smoothing of $\partial(\partial_- V)$) is non-zero. Let $\alpha \in H_1(V, \partial_+ V \cup \partial_- V; Z_2)$ be the dual class. Then it is easy to see that α is represented by a finite collection of locally flat embedded arcs going from $\partial_- V$ to $\partial_+ V$. Now each arc is the core of a 1-handle $I \times D^4$ going from $\partial_- V$ to $\partial_+ V$. Let $P^4 \subset \text{Int } D^4$ be a compact contractible 4-manifold with ∂P the Poincaré homology sphere [F]. Remove $I \times \text{Int } P$ from each of the above 1-handles. This gives a new compact cobordism which is a product along the boundary, and it is again obvious that if the new V has a handle decomposition on $\partial_- V$ so did the original cobordism. Since the obstruction to reducing $\tau(P)$ to a vector bundle rel ∂P is non-zero, it follows that the tangent bundle of the new V reduces to a vector bundle rel the new L.

3. If $\tau(V)$ reduces to a vector bundle rel L, V has a handle decomposition on $\partial_- V$:

The reduction of $\tau(V)$ defines stable reductions of $\tau(\partial_\pm V)$ and by 2.1, reductions of the $\tau(\partial_\pm V)$ themselves. By Lashof and Shaneson [LS], there is a compatible smoothing of $\partial_\pm V \# k(S^2 \times S^2)$, for some k. We may think of $\partial_\pm V \# k(S^2 \times S^2)$ as embedded in outside collar neighborhoods of the $\partial_\pm V$ by first adding trivial 2-handles to $\partial_\pm V \times I$ and then cancelling 3-handles. Thus we have a smoothable manifold W, $\partial_- W = \partial_- V \# k(S^2 \times S^2)$ and $\partial_+ W = \partial_+ V \# k(S^2 \times S^2)$. V is constructed by first adding trivial 2-handles to $\partial_- V \times I$ to reach $\partial_- W$, and then attaching the (smooth) handles of W to reach $\partial_+ W$, and finally attaching the dual three handles to $\partial_+ W$ to get to $\partial_+ V$.

<u>Addendum</u>: We may assume the handles of a given dimension are attached disjointly in order of increasing dimension.

<u>Proof</u>: In the handle decomposition given in the proof above, one can certainly assume the 0, 1, 2 handles are attached before any of the 3, 4, 5 handles, by taking such a handle decomposition for the smooth manifold W. Hence using only general position arguments one can arrange the 0, 1, 2 handles in order on $\partial_- V$ and the dual 0, 1, 2, handles in order on $\partial_+ V$.

<u>Notation</u>: Let V be a compact connected cobordism between four manifolds, and let $H \subset V$ be a 1-handle $I \times D^4$ going from $\partial_- V$ to $\partial_+ V$.

Proposition 3.7: If M is a compact almost smooth 4-manifold, then there is a k such that $M \# k(S^2 \times S^2)$ is an almost handlebody.

Proof: Immediate from Theorems A and D.

We can also add a little to our knowledge of homotopy RP^4's. By Theorem D, all the Cappell-Shaneson RP^4's [CS] and the Finteshel-Stern exotic RP^4 [FS] are homeomorphic to RP^4 mod connected sums with $S^2 \times S^2$'s. Secondly, we note that since $H^3(RP^4;Z_2) \neq 0$ there is an exotic almost smoothing of RP^4. The bundle obstruction to extending this smoothing over the last point is zero, as remarked above (see B2 and the proof thereof). Thus we get a non-trivial smoothing of $RP^4 \# k(S^2 \times S^2)$ by [LS]. Also note that we can assume the smoothing is standard on a neighborhood of RP^2.

4. Disc Bundles.

In [St], R. Stern did a detailed study of the problem of finding a disc bundle inside a given microbundle. He was able to deal with this question except for five dimensional bundles. We offer,

Theorem E: Let X be a 5-dimensional complex. Any 5-dimensional microbundle over X contains a topological disc bundle.

Remark: We do not claim the disc bundle is unique. That involves unknown homotopy groups of Top_4/O_4.

The following is due to Stern for $k > 2$.

Corollary 4.1: Let M^{2k+1} be a closed manifold. Then the tangent microbundle of M splits off a line bundle.

Proof of Corollary: Let $Top(I)_n$, resp. $Top(S)_n$, denote the group of homeomorphisms of I^n, resp. S^n. An n-dimensional microbundle over X contains a disc bundle if and only if $\xi : X \to BTop_n$ lifts to $BTop(I)_n$. Since the restriction map $Top(I)_n \to Top(S)_{n-1}$ is a homotopy equivalence, one has the fibration:

$$* \qquad \mathrm{Top}_{n-1} \to \mathrm{Top(I)}_n \to S^{n-1},$$

and hence the fibration: $S^{n-1} \to \mathrm{BTop}_{n-1} \to \mathrm{BTop(I)}_n$. Thus ξ will split off a line bundle if its Euler class is zero. But for $n = 2k + 1$, the Euler class of $\tau(M)$ is zero.

<u>Proof of Theorem E</u>: From (*) we see that $\mathrm{Top}_4/0_4 \to \mathrm{Top(I)}_5/0_5$ is a homotopy equivalence and that we have the fibration:

$$\mathrm{Top}_4/0_4 \to \mathrm{Top}_5/0_5 \to \mathrm{Top}_5/\mathrm{Top(I)}_5.$$

In particular, since $\pi_4(\mathrm{Top}_5/0_5) = 0$ and $\pi_3(\mathrm{Top}_4/0_4) \to \pi_3(\mathrm{Top}_5/0_5)$ is an isomorphism by our main theorem, we see that $\pi_4(\mathrm{Top}_5/\mathrm{Top(I)}_5) = 0$ and that $\pi_3(\mathrm{Top}_5/0_5) \to \pi_3(\mathrm{Top}_5/\mathrm{Top(I)}_5)$ is trivial. From the map of fibrations:

$$\begin{array}{ccccc} & & \mathrm{Top}_5/0_5 & \longrightarrow & \mathrm{Top}_5/\mathrm{Top(I)}_5 \\ & & \downarrow & & \downarrow \\ & & \mathrm{B0}_5 & \longrightarrow & \mathrm{BTop(I)}_5 \\ & & \downarrow & & \downarrow \\ X & \xrightarrow{\xi} & \mathrm{BTop}_5 & = & \mathrm{BTop}_5 \end{array} ,$$

we see that $\xi|X^3$ lifts to $\mathrm{B0}_5$ since $\pi_i(\mathrm{Top}_5/0_5) = 0$ for $i < 3$, and that the obstruction to getting a lift to $\mathrm{B0}_5$ over X^4 in $H^4(X;\pi_3(\mathrm{Top}_5/0_5))$ maps to zero in $H^4(X;\pi_3(\mathrm{Top}_5/\mathrm{Top(I)}_5)$. Thus $\xi|X^4$ lifts to $\mathrm{BTop(I)}_5$. The lift extends to X since $\pi_4(\mathrm{Top}_5/\mathrm{Top(I)}_5) = 0$.

5. $\pi_i(\mathrm{Top}_4/O_4)$.

Case $i = 2$:

An element $\alpha \in \pi_2(\mathrm{Top}_4/O_4)$ defines an exotic smoothing of $R \times S^2 \times S^1$ as follows: Since the tangent bundle of the standard smoothing is trivial, the classifying map $\tau : R \times S^2 \times S^1 \to B\mathrm{Top}_4$ can be taken to be the constant map to the base point. Define a lift τ_α of τ to BO_4 by $\tau_\alpha = ifp$, where $p : R \times S^2 \times S^1 \to S^2$ is projection, $f : S^2 \to \mathrm{Top}_4/O_4$ represents α and $i : \mathrm{Top}_4/O_4 \to BO_4$ is the inclusion of the fibre when we consider BO_4 as a fibre space over $B\mathrm{Top}_4$. This defines a homotopy class of lifts τ_α and hence a sliced concordance class of smoothings $(R \times S^2 \times S^1)_\alpha$ [LS].

Proposition 5.1: There is a compact 4-manifold V with $\partial V = S^2 \times S^1$ which is h-cobordant rel boundary to $D^3 \times S^1 \# k(S^2 \times S^2)$ and such that the smoothing α of $R \times S^2 \times S^1$ extends to a smoothing of $W = V \cup$ open collar (i.e., the open bicollar of ∂V in W identifies with $R \times S^2 \times S^1$).

Proof: Since $\pi_2(\mathrm{Top}_5/O_5) = 0$, the smoothing α is stably equivalent to the standard smoothing. Thus there is a smoothing β of $R \times S^2 \times S^1 \times I$, $I = [-1,1]$, which is the standard smoothing near $R \times S^2 \times S^1 \times \pm 1$, the smoothing $\alpha \times 1$ on a product neighborhood of $R \times S^2 \times S^1 \times 0$ and with β isotopic to the standard smoothing rel a product neighborhood of the boundary. Identifying $R \times S^2$ with $R^3 - 0$, we can get a smoothing γ of $R^3 \times S^1 \times I$ which is standard near $R^3 \times S^1 \times \pm 1$ and equal to β outside $D^3_\xi \times S^1 \times I$, D^3_ξ a small disc about 0 in R^3.

We can deform the projection $p : (R^3 \times S^1 \times I)_\gamma \to I$ to a smooth map p' transverse to 0 in I, rel the complement of $D^3_\xi \times S^1 \times I$ and a collar neighborhood of the boundary. Let $W = p'^{-1}(0)$. Then W is smooth and the end of W is topologically the same as the end of $R^3 \times S^1$ and has the smoothing α. The composition $qj : W \to R^3 \times S^1$, j the inclusion of W in $R^3 \times S^1 \times I$ and q the projection of $R^3 \times S^1 \times I$ onto $R^3 \times S^1$, is a proper degree one normal map. Let V be the compact topological manifold with $\partial V = S^2 \times S^1$ such that $W = V \cup$ open collar. Then qj restricts to a degree one normal map $h : (V, \partial V) \to (D^3 \times S^1, S^2 \times S^1)$, the identity on the boundary. Following [FQ] or [CS] we can do smooth framed surgery on Int V mod $S^2 \times S^2$'s so that we get a homotopy equivalence of the new V with $D^3 \times S^1 \# k(S^2 \times S^2)$ rel boundary. In fact, as in the proof of Theorem A, we can assume the homotopy equivalence is actually an h-cobordism.

Lemma 5.2: Let α and β be smoothings of W and τ_α and τ_β the corresponding lifts of $\tau : W \to BTop_4$ to BO_4. If $\tau_\alpha \sim \tau_\beta$ on the base point $(p,q) \in S^2 \times S^1$, then $\tau_\alpha \sim \tau_\beta$ on $S^2 \times q$. ($\sim$ means homotopic through lifts.)

Proof: Since V is h-cobordant to $D^3 \times S^1 \# k(S^2 \times S^2)$, the Stiefel-Whitney classes $w_1(V)$ and $w_2(V)$ are zero. It follows that TW_α and hence τW is trivial. Thus we may assume τ sends W to the base point. Again since V is homotopy equivalent rel boundary to $D^3 \times S^1 \# k(S^2 \times S^2)$, the inclusion $i : S^2 \times q \to \partial V \subset W$ is homotopic to the constant map to (p,q). Since $\tau_\alpha \sim \tau_\beta$ on (p,q), $\tau_\alpha i \sim \tau_\beta i$ over τi; i.e., $\tau_\alpha \sim \tau_\beta$ on $S^2 \times q$.

Proposition 5.3: Let V be homotopy equivalent rel boundary to $D^3 \times S^1 \# k(S^2 \times S^2)$ and let $W = V \cup$ open collar. Let $M = V \cup D^3 \times S^1$, identified along their boundaries, and let α be a smoothing of W which is standard on a neighborhood of $(p,q) \in S^2 \times S^1$ in the bicollar. If M is almost smoothable, τ_α is homotopic to the standard lift on a neighborhood of $S^2 \times q$ in the bicollar.

Proof: We identify W with an open neighborhood of V in M. Let β be an almost smoothing of M. Since τW is trivial, if $\tau_\alpha, \tau_\beta : W \to Top_4/O_4 \subset BO_4$ do not land in the same component we can always change τ_α by composition with an element g of Top_4 to achieve this; and then $\tau_\alpha' \sim \tau_\beta$ on the base point, $\tau_\alpha' = g\tau_\alpha$. By 6.2, $\tau_\alpha' \sim \tau_\beta$ on S^2. Since we can take $M_0 = M - (0,q')$, $(0,q') \in D^3 \times S^1$, $q' \neq q$, τ_β extends over $D^3 \times q$. Take the trivialization of τW to be that given by $T(M_0)_\beta$ so that $\tau : M_0 \to$ base point and $\tau_\beta : M_0 \to (1) \subset Top_4/O_4$. Since $\tau_\alpha' \sim \tau_\beta$ on $S^2 \times q$, $\tau_\alpha' | S^2 \times q$ is homotopic to the constant map onto (1). By composing with g^{-1} we see that $\tau_\alpha | S^2 \times q$ is homotopic to a constant map, and since α was standard on a neighborhood of the base point, τ_α must be homotopic to the standard lift on a neighborhood of $S^2 \times q$.

Corollary 5.4: Let $\alpha \in \pi_2(Top_4/O_4)$ and let $M = V \cup D^3 \times S^1$, where V is given by 5.1. If M is almost smoothable $\alpha = 0$.

Proposition 5.5: Let α and M be as in 5.4. If the universal cover of M is smoothable, $\alpha = 0$.

Proof: The obstruction to smoothing M_0 with a given smoothing β in a neighborhood of the base point is a class $\mathcal{O}_\beta \in H^3(M;\pi_2(Top_4/O_4))$. Indeed if β is isotopic to α on a neighborhood of the base point it extends to W, and the obstruction to extending τ_β, and hence β, to M_0 is a class $\mathcal{O}_\beta$ as above. If τ_β corresponds to a different component of Top_4/O_4 than τ_α on the base point, $\tau_\alpha' = g\tau_\alpha$ will be in the same component for some $g \in Top_4$, and hence τ_β will extend over W in any case so that we get an obstruction to smoothing M_0 as above.

If $f : \tilde{M} \to M$ is the universal cover, $f^* : H^3(M;\pi_2(Top_4/O_4)) \to H^3(\tilde{M};\pi_2(Top_4/O_4))$ is an isomorphism since M has the homotopy type of $S^3 \times S^1 \# k(S^2 \times S^2)$. If $\mathcal{O}_\beta$ is non zero, $f^*\mathcal{O}_\beta \neq 0$; but $f^*\mathcal{O}_\beta$ is the obstruction to smoothing M with the pull back smoothing $\tilde{\beta}$ on a neighborhood of the base point in M. Thus if M is smoothable $f^*\mathcal{O}_\beta = 0$ for some β, so $\mathcal{O}_\beta = 0$ and M_0 is smoothable. Hence $\alpha = 0$ by 5.4.

Proposition 5.6: Let α and M be as in 5.5. Then M is homeomorphic to $S^3 \times R \# \infty(S^2 \times S^2)$ and hence smoothable.

Proof: $M = V \cup D^3 \times S^1$ and $M - 0 \times S^1$ is homeomorphic to W. So $\tilde{M} = \tilde{V} \cup D^3 \times R$ and $\tilde{M} - 0 \times R$ is homeomorphic to $\tilde{W}$. Since W is properly h-cobordant to $R^3 \times S^1 \# k(S^2 \times S^2)$, $\tilde{W}$ is properly h-cobordant to $R^4 \# \infty(S^2 \times S^2)$. Since W is smoothable by the pull back of α, Freedman's theorem says $\tilde{W} = R^4 \# \infty(S^2 \times S^2)$. In particular, we can perform topological surgery on W to obtain R^4 and this changes M to a manifold M', the proper homotopy type of $S^3 \times R$. But Siebenmann [F] has shown that such a manifold is homeomorphic to $S^3 \times R$. But then $\tilde{M} = S^3 \times R \# \infty(S^2 \times S^2)$, connected along an embedding f of R^4 in $S^3 \times R$. The Lemma below shows that there is a homeomorphism h of $S^3 \times R$ such that hf is isotopic to the standard embedding of R^4 and hence M is homeomorphic to the standard (smooth) connected sum.

Lemma 5.7: If $f : R^4 \to S^3 \times R$ is any embedding, then there is a smooth embedding $g : R^4 \to S^3 \times R$ and a homeomorphism h of $S^3 \times R$ such that $hf = g$ on D^4.

Proof: We can smoothly identify $S^3 \times R$ with $R^4 - q$, $q \neq 0$, so that f(0) is identified with 0 in R^4. By Kister's theorem [K] there is an ambient homeomorphism k of R^4 with $k(0) = 0$ and $k|D^4 = f|D^4$. Choose $\xi > 0$ such that $q \notin D^4_\xi \cup f(D^4_\xi)$. Then we may assume $k|D^4_\xi = f|D^4_\xi$ and $k(q) = q$. Thus k restricts to a homeomorphism of $S^3 \times R$ so that

$ki = f|D^4_\xi$, where $i : D^4_\xi \to S^3 \times R$ is the smooth embedding so that composed with the inclusion of $S^3 \times R$ in R^4 it is the standard (smooth) embedding of D^4_ξ in R^4. Then $k^{-1}f|D^4$ is isotopic to a smooth embedding g and so using the isotopy extension theorem, we can find a homeomorphism h such that $hf = g$ on D^4.

<u>Main</u> <u>Theorem</u> <u>I</u>: $\pi_2(Top_4/O_4) = 0$.

<u>Proof</u>: This follows immediately from 5.5 and 5.6.

Case $i = 3$:

Let $\alpha \in \pi_3(Top_4/O_4)$, then α defines a smoothing, unique up to sliced concordance, of $S^3 \times R$ which is standard near the base point. We denote this by $(S^3 \times R)_\alpha$. In [LS], it is shown that if α is stably trivial this is the end of a smooth manifold W the proper homotopy type of $R^4 \# k(S^2 \times S^2) = (k(S^2 \times S^2))_0$. By Freedman's classification theorem the underlying topological manifold W is homeomorphic to $R^4 \# k(S^2 \times S^2)$. Since the tangent bundle of the latter is trivial, we can assume $\tau : W \to BTop_4$ is the constant map to the base point, and the standard smoothing β gives a constant lift τ_β to the base point of BO_4. Then W_α defines a lift $\tau_\alpha : W \to Top_4/O_4 \subset BO_4$ of τ. Since the inclusion of $S^3 \times 0 \subset S^3 \times R \subset W$ is homotopically trivial, $\tau_\alpha|S^3$ is homotopic to $\tau_\beta|S^3$.

We wish to show that $\alpha = 0$; but we cannot conclude this directly from the above. That is, if $h : W \to R^4 \# k(S^2 \times S^2)$ is the homeomorphism and $i : S^3 \times R \to W$ is the inclusion, then we do not know that hi is the standard inclusion of the end in $R^4 \# k(S^2 \times S^2)$ and hence we do not know that the pull back by hi of β is the standard smoothing of $S^3 \times R$. On the other hand, τ is homotopic to $\tau'|W$, where τ' is a classifying map for $\overline{W}$, the one point compactification of W. $\overline{W}$ is homeomorphic to $S^4 \# k(S^2 \times S^2)$ and we take τ' to be the constant map to the base point on a neighborhood of the point at ∞. By the covering homotopy property τ_α and τ_β are homotopic to lifts τ'_α and τ'_β of $\tau'|W$, where τ'_β extends to a lift of τ' with τ'_β constant on a neighborhood of ∞. But since the compactification of one end of $S^3 \times R$ is R^4 (with $S^3 \times R = R^4 - 0$) and since $\tau'_\alpha \sim \tau'_\beta$ on S^3, $\tau'_\alpha|S^3 \times R$ extends to a lift of $\tau'|R^4$. But $\tau'|R^4$ has only one homotopy class of lifts which is standard over a base point. Thus $\tau'_\alpha|S^3$ is standard and $\alpha = 0$.

Main Theorem II: $j_* : \pi_3(Top_4/0_4) \to \pi_3(Top/0)$ is an isomorphism.

Proof: The above argument shows the stabilization homomorphism j_* is a monomorphism. On the other hand, Freedman [F] has exhibited an almost smoothed almost parallelizable closed 1-connected manifold of index 8. It follows that the smoothing of the end of this manifold represents a stably non-trivial element of $\pi_3(Top_4/0_4)$. Hence j_* is an isomorphism.

Remark: The base point is irrelevant in the main theorems since $Top_4/0_4$ is homogeneous.

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