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Λ-SPLITTING 4-MANIFOLDS

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GIVEN a homotopy (or homology) equivalence, $f: N \to X$, from a manifold to a Poincaré space, and some "decomposition" of X, we may ask if f is homotopic to a map, g, which restricts to an equivalence over each "piece" of the "decomposition". Finding g is called splitting f. Splitting has been a fruitful technique in classifying manifolds of dimension at least five. Here we carry though a version of 4-dimensional spitting and give applications to 4-dimensional classification problems.

Let (X_1, Y_1) and (X_2, Y_2) be Poincaré pairs of formal dimension $= 4$, and with product neighborhoods $Y_i \times [0, 1]$ of Y_i. Let X_1 and X_2 be connected. Let Y_0 be a component of Y_i for $i = 1$ and 2 with $\pi_1(Y_0) \to \pi_1(X_i)$ an isomorphism. Let $X = X_1 \underset{Y_0}{\cup} X_2$. Let $f: (N, \partial) \to (X, \partial)$ be a $\Lambda = Z[\pi_1(X)]$—equivalence of pairs, i.e. inducing isomorphisms on homology with Λ-coefficients (∂ may be empty), where (N, ∂) is a smooth manifold.

THEOREM 1. *The map f is homotopic* (rel ∂N) *to a map g with $g|g^{-1}(X_i)$ and $g|g^{-1}(Y_0)$ Λ-equivalences, $i = 1$ and* 2.

Proof. Our proof is based on Browder's High-Dimensional Splitting Argument[1]. An elementary argument using $\pi_1(X_i, Y_0) = 0$ enables us to homotop f (we continue to call the map f) so that $f^{-1}(X_i)$ and $f^{-1}(Y_0)$ are connected.

Let $N_i = f^{-1}(X_i)$, $M = f^{-1}(Y_0)$. Choose handle decompositions $\mathcal{H}(N_i, M)$ for N_i relative to M. We may assume $\mathcal{H}(N_i, M)$ contains no 0-handles. By a standard homotopy (see [13]) we may trade first the 1-handles of $\mathcal{H}(N_1, M)$, then the 1-handles of $\mathcal{H}(N_2, M)$. This again uses $\pi_1(X_i, Y_0) = 0$. This results in a new f, N_i, and M where N_i admits a handle decomposition $\mathcal{H}(N_i, M)$ without 1-handles. Now consider:

$$\begin{array}{ccccccccc}
 & & \pi_3(f) & \xrightarrow{\partial} & \pi_2(N_i, M) & \longrightarrow & \pi_2(X, Y_0) & & \\
 & & \downarrow{\scriptstyle h} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2(N_i, M; \Lambda) & \longrightarrow & H_2(N_i, M; \Lambda) & \longrightarrow & H_2(X_i, Y_0; \Lambda) & \longrightarrow & 0
\end{array}$$

The right-hand vertical is an isomorphism by the Hurewicz theorem and the centre vertical an epimorphism since $\pi_1(M_i, N) = 0$. Hence it follows easily that h is an epimorphism.

Let $\{\alpha_j\} \subset \pi_3(f)$ satisfy $h\{\alpha_j\}$ generates $K_2(N_1, M; \Lambda)$. $\{\partial\alpha_j\}$ is easily seen to be represented by relatively imbedded 2-disks. (This because the cores of the 2-handles of $\mathcal{H}(N_1, M)$ generate $\pi_2(N_1, M)$ as $\pi_1(M)$-module. In fact $\mathcal{H}(N_1, M)$ may be modified by handle passing so that $\{\partial\alpha_i\}$ is represented by cores of 2-handles).

According to Wall[5, p. 13], there is a unique regular homotopy class of relative immersions for which handle subtraction yields normal bordism. The following lemma shows that we not need to worry about the relative regular homotopy class of our disks.

LEMMA. *Let $(D^n, \partial) \xrightarrow{f} (k^{2n}, \partial)$ be a relative map of the n-disk into a smooth $2n$-manifold, $n > 1$. Then there is exactly one relative regular homotopy class of relative immersions homotopic to f.*

Proof. According to the Smale–Hirsch classification of immersions, the relative regular homotopy classes of relative immersions homotopic to f are in 1-1 correspondence with the homotopy classes of bundle maps:

$$\begin{array}{ccccc} \tau(S^{n-1}) & \xrightarrow{\text{injection}} & (D^{2n-1} \times S^{n-1}) & \longrightarrow & S^{n-1} \\ \downarrow & & \downarrow & & \\ \tau(D^n) & \xrightarrow{\text{injection}} & (D^{2n} \times D^n) & \longrightarrow & D^n \end{array}$$

and hence with the element of the homotopy group

$$\pi_n\left(\frac{SO(2n)}{SO(n)}, \quad \frac{SO(2n-1)}{SO(n)}\right) \approx \pi_n(SO(2n), SO(2n-1))$$
$$\approx \pi_n(S^{2n-1}) = 0$$

□

Again by a standard homotopy, we may alter f to subtract a tubular neighborhood, U, of the imbedded 2-disks representing $\{\partial\alpha_j\}$ from N_1, and add U to N_2. A calculation shows that $K*(N_1, M; \Lambda = 0, * \neq 3. K*(N_1N_2; \Lambda)' \simeq K*-1(N_2; \Lambda) \simeq k^{5-*}(N_2, M; \Lambda) = 0, * \neq 3$. It follows that $K_2(N_2, M; \Lambda)$ is the only non-zero kernel. By Theorem 2.3[5] $K_2(N_2, M; \Lambda)$ is stably free. Note that N_2 still has a handle decomposition rel M without 1-handles. Introduce some trivial (1-handle, 2-handle) pairs in $\mathscr{H}(N_2, M)$ and trade the 1-handles over to $\mathscr{H}(N_1, M)$. This makes $K_2(N_2, M; \Lambda)$ free. By the method above, represent a basis by imbedded disks and trade a tubular neighborhood by homotoping f to a map g. A calculation[5, Chap. 4] shows that $g: (N_2, M)$ (X_2, Y) is a Λ-equivalence of pairs. Since g induces a map between the Mayer–Vietoris sequences of

$$\left(M \begin{smallmatrix} \nearrow N_1 \searrow \\ \searrow N_2 \nearrow \end{smallmatrix} N\right)$$

and

$$\left(Y \begin{smallmatrix} \nearrow X_1 \searrow \\ \searrow X_2 \nearrow \end{smallmatrix} X\right),$$

inspection shows that $g: N_1 \to X_1$ is also a Λ-equivalence.

□

Note. Versions of Theorem 1 for simple-equivalences and for extending Λ-splittings on a boundary may be proved.

Putting our Λ-splitting theorem together with the "stable" surgery theorem of Cappell and Shaneson[2], we obtain a homology-surgery theorem for simply connected 4-manifolds minus a disk. Let $R(\Sigma^3)$ denote the Rochlin invariant of a homology 3-Sphere Σ^3.

THEOREM 2. *Let $f: K^4 \to Z$ be a degree 1 normal map from a simply connected smooth 4-manifold to a Poincaré space (assume $f|\partial K$ is an integral homology equivalence if $\partial K \neq \phi$). Then $f|: K^4 - D^4 \to Z - D^4$ (f restricted to the complement of an open 4-disk) is normally bordant (rel ∂K) to an integral homology equivalence, $g: P^4 \to Z - D^4$ with $\partial P^4 = \partial K^4 \cup \Sigma^3$, where Σ^3 is an integral homology 3-sphere satisfying*:

(1) $R(\Sigma^3) = [\sigma(Z) - \sigma(K)]/8$ (mod 2) *and*

(2) *$\pi_1(P)$ is in the normal closure of* $\text{inc}_* \, \pi_1(\Sigma^3)$.

Proof. Consider: $K' = K - D^4 \underset{S^3}{\cup} Q \xrightarrow{(f|\partial k) \overset{\text{def}}{=} j} Z - D^4$, where D^4 is the interior of an imbedded 4-disk, Q is a framed bordism of S^3 to a homology 3-sphere ω^3 with $\sigma(Q) = \sigma(Z) - \sigma(K)$ (Q may be taken to be a plumbing construction on $\bigoplus_{t \text{ copies}} \pm E_8$), and k is the canonical map $Q \xrightarrow{k} S^3$. Then $j|\partial K'$ is an integral homology equivalence. The surgery obstruction, $\mathcal{O}(j) = 0 \in \Gamma^4(Z[0] \to Z[0]) = L_4(0) = Z$. It follows from [2] that for some n,

$g \simeq j$ # id satisfy the conclusion of Theorem 1. Let $(N_1, M) = g^{-1}(X_1, Y)$. Since M also bounds $N_2 = g^{-1}(X_2)$, $R(M) = 0$. Set $P = N_1$—(an open tubular neighborhood of an imbedded arc connecting M and ω^3). The normal bordism from $K - D^4$ to P may be constructed as the union of three normal bordisms: $K - D^4$ to $K' - D^4$, $K' - D^4$ to N_1 (make H transverse to $(X_1, Y) \times I$), and N_1 to P. Then $\varsigma^3 = M \# \omega^3$ and $R(\Sigma^3) = R(M) + R(\omega^3) = R(M) = [(\sigma(Z) - \sigma(K))/8] \pmod 2$. Now X is simply connected and $g: N \simeq X$ so by Van Kampen's Theorem. $0 = \pi_1(P) \underset{\pi_1(\Sigma^3)}{*} \pi_1(N - P)$. Therefore the normal closure of $\pi_1(\Sigma^3)$ in each side is the entire group.

□

We now give some applications to the homology-classification of 4-manifolds.

COROLLARY 1A. *The kernel of Rochlin's homomorphism,* $R: \theta_3^H \to Z_2$ (θ_3^H = *homology—h—cobordism classes of homology* 3-*spheres*) *is generated by homology* 3-*spheres* $\Sigma^3 = \partial M^4$, *where* M^4 *is spin and integral-homology-equivalent to* $S^2 \vee S^2$ (*and is a handle body on* $\partial M = \Sigma^3$ *consisting of* 2, 3, *and* 4-*handles only*).

COROLLARY 1B. θ_3^H *is generated by homology* 3-*spheres* $\Sigma^3 = \partial M^4$, *where* M^4 *is integral-homology-equivalent to* S^2 (*and is a handle body on* $\partial M = \Sigma^3$ *consisting of* 2, 3, *and* 4-*handles only*).

Proof of 1A. Let $R(\Sigma^3) = 0$. It is well known that $\Sigma^3 = \partial M^4$, where M^4 is framed and $\sigma(M) = 0$. By [2], there is an integer n such that the surgery problem $M \# n(S^2 \times S^2) \to D^4 \# n(S^2 \times S^2)$ can be solved rel ∂M. Let the solution of $f: N^4 \to D^4 \# n(S^2 \times S^2)$ ($\partial N = \partial M = \Sigma^3$). By Theorem 1, the copies of $S^2 \times S^2$ may be split off inductively. The result is a bordism (homology equivalent to an $(n+1)$-punctured S^4) from ω^3 to n homology 3-spheres (these will be taken as generators) each of which bounds a homology $S^2 \times S^2 - D^4 \simeq S^2 \vee S^2$. The lemma follows.

Proof of 1B. The argument is similar to the one above. We generate Ker(R) by first doing "stable" surgery using $\pm CP^2$ in place of $S^2 \times S^2$, and then split. It then suffices to find a homology 3-sphere Σ^3 with $R(\Sigma^3) = 1$ and $\Sigma^3 = \partial M^4$, $M \simeq S^2$. It is well known that the Poincaré-homology-sphere, P, ($R(P) = 1$) is the boundary of the handle body consisting of the 4-ball with a 2-handle attached with framing $+1$ to the right-handed trefoil knot in ∂ (4-ball). 1B follows.

We obtain some equivalences of certain optimistic conjectures.

CONJECTURE "∂_h". *Every homology* 3-*sphere with Rochlin invariant zero bounds a contractible smooth manifold.*

CONJECTURE "∂_H". *Everyhomology* 3-*sphere with Rochlin invariant zero bounds an acyclic smooth manifold.*

CONJECTURE "S_h". *If* $f: (M^4, \partial) \to (X, \partial)$ *is a degree one normal map from a simply connected smooth manifold to a Poincaré space inducing a homology isomorphism on* ∂ *and if* $\sigma(M) = \sigma(X)$ *then f is normally bordant* (rel ∂) *to a homotopy equivalence.*

CONJECTURE "S_h". *With the preceding hypothesis conclude: f is normally bordant* (rel ∂) *to an integral homology equivalence.*

COROLLARY 2. "∂_h"⇔"S_h" *and* "∂_H"⇔"S_H".

Proof. The implications: "S_h" ⇒ "∂_h" and "S_H" ⇒ "∂_H" are well known. Assume "∂_H". By Theorem 2, normally bord $f|M - D^4$ to a homology equivalence $g: N \to X - D^4$. Then f is normally bordant to a homology equivalence $g \cup c: N \underset{\Sigma^3}{\cup} A \to X$ implying "S_H" ($c: (A, \Sigma^3) \to (D^4, \partial)$ is the collapsing map of an acyclic manifold). If "∂_h" is assumed we may find an A so that $\pi_1(N \underset{\Sigma^3}{\cup} A) = 0$ (by Theorem 2), yielding "S_h".

□

If M is an oriented 4-manifold let $\pitchfork_M$ be its integral intersection form.

where $(\tilde{A}_0, \Sigma_1)$ $(\tilde{A}_0 - D^4, S^3)$, $(\tilde{A}_n, \Sigma_n) \to (A_n - D^4, S^3)$, *and* $(\tilde{A}_i, \partial) \to (A_i - 2D^4, S^3 \amalg S^3)$ *are homology equivalences.*

Proof. By [4], $M \simeq \mathop{\#}\limits_{i=0}^{n} A_i$. Now apply Theorem 1 inductively.

Example. If $\pitchfork_M$ is odd and indefinite, $\pitchfork_M = \bigoplus\limits_{i=0}^{n} \pitchfork_{\pm CP^2}$.

By the techniques of Theorem 1 and the lemma a version of the $(\pi - \pi)$-surgery theorem [5, Chap. 4] follows (easily):

THEOREM 3. *Suppose* $f: (N, \partial) \to (X, Y)$ *is a degree* 1 *normal map from a smooth manifold to a Poincaré pair (over* $\Lambda = Z[\pi_1(X)]$ *coefficients) of formal dimension* $= 4$. *If* $\pi_1(Y) \to \pi_1(X)$ *is an isomorphism, then f is normally cobordant to a* Λ*-homology equivalence of pairs.*

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