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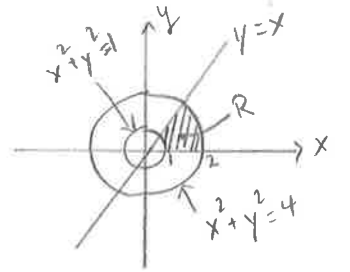
M20550 Calculus III Tutorial
Worksheet 8

1. Evaluate the given integral.

$$\iint_R \arctan\left(\frac{y}{x}\right)\, dA$$

where $R = \{(x, y) : 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}$.



Given the geometry of region R, it's best to compute the double integral using polar coordinates.

In polar, we know $dA = r\,dr\,d\theta$ and

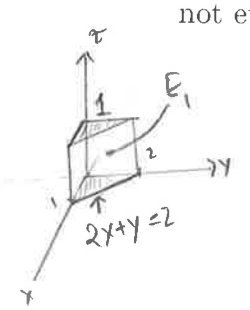
$$\arctan\left(\frac{y}{x}\right) = \arctan(\tan\theta) = \theta \quad \left(\text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right).$$

From the picture of the region R, we have $1 \leq r \leq 2$. To find the upper bound for θ, we need to find θ in (I) quad. such that $y = x$. With $y = x$, we have $r\sin\theta = r\cos\theta \Rightarrow \sin\theta = \cos\theta \Rightarrow \theta = \frac{\pi}{4}$ for θ in (I) quad. So, $0 \leq \theta \leq \frac{\pi}{4}$.

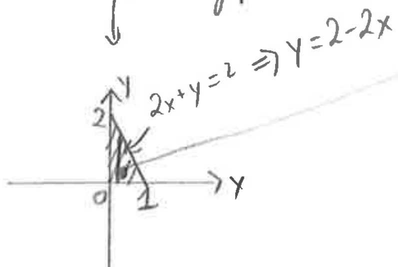
Thus,

$$\iint_R \arctan\left(\frac{y}{x}\right) dA = \int_0^{\pi/4}\int_1^2 \theta\, r\,dr\,d\theta = \int_0^{\pi/4} \frac{1}{2}r^2\theta\Big|_{r=1}^{r=2} d\theta = \int_0^{\pi/4}\frac{3}{2}\theta\,d\theta = \frac{3}{2}\cdot\frac{1}{2}\theta^2\Big|_0^{\pi/4} = \boxed{\frac{3}{64}\pi^2}$$

2. (a) Let E_1 be the solid lies under the plane $z = 1$ and above the region in the xy-plane bounded by $x = 0$, $y = 0$, and $2x + y = 2$. Write the triple integral $\iiint_{E_1} xz\, dV$ but do not evaluate it.



in the xy-plane



We'll use rectangular coordinates to write $\iiint_{E_1} xz\,dV$.

$$\iiint_{E_1} xz\,dV = \iint_R\left(\int_{z=0}^{z=1} xz\,dz\right) dA$$

(now write $dA = dy\,dx$, and the limits for y and x comes from the picture of R)

$$= \boxed{\int_{x=0}^{x=1}\int_{y=0}^{y=2-2x}\int_{z=0}^{z=1} xz\,dz\,dy\,dx}$$

With the order $dy\,dx$:

$0 \leq y \leq 2-2x$

$0 \leq x \leq 1$

$$\left(\text{Another answer is } \int_{y=0}^{y=2}\int_{x=0}^{x=\frac{2-y}{2}}\int_{z=0}^{z=1} xz\,dz\,dx\,dy\right)$$

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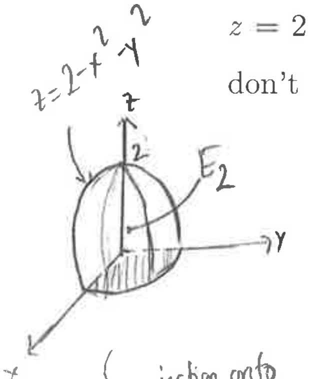
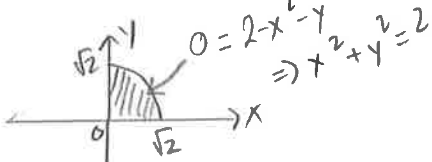
(b) Let E_2 be the solid region in the first octant that lies under the paraboloid $z = 2 - x^2 - y^2$. Write the triple integral $\iiint_{E_2} xz\,dV$ in cylindrical coordinates (you don't need to evaluate it).



projection onto xy-plane



$0 \le r \le \sqrt{2}$

$0 \le \theta \le \frac{\pi}{2}$

In cylindrical coordinates, $dV = r\,dz\,dr\,d\theta$

From the picture of E_2, we see that $0 \le z \le 2 - x^2 - y^2 = 2 - r^2$

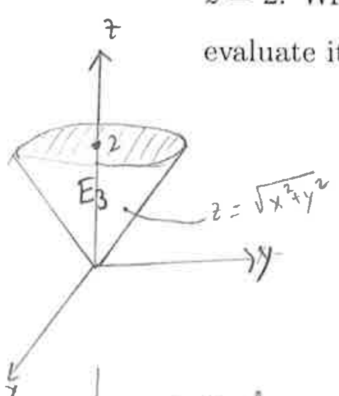
To get the bounds for r and θ, we look at the projection of the solid E_2 onto the xy-plane. We see that

$0 \le r \le \sqrt{2}$ and $0 \le \theta \le \frac{\pi}{2}$.

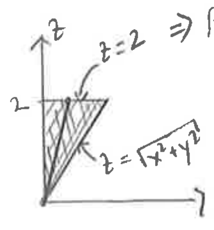
So, $$\iiint_{E_2} xz\,dV = \int_0^{\pi/2}\int_0^{\sqrt{2}}\int_{z=0}^{z=2-r^2} \underbrace{(r\cos\theta)}_{\substack{= \\ x \text{ in cylindrical coord.}}} z\, r\,dz\,dr\,d\theta$$

$$= \boxed{\int_0^{\pi/2}\int_0^{\sqrt{2}}\int_0^{2-r^2} z r^2 \cos\theta\,dz\,dr\,d\theta}$$

(c) Let E_3 be the solid region that lies above the cone $z = \sqrt{x^2+y^2}$ and below the plane $z = 2$. Write the triple integral $\iiint_{E_3} xz\,dV$ in spherical coordinates (you don't need to evaluate it).



cross section of E_3 with the positive yz-plane



$z=2 \Rightarrow \rho\cos\phi = 2$

$\Rightarrow \rho = \frac{2}{\cos\phi}$

$z=\sqrt{x^2+y^2} \Rightarrow \rho\cos\phi = \rho\sin\phi$

$\Rightarrow \cos\phi = \sin\phi$

$\Rightarrow \phi = \frac{\pi}{4}$

$0 \le \rho \le \frac{2}{\cos\phi}$

$0 \le \phi \le \frac{\pi}{4}$

In spherical coord., $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$ and $x = \rho\sin\phi\cos\theta$

$z = \rho\cos\phi$

We can easily see that $0 \le \theta \le 2\pi$. Now, in order to get the bounds for ρ and ϕ, we can look at the cross section of the solid E_3 with the positive yz-plane (see picture on left). Converting the appropriate equations to spherical coord., we get

$0 \le \rho \le \frac{2}{\cos\phi}$ and $0 \le \phi \le \frac{\pi}{4}$

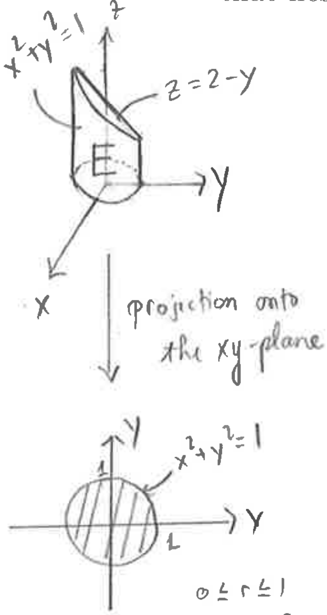
(recall, ϕ is measured from the positive z-axis)

Thus,

$$\iiint_{E_3} xz\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\frac{2}{\cos\phi}} (\rho\sin\phi\cos\theta)(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \boxed{\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\frac{2}{\cos\phi}} \rho^4 \sin^2\phi\cos\phi\cos\theta\,d\rho\,d\phi\,d\theta}$$

3. Write the integral that computes the volume of the part of the solid cylinder $x^2 + y^2 \leq 1$ that lies between the planes $z = 0$ and $z = 2 - y$. Call the solid E



$0 \leq r \leq 1$
$0 \leq \theta \leq 2\pi$

We have $Vol(E) = \iiint_E 1\, dV$. From the picture of E, we see that it will be best to use Cylindrical Coord. to compute the triple integral.

$dV = r\, dz\, dr\, d\theta$

$0 \leq z \leq 2 - y = 2 - r\sin\theta$

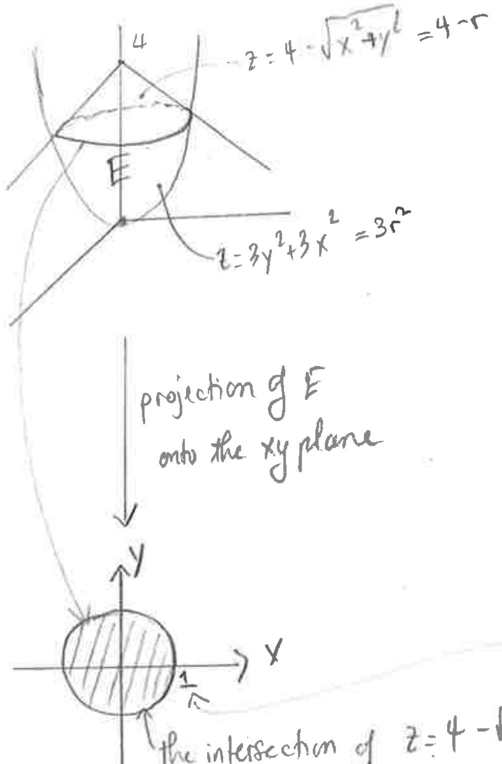
$\left.\begin{matrix} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{matrix}\right\}$ from the picture of the projection on the left.

Thus, $\iiint_E 1\, dV = \int_0^{2\pi}\int_0^1\int_0^{2-r\sin\theta} r\, dz\, dr\, d\theta$

$$= \int_0^{2\pi}\int_0^1 r(2 - r\sin\theta)\, dr\, d\theta = \int_0^{2\pi} \left(r^2 - \frac{1}{3}r^3\sin\theta\right)\Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi}\left(1 - \frac{1}{3}\sin\theta\right)d\theta = \left(\theta + \frac{1}{3}\cos\theta\right)\Big|_0^{2\pi} = 2\pi + \frac{1}{3} - \frac{1}{3} = \boxed{2\pi}$$

4. Set up, but do not solve, the integral that gives the volume of the solid region bounded by the paraboloid $z = 3y^2 + 3x^2$ and the cone $z = 4 - \sqrt{x^2 + y^2}$. Call the solid E



the intersection of $z = 4 - \sqrt{x^2+y^2} = 4 - r$ and $z = 3y^2 + 3x^2 = 3r^2$

Based upon the picture of the solid E, I think it's best to set up the triple integral using cylindrical coordinate

$$Vol(E) = \iiint_E 1\, dV, \quad dV = r\, dz\, dr\, d\theta.$$

We have $3y^2 + 3x^2 \leq z \leq 4 - \sqrt{x^2+y^2}$

$\Rightarrow 3r^2 \leq z \leq 4 - r$

To get the bounds for r and θ, we use the projection of E onto the xy-plane. We have $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$ (see left for details).

So, $\boxed{Vol(E) = \int_0^{2\pi}\int_0^1\int_{3r^2}^{4-r} r\, dz\, dr\, d\theta}$

$\rightarrow 4 - r = 3r^2 \Rightarrow 3r^2 + r - 4 = 0 \Rightarrow (3r+4)(r-1) = 0$

$\Rightarrow r = 1$

or $r = -\frac{4}{3}$ (ignore, want $r > 0$)

So, $0 \leq r \leq 1$
$0 \leq \theta \leq 2\pi$

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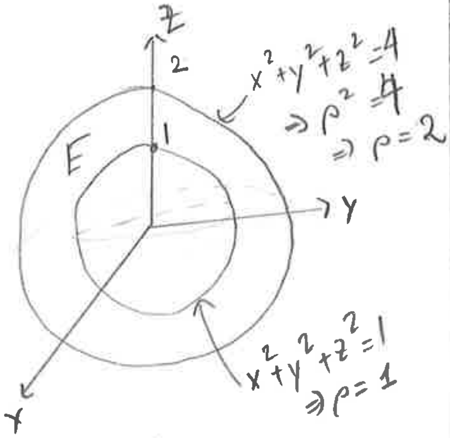
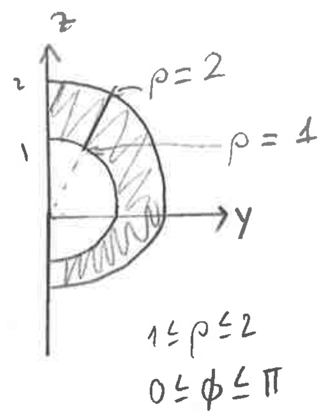
5. Find the mass of $\overbrace{\text{the solid between the spheres } x^2+y^2+z^2=1 \text{ and } x^2+y^2+z^2=4}^{\text{denote } E}$ whose density is $\rho(x,y,z) = x^2+y^2+z^2$.



cross section of E with yz-plane (y positive)



We have $\displaystyle \text{mass} = \iiint_E (\text{density})\, dV$

$$= \iiint_E x^2+y^2+z^2 \, dV$$

Since the solid is between 2 spheres, it's best to use spherical coordinates to compute the triple integral. So,

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

and $x^2+y^2+z^2 = \rho^2$

and from the cross section in left, $1 \le \rho \le 2$ and $0 \le \phi \le \pi$

and we can see that $0 \le \theta \le 2\pi$.

Thus, $$\text{mass} = \int_0^{2\pi}\int_0^{\pi}\int_1^{2} (\rho^2)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{5}\rho^5 \sin\phi \Big|_{\rho=1}^{\rho=2} d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{31}{5} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} -\frac{31}{5}\cos\phi \Big|_{\phi=0}^{\phi=\pi} d\theta$$

$$= \int_0^{2\pi} -\frac{31}{5}(-1-1)\, d\theta$$

$$= \int_0^{2\pi} \frac{62}{5}\, d\theta$$

$$= \frac{62}{5}(2\pi) = \boxed{\frac{124}{5}\pi}$$

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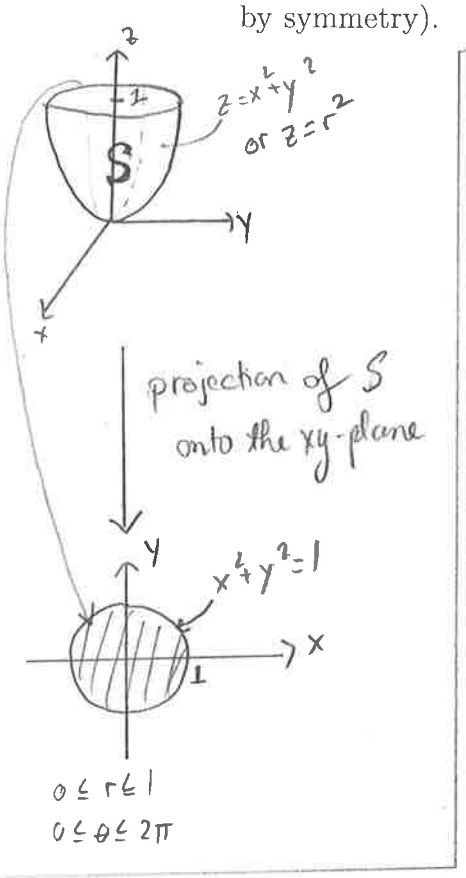
6. Find the center of mass of the solid S bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$ if S has constant density 1 and total mass $\frac{\pi}{2}$. (Hint: $\bar{x}$ and $\bar{y}$ can be found by symmetry).



Because S is symmetric about the z-axis, we get $\bar{x} = 0 = \bar{y}$.

So, we only need to find $\bar{z}$:

$$\bar{z} = \frac{M_{xy}}{\text{mass}} = \frac{2}{\pi}\iiint_S z(\text{density})\, dV$$

$$\Rightarrow \quad \bar{z} = \frac{2}{\pi}\iiint_S z\, dV$$

Now, based upon the picture of S, it's best to compute this triple integral using cylindrical coord.

We see that $x^2+y^2 \le z \le 1 \Rightarrow r^2 \le z \le 1$

And from the projection of S onto the xy-plane, we get

$0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

$$\text{So, } \bar{z} = \frac{2}{\pi}\int_0^{2\pi}\int_0^1\int_{r^2}^1 z\, r\, dz\, dr\, d\theta$$

$$= \frac{2}{\pi}\int_0^{2\pi}\int_0^1 \frac{1}{2}(1-r^4)r\, dr\, d\theta$$

$$= \frac{2}{\pi}\int_0^{2\pi}\int_0^1 \frac{1}{2}r - \frac{1}{2}r^5\, dr\, d\theta$$

$$= \frac{2}{\pi}\int_0^{2\pi}\left(\frac{1}{4} - \frac{1}{12}\right) d\theta$$

$$= \frac{2}{\pi}\,\frac{2}{12}\cdot 2\pi$$

$$= \frac{2}{3}$$

Thus, the center of mass is $\boxed{(0, 0, \frac{2}{3})}$.