Name: SOLUTIONS Date: 10/13/2016

M20550 Calculus III Tutorial
Worksheet 7

1. Find the minimum distance from the parabola $y = x^2$ to the point $(0, 9)$.

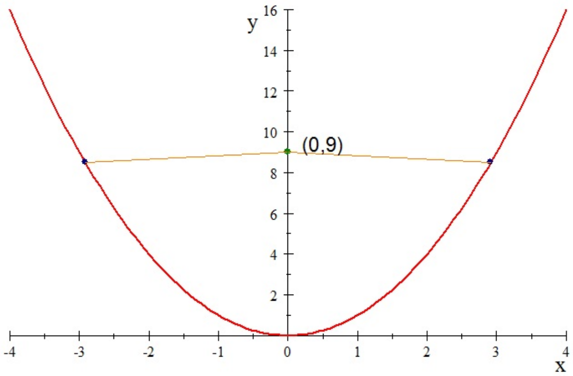
Solution: We want to minimize the function $d(x, y) = \sqrt{x^2 + (y-9)^2}$ subject to $y = x^2$. Since $f(x, y) = x^2 + (y-9)^2 \geq 0$ and $d(x, y) \geq 0$, the minimums of d and f occur at the same points, so to make the algebra simpler, let's minimize f subject to $y = x^2$ instead. Let $g = x^2 - y$. So, since $\nabla f = \langle 2x, 2y - 18\rangle$ and $\nabla g = \langle 2x, -1\rangle$, the system we get is

$$\begin{cases} 2x &= 2\lambda x & \textcircled{1} \\ 2y - 18 &= -\lambda & \textcircled{2} \\ y &= x^2 & \textcircled{3} \end{cases}$$

Equation $\textcircled{1} \iff 2x - 2x\lambda = 0$ has two solutions, either $x = 0$ or $\lambda = 1$.
If $\lambda = 1$, then by $\textcircled{2}$ we have $2y - 18 = -1$, so that $y = \frac{17}{2}$, and plugging this into $\textcircled{3}$ we get that $x = \pm\sqrt{\frac{17}{2}}$. So this case gives us the critical points $\left(\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$ and $\left(-\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$.
If $x = 0$, then by $\textcircled{3}$, $y = 0$, and so the critical point in this case is $(0, 0)$.
Now we check for the minimums:

(x, y)	$f(x, y)$
$(0, 0)$	81
$\left(\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$	$\frac{35}{4}$
$\left(-\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$	$\frac{35}{4}$

So, the minimum value of f is $\dfrac{35}{4}$, and hence the minimum value of d is $\sqrt{\frac{35}{4}}$, and this occurs at the points $\left(\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$ and $\left(-\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$. Said another way, the closest points to $(0, 9)$ on the parabola $y = x^2$ are the points $\left(\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$ and $\left(-\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$ which are a distance of $\sqrt{\frac{35}{4}}$ away.



(Note that you could also do this problem by plugging $y = x^2$ into the equation for d or f and using Calculus I methods.)

2. Maximize the function $f(x, y, z) = xyz$ subject to the constraint $x^2 + 2y^2 + 3z^2 = 9$, assuming that x, y, and z are nonnegative. Explain why the extremum you find is a maximum.

Solution: The gradient of f is

$$\nabla f = \langle yz, xz, xy \rangle .$$

Let $g = x^2 + 2y^2 + 3z^2$, then $\nabla g = \langle 2x, 4y, 6z \rangle$. The system of equations we get by Lagrange multipliers is thus

$$\begin{cases} yz &= 2\lambda x & \textcircled{1} \\ xz &= 4\lambda y & \textcircled{2} \\ xy &= 6\lambda z & \textcircled{3} \\ x^2 + 2y^2 + 3z^2 &= 9 & \textcircled{4} \end{cases}$$

Solving ① for λ gives $\lambda = \dfrac{yz}{2x}$, however, to do this, we have to make sure $x \neq 0$. If $x = 0$, then $f = 0$, but since there are solutions (a, b, c) of $g = 9$ (e.g., $(2, 1, 1)$) such that $f(a, b, c) > 0$, any solution with $x = 0$ will not be a maximum. This same reasoning shows that anything with $y = 0$ or $z = 0$ is not a maximum, so we can assume that $x, y, z > 0$. If we plug λ into ②, we get $xz = 4\dfrac{yz}{2x}y$, and as long as $z \neq 0$ (which is true since $z > 0$) we can divide both sides by z and simplify to get $x^2 = 2y^2$. Plugging λ into ③ we have $xy = 6\dfrac{yz}{2x}z$, and as long as $y \neq 0$ (which is isn't since $y > 0$) we can divide both sides by y and simplify to get $x^2 = 3z^2$. Plugging all this information into ④ gives

$$x^2 + 2y^2 + 3z^2 = 3x^2 = 9 \implies x^2 = 3 \implies x = \pm\sqrt{3}$$

Since $x^2 = 2y^2$ we have $y = \pm\sqrt{\frac{3}{2}}$, and since $x^2 = 3z^2$ we have $z = \pm 1$. Now, since we only want the extrema with x, y, and z nonnegative, we get that the only candidate for the maximum is the extreme point $\left(\sqrt{3}, \sqrt{\frac{3}{2}}, 1\right)$, and $f\left(\sqrt{3}, \sqrt{\frac{3}{2}}, 1\right) = \frac{3}{\sqrt{2}}$. To show it is a maximum (we know it is either a maximum or a minimum) we check to see if there is a point on $x^2 + 2y^2 + 3z^2 = 9$ which makes f smaller than $\frac{3}{\sqrt{2}}$ (meaning it can't be a minimum). The point $(2, 1, 1)$ lies on $x^2 + 2y^2 + 3z^2 = 9$ and $f(2, 1, 1) = 2$ which is smaller than $\frac{3}{\sqrt{2}}$ since $\sqrt{2} < 1.5$. Thus the maximum value of f on $x^2 + 2y^2 + 3z^2 = 9$ is $\frac{3}{\sqrt{2}}$ and occurs at $\left(\sqrt{3}, \sqrt{\frac{3}{2}}, 1\right)$.

3. Minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2z = 6$ and $x + y = 12$ using the method of Lagrange multipliers. Also, explain why the extremum you find is a minimum.

Solution: The gradient of f is

$$\nabla f = \langle 2x, 2y, 2z \rangle .$$

Let $g = x + 2z$ and let $h = x + y$. Then

$$\nabla g = \langle 1, 0, 2 \rangle$$

and

$$\nabla h = \langle 1, 1, 0 \rangle .$$

The system we get is thus

$$\begin{cases} 2x &= \lambda + \mu & \textcircled{1} \\ 2y &= \mu & \textcircled{2} \\ 2z &= 2\lambda & \textcircled{3} \\ x + 2z &= 6 & \textcircled{4} \\ x + y &= 12 & \textcircled{5} \end{cases}$$

③ gives $z = \lambda$. Plug this into ④ to get $\lambda = 3 - \frac{1}{2}x$. ② gives $y = \frac{\mu}{2}$. Plug this into ⑤ to get $\mu = 24 - 2x$. Plug these into ① to get

$$2x = 3 - \frac{1}{2}x + 24 - 2x \quad \Longrightarrow \quad \frac{9}{2}x = 27 \quad \Longrightarrow \quad x = 6.$$

By (5), $y = 6$, and by (4), $z = 0$. So $(6,6,0)$ is our candidate for the minimum, and $f(6,6,0) = 72$. As in problem 1, $(6,6,0)$ is either a minimum or maximum, and we need to rule out it being a maximum. So we are looking for a point (a,b,c) satisfying $g = 6$ and $h = 12$ with $f(a,b,c) > 72$. Theoretically any such point that is not $(6,6,0)$ should work. Consider the point $(0,12,3)$ which satisfies $g = 6$ and $h = 12$. Since $f(0,12,3) = 0 + 144 + 9 = 153 > 72$, we have that $(6,6,0)$ is indeed where the minimum of f subject to $g = 6$ and $h = 12$ occurs. So the minimum value of f subject to $x + 2z = 6$ and $x + y = 12$ is 72 and occurs at the point $(6,6,0)$.

4. Find the maximum value of the function $f(x,y,z) = x + 2y$ on the curve of intersection of the plane $x + y + z = 1$ and the cylinder $y^2 + z^2 = 4$.

Solution: Basically, the problem asks to maximize f subject to two constraints:

$$g(x,y,z) = x + y + z = 1$$
$$h(x,y,z) = y^2 + z^2 = 4$$

We'll do this problem by the method of Lagrange Multipliers: First compute

$$\nabla f(x,y,z) = \langle 1,2,0 \rangle$$
$$\nabla g(x,y,z) = \langle 1,1,1 \rangle$$
$$\nabla h(x,y,z) = \langle 0,2y,2z \rangle$$

We know $\nabla f = \lambda \nabla g + \mu \nabla h$ for some scalars λ, μ. So, along with the two constraints, we have the following system of equations:

$$\begin{cases} 1 &= \lambda & (1) \\ 2 &= \lambda + 2\mu y & (2) \\ 0 &= \lambda + 2\mu z & (3) \\ x + y + z &= 1 & (4) \\ y^2 + z^2 &= 4 & (5) \end{cases}$$

We get $\lambda = 1$ from equation (1). Putting this into equations (2) and (3), we get

$$\begin{cases} 1 &= 2\mu y \\ -1 &= 2\mu z. \end{cases}$$

Adding these two equations, we get $2\mu y + 2\mu z = 0 \implies 2\mu(y+z) = 0$. So, $\mu = 0$ or $y = -z$.

If $\underline{\mu = 0}$, then from equation (2), we have $2 = 1$, a contradiction. So, $\mu \neq 0$.

If $\underline{y = -z}$, then equation (5) yields $2z^2 = 4 \implies z = \pm\sqrt{2}$. So then $y = \mp\sqrt{2}$. And from equation (4), $x = 1 - y - z$. So, $x = 1 - (-\sqrt{2}) - \sqrt{2} = 1$ or $x = 1 - \sqrt{2} - (-\sqrt{2}) = 1$.
So, we obtain the points $(1, -\sqrt{2}, \sqrt{2})$ and $(1, \sqrt{2}, -\sqrt{2})$.

So then,

$$f(1, -\sqrt{2}, \sqrt{2}) = 1 - 2\sqrt{2}$$
$$f(1, \sqrt{2}, -\sqrt{2}) = 1 + 2\sqrt{2}.$$

Thus, the maximum value of f is $1 + 2\sqrt{2}$ on the curve of intersection.

5. Find the volume of the solid S bounded by the surface $z = xe^{xy}$, the planes $x = 2$ and $y = 1$, and the three coordinate planes (i.e. the planes $x = 0$, $y = 0$, $z = 0$).

Solution: Since $z(x, y) = xe^{xy}$ is positive on the rectangular $R = [0, 2] \times [0, 1]$, the volume of the solid S is given by

$$\begin{aligned} \text{Volume}(S) = \iint\limits_R z(x,y)\,dA &= \iint\limits_R xe^{xy}\,dA \\ &= \int_0^2 \int_0^1 xe^{xy}\,dy\,dx \\ &= \int_0^2 \left(e^{xy} \Big|_{y=0}^{y=1} \right) dx \\ &= \int_0^2 (e^x - 1)\,dx \\ &= \left(e^x - x \right)\Big|_{x=0}^{x=2} \\ &= e^2 - 3. \end{aligned}$$

Note, if we chose $dA = dx\,dy$ instead, we would have to use integration by parts for the first integration. But, we can avoid this by choosing the order $dA = dy\,dx$, which is what we did.