Name: SOLUTIONS Date: 09/29/2016

M20550 Calculus III Tutorial
Worksheet 5

1. Find $\dfrac{dz}{dt}$ when $t = 2$, where $z = x^2 + y^2 - 2xy$, $x = \ln(t-1)$ and $y = e^{-t}$.

Solution: We have $z = z\big(x(t), y(t)\big)$. So, by the chain rule, we obtain

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= (2x-2y)\left(\frac{1}{t-1}\right) + (2y-2x)e^{-t}(-1)\\ &= \big(2\ln(t-1) - 2e^{-t}\big)\left(\frac{1}{t-1}\right) - \big(2e^{-t} - 2\ln(t-1)\big)\,e^{-t}.\end{aligned}$$

Hence,

$$\begin{aligned}\left.\frac{dz}{dt}\right|_{t=2} &= \big(2\ln(2-1) - 2e^{-2}\big)\left(\frac{1}{2-1}\right) - \big(2e^{-2} - 2\ln(2-1)\big)\,e^{-2}\\ &= (0 - 2e^{-2})\cdot 1 - (2e^{-2} - 0)e^{-2}\\ &= -2e^{-2} - 2e^{-4}.\end{aligned}$$

2. (a) Let $f(x, y, z) = x^2 - yz$. If $\mathbf{v} = \langle 1, 1, 0\rangle$, find the directional derivative of f in the direction of $\mathbf{v}$ at the point $(1, 2, 3)$.

(b) Interpret your result in part (a) by filling in the blanks and circling the correct word of the statement below:

At the point ____________, the value of the function f is *increasing* / *decreasing* at the rate of ____________ as we move in the direction given by the vector ____________.

Solution: (a) The directional derivative of f in the direction of $\mathbf{v}$ at the point $(1, 2, 3)$, denote $D_{\mathbf{u}}f(1, 2, 3)$ where $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$, is given by

$$D_{\mathbf{u}}f(1, 2, 3) = \nabla f(1, 2, 3) \cdot \mathbf{u}$$

First,

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 1, 1, 0\rangle}{\sqrt{1^2 + 1^2 + 0^2}} = \frac{1}{\sqrt{2}}\langle 1, 1, 0\rangle.$$

Secondly, the gradient of f is given by:

$$\begin{aligned} \nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \\ &= \langle 2x, -z, -y \rangle \\ \implies \nabla f(1,2,3) &= \langle 2, -3, -2 \rangle. \end{aligned}$$

So, now

$$\begin{aligned} D_{\mathbf{u}} f(1,2,3) &= \nabla f(1,2,3) \bullet \mathbf{u} \\ &= \langle 2, -3, -2 \rangle \bullet \frac{1}{\sqrt{2}} \langle 1, 1, 0 \rangle \\ &= \frac{1}{\sqrt{2}} \langle 2, -3, -2 \rangle \bullet \langle 1, 1, 0 \rangle \\ &= \frac{1}{\sqrt{2}} (2 - 3) \\ &= -\frac{1}{\sqrt{2}} \end{aligned}$$

(b) At the point $\underline{(1,2,3)}$, the value of the function f is $\boxed{\text{decreasing}}$ at the rate of $\underline{1/\sqrt{2}}$ as we move in the direction given by the vector $\underline{\langle 1,1,0 \rangle}$.

3. Let $f(x,y) = \ln(xy)$. Find the maximum rate of change of f at $(1,2)$ and the direction in which it occurs.

Solution: It is a fact that f changes the fastest in the direction of its gradient vector and the maximum rate of change is the magnitude of the gradient vector.

With $f(x,y) = \ln(xy)$, we first compute $\nabla f(1,2)$:

$$\begin{aligned} \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle \frac{y}{xy}, \frac{x}{xy} \right\rangle &= \left\langle \frac{1}{x}, \frac{1}{y} \right\rangle \\ \implies \nabla f(1,2) &= \left\langle 1, \frac{1}{2} \right\rangle. \end{aligned}$$

$$\implies |\nabla f(1,2)| = \left| \left\langle 1, \frac{1}{2} \right\rangle \right| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}.$$

So, the maximum rate of change of f at $(1,2)$ is $\dfrac{\sqrt{5}}{2}$ and the direction in which it occurs is $\left\langle 1, \dfrac{1}{2} \right\rangle$.

4. If $h = x^2 + y^2 + z^2$ and $y\cos z + z\cos x = 0$, find $\dfrac{\partial h}{\partial x}$ assuming that x and y are the independent variables.

Solution: We have $h = h\big(x, y, z(x,y)\big)$. So,

$$\frac{\partial h}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} \qquad \text{since } z \text{ is a function of } x.$$

To find $\dfrac{\partial z}{\partial x}$, we use implicit differentiation:

$$\begin{aligned} y\cos z + z\cos x &= 0 \\ \frac{\partial}{\partial x}\big[y\cos z + z\cos x\big] &= \frac{\partial}{\partial x}\big[0\big] \\ -y\sin z\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x}\cos x - z\sin x &= 0 \\ \frac{\partial z}{\partial x}\left(\cos x - y\sin z\right) &= z\sin x \\ \frac{\partial z}{\partial x} &= \frac{z\sin x}{\cos x - y\sin z} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial h}{\partial x} &= 2x + 2z\left(\frac{z\sin x}{\cos x - y\sin z}\right) \\ \implies \frac{\partial h}{\partial x} &= 2x + \frac{2z^2\sin x}{\cos x - y\sin z}. \end{aligned}$$

5. A cylinder containing an incompressible fluid is being squeezed from both ends. If the length of the cylinder is *decreasing* at a rate of 3m/s, calculate the rate at which the radius is changing when the radius is 2m and the length is 1m. (Note: An incompressible fluid is a fluid whose volume does not change.)

Solution: Let V be the volume of the cylinder, r be the radius of the cylinder, and l be its length. Then, $V = \pi r^2 l$. So, $V = V\big(r(t), l(t)\big)$.

By assumptions, we have $\dfrac{dl}{dt} = -3$ and incompressibility of the fluid implies $\dfrac{dV}{dt} = 0$.

We want to find $\dfrac{dr}{dt}$ at the instant when $r = 2$ and $l = 1$. We have

$$\begin{aligned}\frac{dV}{dt} &= \frac{d}{dt}\left[\pi r^2 l\right]\\ 0 &= 2\pi r l\frac{dr}{dt} + \pi r^2\frac{dl}{dt}. \quad \text{And we know } \frac{dl}{dt} = -3; \text{ so}\\ 0 &= 2\pi r l\frac{dr}{dt} - 3\pi r^2\\ \frac{dr}{dt} &= \frac{3r}{2l}.\end{aligned}$$

Hence, when $r = 2, l = 1$, we get $\dfrac{dr}{dt} = \dfrac{3 \cdot 2}{2 \cdot 1} = 3\text{m/s}$.

6. Let $r = r(x, y)$, $x = x(s, t)$, and $y = y(t)$. Given that

$$\begin{array}{lll} x(1,0) = 2, & x_s(1,0) = -1, & x_t(1,0) = 7,\\ y(0) = 3, & y(1) = 0 & y'(0) = 4,\\ r(2,3) = -1, & r_x(2,3) = 3, & r_y(2,3) = 5,\\ r_x(1,0) = 6, & r_y(1,0) = -2, & \end{array}$$

calculate $\dfrac{\partial r}{\partial t}$ at $s = 1, t = 0$.

Solution: We have $r = \big(x(s,t), y(t)\big)$. So, from the chain rule, we get

$$\begin{aligned}\frac{\partial r}{\partial t} &= \frac{\partial r}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial r}{\partial y}\frac{dy}{dt}\\ &= r_x x_t + r_y y'\\ &= r_x(x,y)x_t(s,t) + r_y(x,y)y'(t).\end{aligned}$$

When $s = 1$ and $t = 0$, we have $x = x(1,0) = 2$ and $y = y(0) = 3$. So,

$$\begin{aligned}\left.\frac{\partial r}{\partial t}\right|_{s=1,\,t=0} &= r_x(2,3)x_t(1,0) + r_y(2,3)y'(0)\\ &= (3)(7) + (5)(4)\\ &= 41.\end{aligned}$$

7. Suppose that over a certain region of space the electrical potential V is given by

$$V(x, y, z) = 5x^2 - 3xy + xyz.$$

(a) Find the rate of change of the potential at $P(1,1,0)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

(b) In which direction does V decrease most rapidly at P?

(c) What is the maximum rate of change at P?

Solution: (a) We have $\mathbf{v} = \langle 1, 1, -1\rangle$. So, $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{1}{\sqrt{3}}\langle 1, 1, -1\rangle$. We want to find $D_{\mathbf{u}}V(1,1,0)$, the directional derivative of V in the direction of $\mathbf{v}$ at the point $P(1,1,0)$.

First, $\nabla V = \langle 10x - 3y + yz, -3x + xz, xy\rangle \implies \nabla V(1,1,0) = \langle 7, -3, 1\rangle$. So,

$$\begin{aligned} D_{\mathbf{u}}V(1,1,0) &= \nabla V(1,1,0) \cdot \mathbf{u} \\ &= \langle 7, -3, 1\rangle \cdot \frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle \\ &= \frac{1}{\sqrt{3}}(7 - 3 - 1) \\ &= \frac{3}{\sqrt{3}} = \sqrt{3}. \end{aligned}$$

(b) At P, V decreases most rapidly in the direction of $-\nabla V(1,1,0)$ which is $\langle -7, 3, -1\rangle$.

(c) The maximum rate of change at P is given by $|\nabla V(1,1,0)| = |\langle 7, -3, 1\rangle| = \sqrt{59}$.

8. Find **<u>all</u>** points at which the direction of fastest change of the function $f(x,y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

Solution: We know the direction of fastest change of f at a point (x,y) is given by the direction of $\nabla f(x,y) = \langle 2x - 2, 2y - 4\rangle$. So, we want to find all pairs (x,y) such that $\langle 2x - 2, 2y - 4\rangle = k\langle 1, 1\rangle$ for any constant k. We obtain the system of equations

$$\begin{cases} 2x - 2 &= k \\ 2y - 4 &= k \end{cases}$$

Then, $2x - 2 = 2y - 4 \implies y = x + 1$. Thus, all the wanted pairs (x,y) are $(x, x+1)$, where x admits any value in the domain. This is exactly all the points on the line $y = x + 1$ in the domain of f.

9. (a) Find an equation for the tangent line (in vector or parametric form) at the point $(2,2,1)$ to the curve of intersection of the two surfaces $g(x,y,z) = 2x^2 + 2y^2 + z^2 = 17$ and $h(x,y,z) = x^2 + y^2 - 3z^2 = 5$.

(b) Suppose $f(x, y, z)$ is a function with $\nabla f = \langle 1, 0, 0 \rangle$ at the point $(2, 2, 1)$. Starting at $(2, 2, 1)$, which direction should one travel along the curve of intersection in order to increase f? (*Note: You can give a tangent vector to the curve at* $(2, 2, 1)$ *that points in the desired direction.*)

Solution: (a) First, we want to find a parametrization of this curve of intersection. All points on this curve should satisfy

$$\begin{cases} 2x^2 + 2y^2 + z^2 = 17 & \implies 2(x^2 + y^2) + z^2 = 17 \quad (1) \\ x^2 + y^2 - 3z^2 = 5 & \implies x^2 + y^2 = 5 + 3z^2 \quad (2) \end{cases}$$

Rewriting the first equation using the second equation, we get

$$2(5 + 3z^2) + z^2 = 17 \implies 7z^2 = 7 \implies z = \pm 1.$$

We will be writing an equation of the tangent line to this curve at the point $(2, 2, 1)$. So, we only consider the case when $z = 1$. Now, with $z = 1$, equation (2) above gives $x^2 + y^2 = 8$. On the xy-plane, this is just a circle center at the origin with radius $\sqrt{8}$. So, a parametrization of the curve of intersection between the two surfaces is given by

$$\mathbf{r}(t) = \left\langle \sqrt{8}\cos t, \sqrt{8}\sin t, 1 \right\rangle.$$

Now, we see that $t = \dfrac{\pi}{4}$ corresponds to the point $(2, 2, 1)$. Thus, a parallel vector to the tangent line at $(2, 2, 1)$ is given by

$$\mathbf{r}'\left(\frac{\pi}{4}\right) = \langle -2, 2, 0 \rangle.$$

And so, a vector equation of this tangent line is

$$\langle x, y, z \rangle = \langle 2, 2, 1 \rangle + t \langle -2, 2, 0 \rangle.$$

OR

(a)′ There is no need to parametrize the intersection curve. Both ∇g and ∇h are perpendicular to $\mathbf{r}'$ so $\nabla g \times \nabla h$ is parallel to $\mathbf{r}'$. We get

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4x & 4y & 2z \\ 2x & 2y & -6z \end{vmatrix} = \langle -28yz, 28xz, 0 \rangle$$

And so, a vector equation of this tangent line is

$$\langle x, y, z \rangle = \langle 2, 2, 1 \rangle + t \langle -56, 56, 0 \rangle.$$

Of course we can divide by 28 to get the formula in solution (a) or by 56 and get

$$\langle x, y, z \rangle = \langle 2, 2, 1 \rangle + t \langle -1, 1, 0 \rangle .$$

(b) There are two ways to go from $(2, 2, 1)$ on this curve. One is in the direction of the tangent vector $\langle -2, 2, 0 \rangle$. The other is going opposite direction to $\langle -2, 2, 0 \rangle$, which is going in the same direction as $\langle 2, -2, 0 \rangle$.

We want to follow a way such that the directional derivative of f at $(2, 2, 1)$ would be positive in that direction because we want the value of f to increase. We see that the directional derivative of f at $(2, 2, 1)$ in the direction of $\langle 2, -2, 0 \rangle$ is positive since

$$\nabla f(2, 2, 1) \bullet \langle 2, -2, 0 \rangle = \langle 1, 0, 0 \rangle \bullet \langle 2, -2, 0 \rangle = 2 > 0.$$

Thus, we want to travel along the curve of intersection in the direction of $\langle 2, -2, 0 \rangle$ in order to increase f.