Name: SOLUTIONS Date: 09/22/2016

M20550 Calculus III Tutorial
Worksheet 4

1. Find and sketch the domain of the function

$$f(x, y) = \frac{\ln(x^2 + 4y^2 - 4)}{9 - x^2}.$$

Solution: The domain of the function is the set of pairs (x, y) we can plug into the function. Since the function is a fraction, the denominator cannot be zero. Thus we have that

$$x^2 \neq 9 \Leftrightarrow x \neq \pm 3.$$

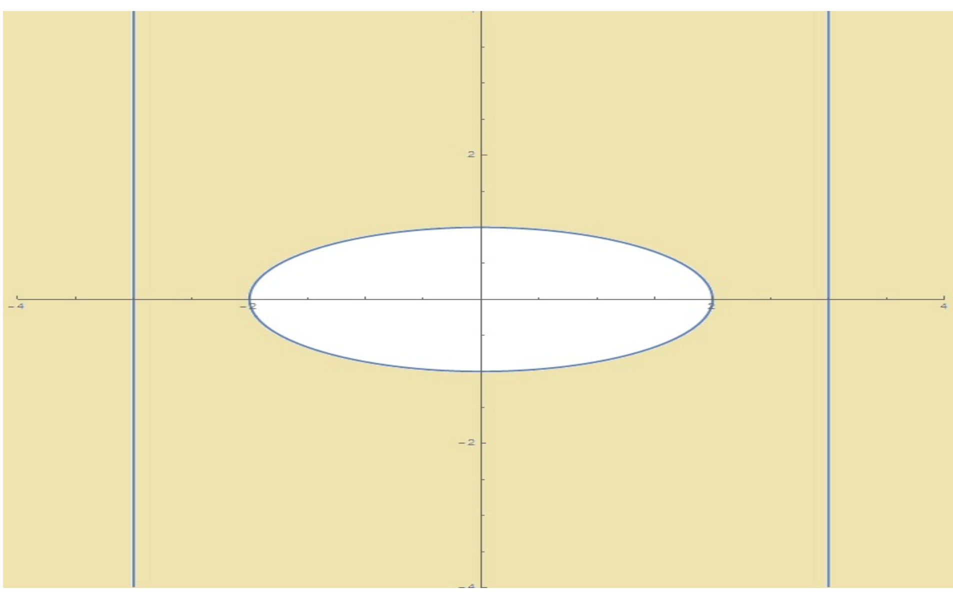
Furthermore, in the numerator, the input of ln must be positive, thus

$$x^2 + 4y^2 - 4 > 0 \Leftrightarrow x^2 + 4y^2 > 4.$$

So, as a set, the domain is

$$\text{domain}(f) = \left\{ (x, y) | \, x \neq \pm 3 \text{ and } x^2 + 4y^2 > 4 \right\}.$$

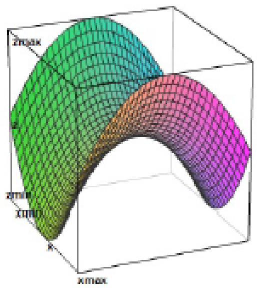
Graphically, $x \neq \pm 3$ removes the vertical lines $x = -3$ and $x = 3$ from the domain, and $x^2 + 4y^2 > 4$ or $\frac{x^2}{4} + y^2 > 1$ says that we can only take points outside of the ellipse $\frac{x^2}{4} + y^2 > 1$.



The yellow shaded region is the domain (the blue lines are removed from the domain).

2. Select the correct graph and the correct contour plot of level curves for the function

$$f(x,y) = x^2 - y^2$$

I.


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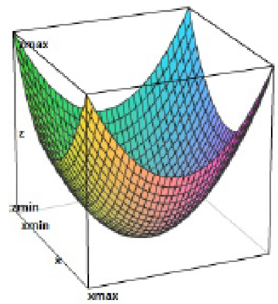

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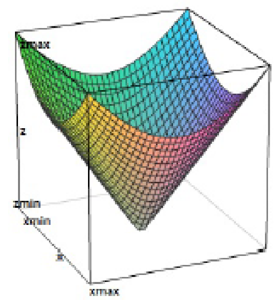
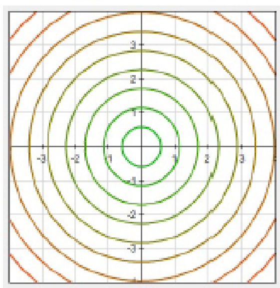


A.


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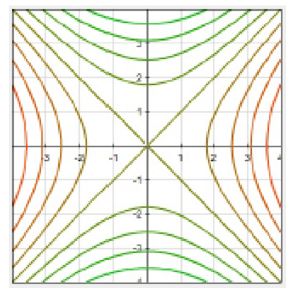
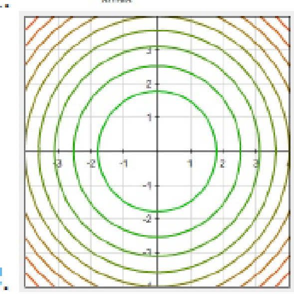


C.


(a) I and B

(b) I and A

(c) II and A

(d) II and C

(e) III and C

Solution: Let's first determine the level curves for f. These are the family of curves with the equation $x^2 - y^2 = k$, where k is a constant. These equations are obviously not equations of circles. So, we eliminate choices A and C. Thus, B must be the level curves (or contour plot) for f. Here, $x^2 - y^2 = k$ with k is a constant give us a family of hyperbolas. Based upon the level curves for f, we easily see that the graph of f cannot be II or III. It must be I.

3. Evaluate the following limit

$$\lim_{(x,y)\to(0,0)} \frac{y + xe^{-y^2}}{1 + x^2}.$$

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{y + xe^{-y^2}}{1 + x^2} = \frac{0 + 0 \cdot e^0}{1 + 0} = \frac{0}{1} = 0.$$

4. Show that the following limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}.$$

Solution: Let $f(x,y) = \dfrac{x^2y}{x^4+y^2}$. We will show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist by showing that f approaches two different values as (x,y) approaches $(0,0)$ along two different paths.

First, let (x,y) approach $(0,0)$ along the x-axis, i.e. $y=0$. We have $f(x,0) = \dfrac{0}{x^4+0} = 0$. So,

$$f(x,y) \longrightarrow 0 \text{ as } (x,y) \longrightarrow (0,0) \text{ along the } x\text{-axis.}$$

Next, let (x,y) approach $(0,0)$ along the curve $y = x^2$. We have $f(x,x^2) = \dfrac{x^2(x^2)}{x^4+(x^2)^2} = \dfrac{x^4}{2x^4} = \dfrac{1}{2}$. So,

$$f(x,y) \longrightarrow \frac{1}{2} \text{ as } (x,y) \longrightarrow (0,0) \text{ along the curve } y = x^2.$$

Since f admits two different limits along two different paths, the limit does not exist.

5. Find the second partial derivative g_{xy} of the function

$$g(x,y) = x^3y^2 + e^{xy}.$$

Solution: First, we find g_x by regarding y as a constant in $g(x,y) = x^3y^2 + e^{xy}$. We have,

$$g_x = 3x^2y^2 + ye^{xy}.$$

Now, $g_{xy} = (g_x)_y$. So, we regard x as a constant in $g_x = 3x^2y^2 + ye^{xy}$ and get

$$g_{xy} = 6x^2y + 1\cdot e^{xy} + ye^{xy}(x).$$

or

$$g_{xy} = 6x^2y + e^{xy} + xye^{xy}.$$

6. Let $z = z(x,y)$ be defined implicitly as a function of x and y by the equation

$$x^2e^y = -z\cos(yz).$$

Find $\dfrac{\partial z}{\partial x}$ at the point $x = 1$, $y = 0$, and $z = -1$.

Solution: We're going to use implicit differentiation. And because we're looking for $\dfrac{\partial z}{\partial x}$, y is going to be treated as a constant. And here, z is understood to be a function of x. We have

$$\frac{\partial}{\partial x}\left[x^2 e^y\right] = \frac{\partial}{\partial x}\left[-z\cos(yz)\right]$$

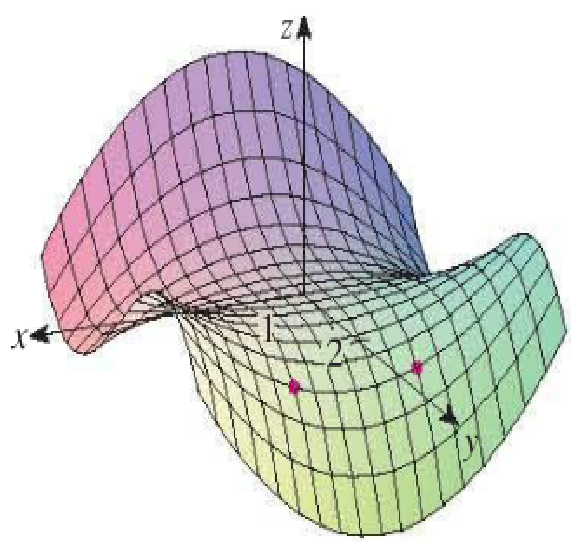
$$2xe^y = -1\frac{\partial z}{\partial x}\cos(yz) + -z\left(-\sin(yz)\right)\left(y\frac{\partial z}{\partial x}\right)$$

$$2xe^y = -\frac{\partial z}{\partial x}\cos(yz) + yz\sin(yz)\frac{\partial z}{\partial x} \qquad (*)$$

We want to find $\dfrac{\partial z}{\partial x}$ at the point $x = 1$, $y = 0$, and $z = -1$. Plugging these values for x, y, z into equation $(*)$, we then find out the value of $\dfrac{\partial z}{\partial x}$:

$$2(1)e^0 = -\frac{\partial z}{\partial x}\cos(0) + 0 \quad \Longrightarrow \quad 2 = -\frac{\partial z}{\partial x} \quad \Longrightarrow \quad \frac{\partial z}{\partial x} = -2.$$

7. The graph of f is shown below



Determine the sign of

(a) $f_x(1, 2)$

(b) $f_y(1,2)$
(c) $f_x(-1,2)$
(d) $f_y(-1,2)$

Solution:

(a) $f_x(1,2)$ gives the rate of change of f as we move from the point $(1,2)$ in positive x-direction with y being fixed. We see that f increases in this direction. So, $f_x(1,2)$ is positive.

(b) $f_y(1,2)$ gives the rate of change of f as we move from the point $(1,2)$ in positive y-direction with x being fixed. We see that f decreases in this direction. So, $f_x(1,2)$ is negative.

(c) $f_x(-1,2)$ gives the rate of change of f as we move from the point $(-1,2)$ in positive x-direction with y being fixed. We see that f decreases in this direction. So, $f_x(-1,2)$ is negative.

(d) $f_y(-1,2)$ gives the rate of change of f as we move from the point $(-1,2)$ in positive y-direction with x being fixed. We see that f decreases in this direction. So, $f_x(-1,2)$ is negative.

8. The paraboloid $z = 6 - x - x^2 - 2y^2$ intersects the plane $x = 1$ in a parabola. Use the geometry of partial derivative to find the **slope** for the tangent line to this parabola at the point $(1,2,-4)$.

Solution: We have $z = f(x,y) = 6 - x - x^2 - 2y^2$. The vertical plane $x = 1$ intersects the graph of f in a parabola (the x-direction is fixed). Thus, $f_y(1,2)$ gives the slope of the tangent line of this parabola at the point $(1,2,-4)$. We have

$$f_y = -4y \qquad \Longrightarrow \qquad f_y(1,2) = -4(2) = -8.$$

Thus, the required slope is -8.

9. *Here is a challenge:* Refer to the graph of f in number 7, what is the sign of $f_{xx}(1,2)$?

Solution: We have $f_{xx} = \dfrac{\partial}{\partial x}(f_x)$. So, f_{xx} is the rate of change of f_x in the x-direction with y being fixed. Note that $f_x(1,2)$ is positive. And as we move in the positive x-direction, the surface becomes steeper and so the slope becomes bigger, i.e. the value of f_x is increasing in this direction. So, $f_{xx}(1,2)$ is positive.