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M20550 Calculus III Tutorial
Worksheet 2

1. Find an equation of the plane passes through the point $(1, 1, -7)$ and perpendicular to the line $x = 1 + 4t$, $y = 1 - t$, $z = -3$.

> ***Solution:*** To write an equation of a plane, we need one point on the plane and a normal vector (a vector that is perpendicular to the plane).
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> In this problem, we have the point $(1, 1, -7)$ on the plane. Now, we need to find a normal vector. We know our plane is perpendicular to the line $x = 1 + 4t$, $y = 1 - t$, $z = -3$. So, the parallel vector to this line, which is $\mathbf{v} = \langle 4, -1, 0 \rangle$, can be used as the normal vector to our plane.
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> Finally, an equation of the plane with normal vector $\langle 4, -1, 0 \rangle$ passing through $(1, 1, -7)$ is given by
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> $$\begin{aligned} \langle 4, -1, 0 \rangle \cdot \langle x, y, z \rangle &= \langle 4, -1, 0 \rangle \cdot \langle 1, 1, -7 \rangle \\ \implies 4x - y &= 3. \end{aligned}$$

2. Let ℓ be the line of intersection of the planes given by equations $x - y = 1$ and $x - z = 1$. Find an equation for ℓ in the form $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$.

> ***Solution:*** To write an equation of the line ℓ, we need to find one point on ℓ and a parallel vector to ℓ.
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> Since ℓ is the line of intersection of two planes, to find a point on ℓ, we need to find a point that contained in both planes. A point on both planes can be found by setting $x = 1$, so $y = z = 0$. And we get the point $(1, 0, 0)$ on ℓ.
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> A normal vector for the first plane is $\langle 1, -1, 0 \rangle$ and a normal vector for the second plane is $\langle 1, 0, -1 \rangle$. A parallel vector of ℓ is a vector perpendicular to the normal vectors of both planes. Thus, a parallel vector of ℓ is given by
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> $$\langle 1, -1, 0 \rangle \times \langle 1, 0, -1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle .$$
>
> Hence, the vector equation of ℓ is
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> $$\mathbf{r}(t) = \langle 1, 0, 0 \rangle + t \langle 1, 1, 1 \rangle .$$

3. A particle moves in space in such a way that at time t ($t \geq 0$), its position is given by the vector-valued function $\mathbf{r}(t) = \langle t^2+1, 2t^2-1, 2-3t^2 \rangle$.

   (a) At what time(s) does the particle hit the plane $2x+2y+3z=3$?

   (b) Find the point of intersection, if any.

Solution: (a) We have $\mathbf{r}(t) = \langle t^2+1, 2t^2-1, 2-3t^2 \rangle$. So the x, y, z-coordinates of the particle are given by:

$$x = t^2+1, \qquad y = 2t^2-1, \qquad z = 2-3t^2.$$

At the instant the particle hits the plane, the x, y, z-coordinates of the particle have to satisfy the equation $2x+2y+3z=3$. Thus, we get the equation

$$\begin{aligned} 2(t^2+1)+2(2t^2-1)+3(2-3t^2) &= 3 \\ 2t^2+2+4t^2-2+6-9t^2 &= 3 \\ -3t^2+6 &= 3 \\ t^2 &= 1 \\ t &= 1 \quad \text{or} \quad t=-1 \end{aligned}$$

Therefore, the particle hits the plane $2x+2y+3z=3$ at time $t=1$.

(b) When $t=1$, We have $\mathbf{r}(1) = \langle 1^2+1, 2(1)^2-1, 2-3(1)^2 \rangle = \langle 2, 1, -1 \rangle$. Thus, the point of intersection is $(2,1,-1)$.

4. Find an equation of the tangent line to the space curve $\mathbf{r}(t) = \langle 2t^3, 3t, 3t^2 \rangle$ at the point $(-2,-3,3)$.

Solution: First, we want to find t corresponds to the point $(-2,-3,3)$. t corresponds to $(-2,-3,3)$ must satisfy the equations

$$2t^3 = -2, \quad 3t = -3, \quad 3t^2 = 3.$$

From the second equation, we know $t=-1$.

Next, we want to find $\mathbf{r}\mathbf{r}'(-1)$, the tangent vector at $t=-1$. The derivative of $\mathbf{r}(t)$ is given by $\mathbf{r}'(t) = \langle 6t^2, 3, 6t \rangle$. So the tangent vector at $t=-1$ is $\mathbf{r}'(-1) = \langle 6, 3, -6 \rangle$.

Then, the vector equation of the tangent line at $(-2,-3,3)$ is

$$\langle x, y, z \rangle = \langle -2, -3, 3 \rangle + t \langle 6, 3, -6 \rangle.$$

5. Find $\mathbf{r}(t)$ if $\mathbf{r}''(t) = e^t\mathbf{i}$, $\mathbf{r}(0) = 2\mathbf{i}+3\mathbf{j}+2\mathbf{k}$, and $\mathbf{r}'(0) = \mathbf{i}+\mathbf{j}+\mathbf{k}$.

Solution:

$$\mathbf{r}'(t) = \int \mathbf{r}\mathbf{r}''(t)\,dt = \int \left\langle e^t, 0, 0\right\rangle\,dt = \left\langle e^t, 0, 0\right\rangle + \mathbf{c}.$$

To find $\mathbf{c}$, we use the information $\mathbf{r}'(0) = \langle 1, 1, 1\rangle$. From the above, we have $\mathbf{r}'(0) = \langle e^0, 0, 0\rangle + \mathbf{c}$. So, $\langle e^0, 0, 0\rangle + \mathbf{c} = \langle 1, 1, 1\rangle \implies \mathbf{c} = \langle 1, 1, 1\rangle - \langle e^0, 0, 0\rangle = \langle 0, 1, 1\rangle$. Thus, we get

$$\mathbf{r}'(t) = \left\langle e^t, 0, 0\right\rangle + \langle 0, 1, 1\rangle \implies \mathbf{r}'(t) = \left\langle e^t, 1, 1\right\rangle.$$

Then

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \left\langle e^t, 1, 1\right\rangle dt = \left\langle e^t, t, t\right\rangle + \mathbf{d}.$$

To find $\mathbf{d}$, we use the information $\mathbf{r}(0) = \langle 2, 3, 2\rangle$. We have $\mathbf{r}(0) = \langle e^0, 0, 0\rangle + \mathbf{d} = \langle 2, 3, 2\rangle$. So, $\mathbf{d} = \langle 2, 3, 2\rangle - \langle e^0, 0, 0\rangle = \langle 1, 3, 2\rangle$.

Finally, we get

$$\mathbf{r}(t) = \left\langle e^t, t, t\right\rangle + \langle 1, 3, 2\rangle \implies \mathbf{r}(t) = \left\langle e^t + 1, t + 3, t + 2\right\rangle.$$

6. Let P be a plane with normal vector $\langle -2, 2, 1\rangle$ passing through the point $(1, 1, 1)$. Find the distance from the point $(1, 2, -5)$ to the plane P.

Solution: Let's make a vector $\mathbf{b}$ from the point $(1, 1, 1)$ to the point $(1, 2, -5)$:

$$\mathbf{b} = \langle 1 - 1, 2 - 1, -5 - 1\rangle = \langle 0, 1, -6\rangle.$$

Then, the distance D from the point $(1, 2, -5)$ to the plane P is given by

$$D = |\text{comp}_{\mathbf{n}}\mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{|\langle -2, 2, 1\rangle \cdot \langle 0, 1, -6\rangle|}{|\langle -2, 2, 1\rangle|} = \frac{|-4|}{\sqrt{(-2)^2 + 2^2 + 1^2}} = \frac{4}{3}.$$

7. Find an equation of the plane that passes through the point $(1, 2, 3)$ and contains the line $\frac{1}{3}x = y - 1 = 2 - z$.

Solution: For this problem, in order to find a normal vector of the plane, we first need to find two vectors on the plane then take their cross product.

One vector that lies on the plane is a parallel vector of the line $\frac{1}{3}x = y - 1 = 2 - z$ (because this line is contained in the plane). Note that $\frac{1}{3}x = y - 1 = 2 - z \iff$

$\dfrac{x-0}{3} = \dfrac{y-1}{1} = \dfrac{z-2}{-1}$. So, a parallel vector of this line is $\mathbf{v_1} = \langle 3, 1, -1 \rangle$. Thus, we have $\mathbf{v_1} = \langle 3, 1, -1 \rangle$ lies on the plane.

To get another vector on the plane, we take one point on the line and make a vector with the point on the plane $(1, 2, 3)$. One point on the line $\dfrac{x-0}{3} = \dfrac{y-1}{1} = \dfrac{z-2}{-1}$ is $(0, 1, 2)$. So, we get the second vector $\mathbf{v_2}$ on the plane, $\mathbf{v_2} = \langle 1-0, 2-1, 3-2 \rangle = \langle 1, 1, 1 \rangle$.

Then, a normal vector is given by

$$\mathbf{v_1} \times \mathbf{v_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \langle 2, -4, 2 \rangle .$$

So, the equation of the required plane is

$$\begin{aligned} &\langle 2, -4, 2 \rangle \cdot \langle x, y, z \rangle = \langle 2, -4, 2 \rangle \cdot \langle 1, 2, 3 \rangle \\ &\implies 2x - 4y + 2z = 0 \\ &\implies x - 2y + z = 0 \end{aligned}$$

8. Find a vector function that represents the curve of intersection of the cylinder $x^2+y^2 = 9$ and the plane $x + y - z = 5$.

Solution: To find a vector function that represents the curve of intersection, we need to be able to describe x, y, z in terms of t for this curve.

On the xy-plane, $x^2 + y^2 = 9$ represents a circle centers at the origin with radius 3. So, we can write the parametric equations for this circle as follows:

$$x = 3\cos t, \qquad y = 3\sin t, \qquad 0 \le t \le 2\pi.$$

And from the equation of the plane, we get

$$z = x + y - 5 \implies z = 3\cos t + 3\sin t - 5, \qquad 0 \le t \le 2\pi.$$

So, a vector function that represents the curve of intersection is given by

$$\mathbf{r}(t) = (3\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + (3\cos t + 3\sin t - 5)\,\mathbf{k}, \qquad 0 \le t \le 2\pi.$$