1.(6pts) Find the integral $\iint_H \langle x, y, z\rangle \bullet d\mathbf{S}$ where H is the part of the upper hemisphere of $x^2+y^2+z^2=a^2$ above the plane $z=\frac{a}{2}$ and the normal points up.

Useful Facts: $\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$ and $d\mathbf{S}=\pm a\sin\phi\,\langle a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi\rangle\, dA$. As part of the problem, you need to decide which sign you need.

(a) $-\frac{7a^3}{2}$ (b) $\frac{7a^3}{2}$ (c) $-\pi a^3$ (d) πa^3 (e) $\frac{\pi a^3}{2}-\frac{\pi a^2}{2}$

2.(6pts) Let $z=f(x,y)$ and suppose $f(2,1)=3$. Suppose $x=g(u,t)$ and $y=h(u,t)$ with $g(-1,3)=2$ and $h(-1,3)=1$. Then z is a function of u and t. Find $\frac{\partial z}{\partial u}$ at the point $(-1,3)$ if $\text{grad} f=\langle 4,5\rangle$ at $(2,1)$, $\text{grad} f=\langle 1,-1\rangle$ at $(-1,3)$ and $g_u(u,t)=3$ at $(-1,3)$, $h_u(u,t)=-2$ at $(-1,3)$. Which number below is $z_u(u,t)$ at $(-1,3)$?

(a) 0 (b) 2 (c) 1 (d) 3 (e) 4

3.(6pts) Compute $\displaystyle\iint_D y\,dA$ where D is the upper half of the disk of radius a centered at the origin.

(a) $\dfrac{2a^3}{3}$ (b) $\dfrac{a^3}{3}$ (c) $\dfrac{2a^2}{3}$ (d) $\dfrac{4a^2}{3}$ (e) $\dfrac{4a^3}{3}$

4.(6pts) The two level surfaces $f(x,y,z) = x^2y - xyz + z^2 = 7$ and $g(x,y,z) = x^2 + y^2 + z^2 = 14$ intersect at the point $(3,1,2)$. Which equation below is an equation for the tangent line to the curve of intersection at the point $(3,1,2)$?

(a) $t\,\langle 1,1,0\rangle + \langle 1,-1,2\rangle$

(b) $(1-t)\,\langle 3,1,2\rangle + t\,\langle 1,-1,-1\rangle$

(c) $\langle 3,1,2\rangle + t\,\langle 1,-1,-1\rangle$

(d) $(1-t)\,\langle 3,1,2\rangle + t\,\langle 2,1,2\rangle$

(e) $\langle 3,1,2\rangle + t\,\langle 2,1,2\rangle$

5.(6pts) Which function below is a potential function for the field

$$\mathbf{F} = \left\langle yz^2 + 2xyz, xz^2 + x^2z, 2xyz + x^2y - 2z \right\rangle$$

(a) $f(x, y, z) = xyz^2 + x^2yz - \frac{1}{2}z^4$

(b) $f(x, y, z) = x^3y^2z + x^2 - xyz^2$

(c) $f(x, y, z) = 2xyz^2 - \frac{1}{2}xy^2z - y^2$

(d) $f(x, y, z) = xyz^2 + \frac{1}{2}x^3y^2z + x^2$

(e) $f(x, y, z) = xyz^2 + x^2yz - z^2$

6.(6pts) Find the area of the parallelogram with vertices $(1, 2)$, $(3, 5)$, $(4, 3)$ and $(6, 6)$.

(a) 8 (b) 7 (c) 11 (d) 10 (e) 9

7.(6pts) Find the directional derivative of $f(x,y) = x^2 - xy + y^3$ at the point $(2,1)$ in the same direction as the vector $\mathbf{u} = \langle 1, -1 \rangle$.

(a) $\dfrac{1}{\sqrt{10}}\langle 3,1\rangle$ (b) 0 (c) $\sqrt{2}$ (d) $\dfrac{1}{\sqrt{10}}\langle 1,-3\rangle$ (e) 2

8.(6pts) Maximize the function $x - 2y + z$ subject to the requirement that the points lie on the surface $x^2 + 4y^2 + z^2 = 3$.

(a) 3 (b) The function has no maximum value on the surface.

(c) 4 (d) $\dfrac{3}{2}$ (e) 2

9.(6pts) Which iterated integral below gives the same number as the iterated integral
$$\int_0^2 \int_{\sqrt[4]{8y}}^2 \sqrt{x^5+1}\, dx\, dy$$

(a) $\displaystyle\int_0^2 \int_0^{x^4/8} \sqrt{x^5+1}\, dy\, dx$ (b) $\displaystyle\int_0^2 \int_0^{8x^4} \sqrt{x^5+1}\, dy\, dx$ (c) $\displaystyle\int_0^{16} \int_0^{8x^4} \sqrt{x^5+1}\, dy\, dx$

(d) $\displaystyle\int_0^{x^4/8} \int_0^2 \sqrt{x^5+1}\, dy\, dx$ (e) $\displaystyle\int_0^{16} \int_0^{x^4/8} \sqrt{x^5+1}\, dy\, dx$

10.(6pts) Find the integral $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the square in the xy-plane starting at the origin, going to $(0,2)$, then to $(2,2)$, then to $(2,0)$ and finally back to the origin, and $\mathbf{F} = \left\langle \sqrt{8-x^2} - xy, y \right\rangle$

(a) 3 (b) -2 (c) -3 (d) -4 (e) 0

11.(6pts) Let $\mathbf{F} = \left\langle x, -2y, x^2 + z \right\rangle$. Compute $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$ over the part of the surface $x^2 + y^2 + zx^2 + zy^2 + z^3 = 9$ which lies above the xy plane. Note that S together with the disk D $x^2 + y^2 \leqslant 9$, $z = 0$ bounds a solid E. Try integrating over a different surface.

(a) $\dfrac{49\pi^2}{2}$ (b) $\dfrac{81\pi}{4}$ (c) 0 (d) $-\dfrac{\pi}{3}$ (e) $\dfrac{7\pi}{4}$

12.(6pts) Let S be the surface $\mathbf{r}(u, v) = \left\langle uv^2, uv, u^2v \right\rangle$ and note that $\mathbf{r}(2, 1) = \langle 2, 2, 4 \rangle$. Which equation below is an equation of the tangent plane to S at $\langle 2, 2, 4 \rangle$?

(a) $2x - 6y + z = -4$ (b) $x + y + 4z = 20$ (c) $2x + y + 2z = 14$

(d) $x - y + 4z = 16$ (e) $2x - 2y + z = 0$

13.(6pts) Find the area of the piece of the cylinder over $y = x^3$, $0 \leqslant x \leqslant 1$ above the plane $z = 0$ and below the graph of the cylinder $z = 36x^3$.

(a) $\pi\sqrt[3]{36}$ (b) 108π (c) $\frac{2}{3}\left(\sqrt{1000} - 1\right)$ (d) $12\pi\left(\sqrt{1000}\right)$ (e) $\sqrt[3]{36} - 1$

14.(6pts) If C is the curve $\mathbf{r}(t) = \left\langle (\sin 2t)\sqrt{4 - t^2}, \left(t - \frac{\pi}{2}\right), 2 + \sin(t) \right\rangle$, $0 \leqslant t \leqslant \frac{\pi}{2}$ find $\int_C \nabla f \bullet d\mathbf{r}$ where $f(x, y, z) = x^2y^3 + zy + \frac{xy}{z}$.

(a) 1 (b) 0 (c) π (d) $\frac{16}{5}$ (e) -2

15.(6pts) Let $\mathbf{r}(u,v) = \langle u^2 + uv, v^2 + uv \rangle$. Compute the Jacobian of this transformation.

(a) $u^2 + v^2 - 4uv$ (b) $4uv + 2u^2 + 2v^2$ (c) $4uv - 2u^2 + 2v^2$

(d) $4uv + 2u^2 - 2v^2$ (e) $(u^2 + v^2)uv$

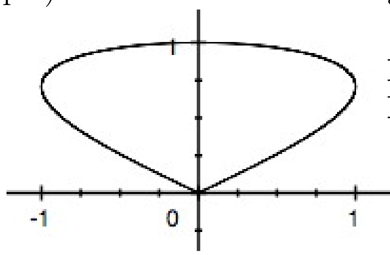
16.(6pts) Find the moment about the yz plane of a thin sheet of unit density bent in the shape of a surface $\mathbf{r}(u,v) = \langle u^3, v^4, uv \rangle$ for $-1 \leqslant u \leqslant 1$ and $-1 \leqslant v \leqslant 1$. Suppose T is the square with vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$.

(a) $\iint_T u^3 \sqrt{u^6 + v^8 + u^2v^2}\, dA$ (b) $\iint_T u^3 \sqrt{16v^8 + 9u^6 + 144u^4v^6}\, dA$

(c) $\iint_T v^4 \sqrt{16v^8 - 9u^6 + 144u^4v^6}\, dA$ (d) $\iint_T u^3 \sqrt{16v^8 - 9u^6 + 144u^4v^6}\, dA$

(e) $\iint_T v^4 \sqrt{16v^8 + 9u^6 + 144u^4v^6}\, dA$

17.(6pts) Find the area of the region enclosed by the curve $x = \sin(2t)$, $y = \sin(t)$, $0 \leqslant t \leqslant \pi$.



Fact: $\sin(2t) = 2\sin(t)\cos(t)$, $\cos(2t) = \cos^2(t) - \sin^2(t)$.
Hint: Use Green's Theorem.

(a) $\dfrac{8-2\pi}{5}$ (b) $\dfrac{2\pi}{3}$ (c) $\dfrac{2\pi-1}{5}$ (d) $\dfrac{4}{3}$ (e) $\dfrac{4\pi^2}{30}$

18.(6pts) Which number below is $\displaystyle\iint_S \langle xyz, xyz, xyz\rangle \cdot d\mathbf{S}$ where S consists of the six faces of the cube with sides of length 2 in the first octant and with one vertex at the origin and with normal vector pointing out of the cube?

(a) -4 (b) 6 (c) 12 (d) 24 (e) 1

19.(6pts) Let C be the intersection of the cylinder over the triangle T with vertices $(0,0)$, $(1,0)$ and $(0,1)$ with the sphere $x^2+y^2+z^2=9$. Orient T counterclockwise. Which integral below is equal to $\int_C \langle y, z, x\rangle \cdot d\mathbf{r}$?

(a) $\iint_T \frac{x-y}{\sqrt{9-x^2-y^2}} - 1\ dA$ (b) $\iint_T \frac{-x+y}{\sqrt{9-x^2-y^2}} - 1\ dA$ (c) $\iint_T \frac{-x-y}{\sqrt{9-x^2-y^2}} - 1\ dA$

(d) $\iint_T \frac{x-y}{\sqrt{9-x^2-y^2}} + 2\ dA$ (e) $\iint_T \frac{-x-y}{\sqrt{9-x^2-y^2}} + 2\ dA$

20.(6pts) Let $\mathbf{r}(t) = \left\langle t^2, (t-1)^3 \right\rangle$, $0 \leqslant t \leqslant 2$. Which parametrized surface below is the result of rotating this curve about the y axis?

(a) $\left\langle t^2, (t-1)^3\cos(\theta), t^2\sin(\theta)\right\rangle$; $0 \leqslant t \leqslant 2$, $0 \leqslant \theta \leqslant 2\pi$

(b) $\left\langle t^2\cos(\theta), (t-1)^3, t^2\sin(\theta)\right\rangle$; $0 \leqslant t \leqslant 2$, $0 \leqslant \theta \leqslant 2\pi$

(c) $\left\langle t^2, (t-1)^3\cos(\theta), t^2\sin(\theta)\right\rangle$; $0 \leqslant t \leqslant 2$, $0 \leqslant \theta \leqslant \pi$

(d) $\left\langle t^2\cos(\theta), (t-1)^3, t^2\sin(\theta)\right\rangle$; $0 \leqslant t \leqslant 2$, $0 \leqslant \theta \leqslant \pi$

(e) $\left\langle t^2\cos(\theta), (t-1)^3\cos(\theta), t^2\sin(\theta)\right\rangle$; $0 \leqslant t \leqslant 2$, $0 \leqslant \theta \leqslant \pi$

21.(6pts) Which point below is a local maximum for the function $f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$. Note $f_{xx} = 12x$, $f_{xy} = f_{yx} = 12y$ and $f_{yy} = 12x - 18y$. The function f has exactly four critical points listed as answers (a)-(e).

(a) $(5,0)$ (b) $(3,4)$ (c) $(-3,-4)$

(d) The function had no local maxima. (e) $(-5,0)$

22.(6pts) Find the length of the curve given by $\mathbf{r}(t) = \left\langle \frac{1}{3}t^3, \frac{1}{2}t, \frac{1}{2}t^2 \right\rangle$ between the points $\langle 0,0,0\rangle$ and $\langle 72,3,18\rangle$.

(a) 34 (b) 50 (c) 19 (d) 75 (e) 86

23.(6pts) Which number below is the cosine of the angle of intersection of the two planes $2x + 3y + z = 10$ and $3x + 2y - z = 4$.

(a) $\frac{4}{5}$ (b) $-\frac{3}{16}$ (c) $\frac{11}{14}$

(d) $-\frac{2}{3}$ (e) The two planes do not intersect.

24.(6pts) Let $\mathbf{F} = \left\langle x + e^{\sin(yz)}, y + \sin(x^2 + z), \cos(xyz) \right\rangle$. Then $\operatorname{div} \mathbf{F}$ is a function, $\operatorname{grad}(\operatorname{div} \mathbf{F})$ is a field and so $\operatorname{curl}\big(\operatorname{grad}(\operatorname{div} \mathbf{F})\big)$ is a field. Which field below is it?

(a) $\langle -x, 2y, -z \rangle$ (b) $\langle 1, 1, \sin(xyz) \rangle$ (c) $\langle -1, 2, -1 \rangle$

(d) $\langle x, y, z \rangle$ (e) $\langle 0, 0, 0 \rangle$

25.(6pts) Compute the divergence of $\mathbf{F}$, $\nabla \bullet \mathbf{F}$ where $\mathbf{F} = \nabla f$ with $f(x, y, z) = x^2z + xyz + y^3$.

(a) $2xz + xz$

(b) $3xz + yz + 3y^2$

(c) 0

(d) $3xz + yz + 3y^2 + x^2 + xy$

(e) $6y + 2z$

1. Solution. In spherical coordinates, H is given by $\rho = a$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \phi \leqslant \dfrac{\pi}{3}$, so H is parametrized by $\mathbf{r}(\theta, \phi) = \langle a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi\rangle$ so a normal vector is given by

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\theta\sin\phi & a\cos\theta\sin\phi & 0 \\ a\cos\theta\cos\phi & a\sin\theta\cos\phi & -a\sin\phi \end{vmatrix} =$$

$$\left\langle -a^2\cos\theta\sin^2\phi, -a^2\sin\theta\sin^2\phi, -a^2\sin\phi\cos\phi \right\rangle = a\sin\phi\,\langle a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi\rangle.$$

The region in the θ-ϕ plane is T given by $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \phi \leqslant \dfrac{\pi}{3}$

Hence $\displaystyle\iint_H \langle x, y, z\rangle \bullet d\,\mathbf{S} =$

$$\iint_T \langle a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi\rangle \bullet a\sin\phi\,\langle a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi\rangle\, dA =$$

$$\iint_T a(\sin\phi)a^2\, dA = a^3\iint_T \sin\phi\, dA = a^3\int_0^{2\pi}\int_0^{\pi/3}\sin\phi\, d\phi\, d\theta = -a^3\int_0^{2\pi}\cos\phi\Big|_0^{\pi/3}\, d\theta =$$

$$a^3 2\pi\frac{1}{2}.$$

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2. Solution. Let $\mathbf{r} = \langle g, h\rangle$. Then $g_u(u,t) = \nabla f \bullet \mathbf{r}_u$ at $(-1, 3)$ so the answer is $\langle 4, 5\rangle \bullet \langle 3, -2\rangle = 2$.

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3. Solution. In polar coordinates, D is $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant \pi$ so $\displaystyle\iint_D y\, dA = \int_0^a\int_0^\pi r\sin(\theta) r\, d\theta\, dr =$

$$\left(\int_0^a r^2\, dr\right)\left(\int_0^\pi \sin(\theta)\, d\theta\right) = \frac{a^3}{3}\cdot\left(-\cos(\theta)\Big|_0^\pi\right) = -\frac{a^3}{3}(-1-(1)) = \frac{2a^3}{3}$$

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4. Solution. $\nabla f = \left\langle 2xy - yz, x^2 - xz, -xy + 2z \right\rangle$ and $\nabla g = \langle 2x, 2y, 2z\rangle$ so at the point in question $\nabla f = \langle 4, 3, 1\rangle$ and $\nabla g = \langle 6, 2, 4\rangle$ so a vector on the tangent line is $\nabla f \times \nabla g = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 1 \\ 6 & 2 & 4 \end{vmatrix} = \langle 12 - 2, -(16 - 6), 8 - 18\rangle = \langle 10, -10, -10\rangle$ A parallel vector is $\langle 1, -1, -1\rangle$.
Hence $\langle 3, 1, 2\rangle + t\,\langle 1, -1, -1\rangle$.

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5. Solution. $f_x = yz^2 + 2xyz$ so $f = xyz^2 + x^2yz + g(y, z)$.
$f_y = xz^2 + x^2z + g_y = xz^2 + x^2z$ so $g_y = 0$ and $g(y, z) = h(z)$.
$f_z = 2xyz + x^2y + h' = 2xyz + x^2y - 2z$ so $h' = -2z$ and $h = -z^2$.
$f(x, y, z) = xyz^2 + x^2yz - z^2$

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6. Solution. $(1, 2) + \langle 2, 3\rangle = (3, 5)$, $(1, 2) + \langle 3, 1\rangle = (4, 3)$. Area is $\pm \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix} =$ $\langle 0, 0, 2 - 9\rangle = -7$ so 7.

7. Solution. First $\nabla f = \left\langle 2x - y, 3y^2 - x \right\rangle$ and $\nabla f(2,1) = \langle 3, 1\rangle$. The unit vector is the direction of $\mathbf{u}$ is $\frac{1}{\sqrt{2}}\langle 1, -1\rangle$ so $D_{\mathbf{u}} f(2,1) = \langle 3, 1\rangle \bullet \frac{1}{\sqrt{2}}\langle 1, -1\rangle = \frac{2}{\sqrt{2}} = \sqrt{2}$.

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8. Solution. If $f = x - 2y + z$ and $g = x^2 + 4y^2 + z^2$ then Lagrange says the maxima and minima occur at points where $\nabla f = \lambda \nabla g$ and $g = 3$. Hence $\langle 1, -2, 1\rangle = \lambda \langle 2x, 8y, 2z\rangle$ or $\langle x, y, z\rangle = \left\langle \frac{1}{2\lambda}, -\frac{1}{4\lambda}, \frac{1}{2\lambda}\right\rangle$. Hence $g(x,y,z) = \left(\frac{1}{2\lambda}\right)^2 + 4\left(\frac{1}{4\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 3$ or $3\frac{1}{4\lambda^2} = 3$ or $\lambda = \pm\frac{1}{2}$. Hence there are eight points $\left(\pm 1, \pm\frac{1}{2}, \pm 1\right)$. The maximum occurs when all three terms are positive so $f = 1 - 2(-1/2) + 1 = 2 + 1 = 3$.

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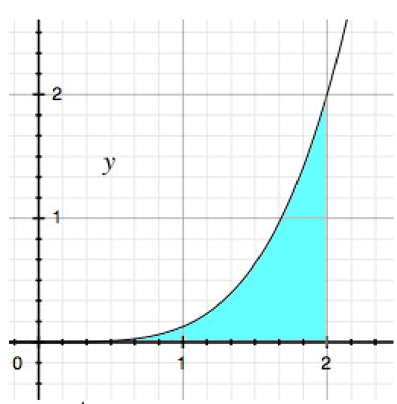
9. Solution. Write the iterated integral as a double integral over the region D given by $\sqrt[4]{8y} \leqslant x \leqslant 2$; $0 \leqslant y \leqslant 2$.



Setting up the other way, $\displaystyle\int_0^2 \int_0^{x^4/8} \sqrt{x^5 + 1}\, dy\, dx$.

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10. Solution. By Green's Theorem $\int_{\partial S} M dx + N dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$ where S is the square and $\mathbf{F} = \langle M, N\rangle$. The curve ∂S is $-C$ so if we compute the double integral we will get the negative of the answer. We get $\iint_S (0) - (-y)\, dA = \int_0^2 \int_0^2 y\, dx\, dy = 2\int_0^2 y\, dy = y^2\big|_0^2 = 4$.

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11. Solution. $\iint_S x\, dx - 2y\, dy + (x^2 + z)\, dz + \iint_D x\, dx - 2y\, dy + (x^2 + z)\, dz = \iiint_E 0\, dV$ where the normal vector to D points down. $\mathbf{F} = \left\langle x, -2y, x^2 + z\right\rangle$. To parametrize D use $\mathbf{r}(x, y) = \langle x, y, 0\rangle$ with normal vector $\langle 0, 0, -1\rangle$. $\iint_D x\, dx - 2y\, dy + (x^2 + z)\, dz = \iint_D -x^2\, dA = \int_0^3 \int_0^{2\pi} r^2 \cos^2(\theta) r\, d\theta\, dr = \left(\int_0^3 r^3\, dr\right) \cdot \left(\int_0^{2\pi} \cos^2(t)\, dt\right) = \frac{81\pi}{4}$.

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12. Solution. $\frac{\partial \mathbf{r}}{\partial u} = \left\langle v^2, v, 2uv\right\rangle$ and $\frac{\partial \mathbf{r}}{\partial v} = \left\langle 2uv, u, u^2\right\rangle$ so at $u = 2$, $v = 1$,

$$\mathbf{r}_u \times \mathbf{r}_v = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v^2 & v & 2uv \\ 2uv & u & u^2 \end{vmatrix} = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 4 \\ 4 & 2 & 4 \end{vmatrix} = \langle (4-8), -(4-16), (2-4)\rangle = \langle -4, 12, -2\rangle$$

So $\langle -4, 4, -2\rangle \bullet \langle x-1, y-2, z-4\rangle = 0$ or $-4x + 4y - 2z = -4$.

13. Solution. The surface is parametrized by $\mathbf{r}(x,z) = \left\langle x, x^3, z \right\rangle$ over the region T given by $0 \leqslant x \leqslant 1$ and $0 \leqslant z \leqslant 36x^3$. The normal vector is $\det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3x^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left\langle 3x^2, -1, 0 \right\rangle$ so $\iint_S 1 dS = \iint_T \sqrt{1+9x^4}\, dA = \int_0^1 \int_0^{36x^3} \sqrt{1+9x^4}\, dz\, dx = 36 \int_0^1 (x^3)\sqrt{1+9x^4}\, dx$. Let $u = 1 + 9x^4$ so $du = 36x^3\, dx$ so $36 \int_0^1 x^3 \sqrt{1+9x^4}\, dx = \int_1^{10} \sqrt{u}\, du = \frac{2}{3}\left(\sqrt{10}^3 - 1\right) = \frac{2}{3}\left(\sqrt{10}^3 - 1\right)$

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14. Solution. By the Fundamental Theorem, the answer is $f\big(\mathbf{r}(b)\big) - f\big(\mathbf{r}(a)\big)$ and $\mathbf{r}\left(\frac{\pi}{2}\right) = \langle 0, 0, 3\rangle$, $\mathbf{r}(0) = \left\langle 0, -\frac{\pi}{2}, 2\right\rangle$. Now $f\left(\mathbf{r}\left(\frac{\pi}{2}\right)\right) = f\,(0,0,3) = 0$; $f\,(\mathbf{r}(0)) = f\left(0, -\frac{\pi}{2}, 2\right) = -\pi$ so the answer is $0 - (-\pi) = \pi$.

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15. Solution. The Jacobian is $\det \begin{vmatrix} 2u & + \\ v & u \end{vmatrix} = (2u+v)(2v+u) - (u)(v) = (4uv + 2u^2 + 2v^2 + uv) - (uv) = 4uv + 2u^2 + 2v^2$.

16. Solution. The short answer is $\iint_S x\,dS$. A normal vector is $\det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3u^2 & 0 & v \\ 0 & 4v^3 & u \end{vmatrix} = \left\langle -4v^4, -3u^3, 12u^2v^3 \right\rangle$ and its length is $\sqrt{16v^8 + 9u^6 + 144u^4v^6}$ so $\iint_S x\,dS = \iint_T u^3\sqrt{16v^8 + 9u^6 + 144u^4v^6}\,dA.$

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17. Solution. $\mathbf{r}' = \langle 2\cos(2t), \cos(t)\rangle$ The area is $\iint_A dA = \int_0^\pi \langle 0, x\rangle \cdot \langle 2\cos(2t), \cos(t)\rangle\,dt = \int_0^\pi \sin(2t)\cos(t)\,dt = 2\int_0^\pi \sin(t)\cos^2(t)\,dt = -2\frac{\cos^3(y)}{3}\Big|_0^\pi = -\big((-2/3) - (2/3)\big) = \frac{4}{3}.$

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18. Solution. By the Divergence Theorem, $\iint_S \langle xyz, xyz, xyz\rangle \cdot d\mathbf{S} = \iint_U \nabla \cdot \mathbf{F}\,dV = \int_0^2\int_0^2\int_0^2 (yz + xz + xy)\,dx\,dy\,dz = \int_0^2\int_0^2 (2yz + 2z + 2y)\,dy\,dz = \int_0^2 (4z + 4z + 4)\,dz = 4\int_0^2 (2z+1)\,dz = 4\left(z^2 + z\right)\Bigg|_0^2 = 24$

19. Solution. By Stoke's Theorem, the answer is $\iint_S curl\,\mathbf{F}\cdot d\mathbf{S}$ where S is the part of the sphere lying over the triangle T. The part of the sphere we have is the graph of $z = \sqrt{9-x^2-y^2}$ over the triangle T. So , $d\,\mathbf{S} = \left\langle \frac{x}{\sqrt{9-x^2-y^2}}, \frac{y}{\sqrt{9-x^2-y^2}}, 1 \right\rangle dA$

$$\det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \langle -1, -(1), -1 \rangle = \langle -1, -1, -1 \rangle$$

so we need to integrate

$$\iint_T \frac{-x-y}{\sqrt{9-x^2-y^2}} - 1\ dA$$

over the triangle T.

20. Solution. $\mathbf{R}(t,\theta) = \left\langle t^2\cos(\theta), (t-1)^3, t^2\sin(\theta) \right\rangle$; $0 \leqslant t \leqslant 2$, $0 \leqslant \theta \leqslant 2\pi$

21. Solution. The critical points are located at the zeros of the gradient: $f_x = 6x^2 + 6y^2 - 150$, $f_y = 12xy - 9y^2$. Since $12xy - 9y^2 = 3y(4x - 3y$ either $y = 0$ or $4x - 3y = 0$ so $y = \frac{4}{3}x$. Note $f_x = 0$ is the same as $x^2 + y^2 = 25$.

If $y = 0$, $x = \pm 5$. If $y = \frac{4}{3}x$ $x^2 + \frac{16}{9}x^2 = 25$ or $\frac{25}{9}x^2 = 25$ so $x = \pm 3$ and the critical points are $(5,0)$, $(-5,0)$ $(3,4)$ and $(-3,-4)$.

The Hessian is $\begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 12x & 12y \\ 12y & 12x - 18y \end{bmatrix} = 6\begin{bmatrix} 2x & 2y \\ 2y & 2x - 3y \end{bmatrix}$ whose determinant is $4x^2 - 6xy - 4y^2 = 2(2x^2 - 3xy - 2y^2)$ so the type of critical point is determined by $h(x,y) = 2x^2 - 3xy - 2y^2$.

$h(5,0) = 25 = h(-5,0) > 0$, $h(3,4) = h(-3,-4) = 18 - 24 - 18 = -24 < 0$. Negative Hessian means saddle point so $(pm5, 0)$ are the only possibilities. $f_{xx}(5,0) = 60 > 0$ so $(5,0)$ is local minimum; $f_{xx}(-5,0) = -60 < 0$ so $(-5,0)$ is local maximum.

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22. Solution. The parametrized curve is $\mathbf{r}$, $0 \leqslant t \leqslant 6$. Hence the length is $\int_0^6 |\mathbf{r}'(t)|\,dt$. Now $\mathbf{r}'(t) = \left\langle t^2, \frac{1}{2}, t \right\rangle$ so $|\mathbf{r}'(t)| = \sqrt{t^4 + \frac{1}{4} + t^2} = t^2 + \frac{1}{2}$ so $\int_0^6 |\mathbf{r}'(t)|\,dt = \frac{1}{3}t^3 + \frac{1}{2}t\Big|_0^6 = 75.$

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23. Solution. The angle is the angle between the two normal vectors $\langle 2,3,1\rangle$ and $\langle 3,2,-1\rangle$ and so $\cos\theta = \frac{11}{14}$.

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24. Solution. The curl of a gradient is $\langle 0, 0, 0\rangle$ and we have a gradient.

25. Solution. $\mathbf{F} = \nabla f = \left\langle 2xz + yz, xz + 3y^2, x^2 + xy \right\rangle$ so $\nabla \bullet \mathbf{F} = (2z) + (6y) + (0) = 6y + 2z$.