1.(6pts) Which integral computes the area of the quarter-disc of radius a centered at the origin in the first quadrant?

(a) $\int_0^{\pi/2}\int_0^a r dr\, d\theta$ (b) $\int_0^{\pi/4}\int_0^a r dr\, d\theta$ (c) $\int_0^{2\pi}\int_0^a dr\, d\theta$

(d) $\int_0^{\pi/2}\int_0^a dr\, d\theta$ (e) $\int_0^{\pi/2}\int_{-a}^a r dr\, d\theta$

2.(6pts) Find the moment about the y axis of a triangle with vertices $(0,0),(1,0),(0,1)$ and density distribution $\rho(x,y)=x^2$.

(a) 1/20 (b) 3/5 (c) 1/5

(d) 1/62 (e) 1/12

3.(6pts) Compute the triple integral $\iiint_V \frac{1}{y+x} dV$ over the region V bounded by $x=y(2-y)$, $x=(2-y)^2$, $z=1$ and $z=1+y+x$.

(a) $\frac{1}{3}$ (b) $\frac{3}{8}$ (c) $\frac{1}{6}$

(d) $\frac{5}{9}$ (e) $\frac{1}{8}$

4.(6pts) Consider the change of variables $x=2u+v$, $y=u-v^2$. Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

(a) $-4v-1$ (b) $4v-1$ (c) $2x+y$

(d) $1-y$ (e) $2u+v$

5.(6pts) Use the change of variables $x=2u, y=u+v$ to rewrite the following integral in terms of u and v:

$$\int_0^2\int_{x/2}^{x/2+1} y^2\, dy\, dx$$

(a) $\int_0^1\int_0^1 2(u+v)^2\, du\, dv$ (b) $\int_0^1\int_0^1 (u+v)^2\, du\, dv$ (c) $\int_0^2\int_0^1 2(u+v)^2\, du\, dv$

(d) $\int_0^1\int_0^2 y^2\, du\, dv$ (e) $\int_0^1\int_0^1 2y^2\, du\, dv$

6.(6pts) Evaluate the line integral $\int_C xy\, ds$ where C is the right half of the circle $x^2 + y^2 = 16$ starting at $(0, -4)$.

(a) 0 (b) π (c) 2π

(d) $\pi/2$ (e) $-\pi/3$

~~**7.**(6pts) Find the function p such that~~

$$\cancel{\nabla p = \langle y^3 \sin(z), 3xy^2 \sin(z), xy^3 \cos(z)\rangle}$$

~~and $p(0,0,0) = 0$. What is $p\left(3, 2, \frac{\pi}{2}\right)$?~~

(a) 24 (b) 48 (c) 0

(d) -12 (e) 1

~~**8.**(6pts) Evaluate~~ $\oint_C \left(3y - e^{\sin(x^2)} \cos(3x^2)\right) dx + \left(5x + \arctan(y^2)\right) dy$ where C is the circle ~~$x^2 + y^2 = 4$ with counter-clockwise orientation.~~

(a) 8π (b) 2 (c) 4π

(d) $-\pi^2$ (e) $4\pi^2 - 2$

9.(6pts) Let E be the solid determined by $x^2 + y^2 \leqslant z^2, x^2 + y^2 + z^2 \leqslant 9$ and $z \geqslant 0$. Which of the following expresses

$$\iiint_E \sqrt{x^2 + y^2 + z^2}\, dV$$

as iterated integrals correctly?

Hints: Depending on how you do your calculations one of the following formulas may be helpful: $\arccos\left(\frac{\sqrt{2}}{2}\right) = \pi/4 = \arctan(1)$.

(a) $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^3 \rho^3 \sin\phi\, d\rho\, d\phi\, d\theta$ (b) $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^3 \rho^3\, d\rho\, d\phi\, d\theta$ (c) $\int_0^{2\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^3 \rho^3 \sin\phi\, d\rho\, d\phi\, d\theta$

(d) $\int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{2\cos\phi}^3 \rho^3 \sin\phi\, d\rho\, d\phi\, d\theta$ (e) $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_{2\cos\phi}^3 \rho^3 \sin\phi\, d\rho\, d\phi\, d\theta$

10.(10pts) Compute the mass of the disc D centered at the origin with radius 1 and density $\rho(x, y) = 1 + y$.

11.(10pts) A tetrahedron has vertices at the points $(0,0,0)$, $(1,0,0)$, $(1,1,0)$ and $(1,0,1)$. Compute the mass of the tetrahedron if the density function is $\rho(x,y,z) = \cos\left(\frac{\pi}{2}z\right)$.

12.(10pts) Let R denote the parallelogram in the xy-plane bounded by the lines $y = 1-x$, $y = -x$, $y = 2x$ and $y = 2x+3$. Compute the integral

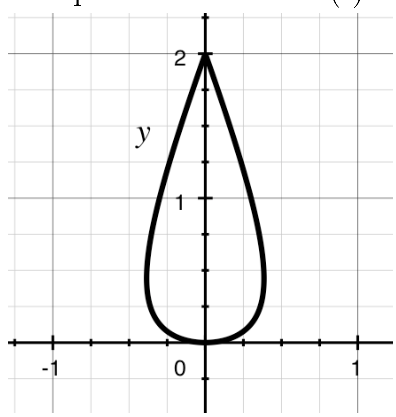
$$\iint_R (x+y)^2\, dA$$

using a suitable change of variables.

13. (10pts)

(a) (4pts) Let C be a curve which is the boundary of a bounded region D in the plane. Suppose the density of D, $\mu(y)$ is a function of y. Show that the moment about the x axis of this region with density μ can be found by $\displaystyle -\oint_C xy\mu(y)\,dy$ where C is oriented as in Green's Theorem.

(b) (2pts) Here is a graph of the parametric curve $\mathbf{r}(t) = \langle t^3 - t, t^4 + t^2 \rangle$, $-1 \leqslant t \leqslant 1$.



Is the orientation given by this parametrization the same as the one required by Green's Theorem?

(c) (6pts) Suppose the density of this region is 1. Show that the moment about the x-axis is

$$-2\int_{-1}^{1} t^4(t^4-1)(2t^2+1)\,dt$$

1. Solution. By inspection. Bounds are $r = 0..a$, $\theta = 0..\pi/2$, integrand is $1dA = rdr\,d\theta$.

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2. Solution. The coordinates of the center of mass are $\bar{x} = M_y/M$, $\bar{y} = M_x/M$, so we compute each of these quantities. The integrals that appear are easy to compute, since they are just polynomials in x and y.

$$M = \iint\limits_{\Delta} \rho(x,y)dA = \int\limits_0^1 \int\limits_0^{1-x} x^2 dy\,dx = \int\limits_0^1 x^2(1-x)\,dx = \int\limits_0^1 (x^2 - x^3)\,dx = \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{12};$$

$$M_x = \iint\limits_{\Delta} y\rho(x,y)\,dA = \int\limits_0^1 \int\limits_0^{1-x} yx^2 dy\,dx = \int\limits_0^1 \frac{y^2}{2}x^2\Big|_0^{1-x} dx = \frac{1}{2}\int\limits_0^1 (1-x)^2x^2\,dx$$

$$= \frac{1}{2}\int\limits_0^1 x^2 - 2x^3 + x^4\,dx = \frac{1}{2}\left(\frac{x^3}{3} - 2\frac{x^4}{4} + \frac{x^5}{5}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{1}{60}(10 - 15 + 6) = \frac{1}{60};$$

$$M_y = \iint\limits_{\Delta} x\rho(x,y)\,dA = \int\limits_0^1 \int\limits_0^{1-x} x^3 dy\,dx = \int\limits_0^1 x^3y\Big|_0^{1-x} dx = \int\limits_0^1 (x^3 - x^4)\,dx = \frac{x^4}{4} - \frac{x^5}{5}\Big|_0^1 = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

Thus, $\bar{x} = (1/60)/(1/12) = 1/5$, $\bar{y} = (1/20)/(1/12) = 3/5$.

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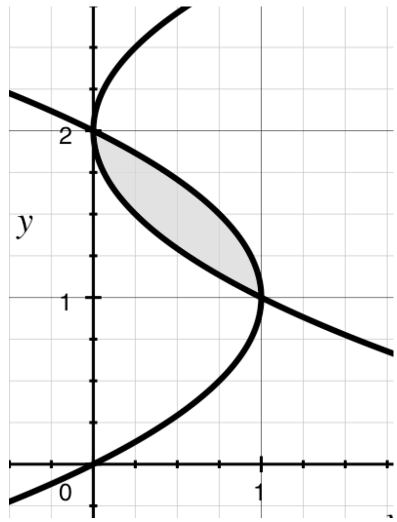
3. Solution.

The curves $x = y(2-y)$, $x = (2-y)^2$ intersect at $y(2-y) = (2-y)^2$ or $y = 2$ and $y = 2 - y$ so $y = 1$.



Thus

$$\iiint_V \frac{1}{y+x} dV = \int_1^2 \int_{(2-y)^2}^{y(2-y)} \int_1^{1+y+x} \frac{1}{y+x}\, dz\, dx\, dy$$
$$= \int_1^2 \int_{(2-y)^2}^{y(2-y)} \frac{1+y+z-1}{y+z} dx\, dy = \int_{1/2}^1 \int_{(2-y)^2}^{y(2-y)} dx\, dy =$$
$$= \int_1^2 \left(y(2-y) - (2-y)^2\right) dy = \int_1^2 \left(2y - y^2 - (4 - 4y + y^2)\right) dy$$
$$= \int_1^2 \left(-4 + 6y - 2y^2\right) dy = -4y + 3y^2 - \frac{2}{3}y^3 \Big|_1^2$$
$$= \left(-8 + 12 - \frac{16}{3}\right) - \left(-4 + 3 - \frac{2}{3}\right) = -\frac{4}{3} - \left(-\frac{5}{3}\right) = \frac{1}{3}$$

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4. Solution. The Jacobian is

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = 2(-2v) - 1 = -4v - 1.$$

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5. Solution. The domain of integration in terms of u, v is $0 \leqslant u \leqslant 1$ and $0 \leqslant v \leqslant 1$, since the equations $y = x/2$ and $y = x/2 + 1$ are $v = 0$ and $v = 1$ in terms of u, v. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = 2 \cdot 1 - 0 = 2.$$

So the integral is

$$\int_0^1 \int_0^1 (u + v)^2 \cdot 2\, du\, dv.$$

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6. Solution. We can parametrize C as $\langle 4\cos t, 4\sin t\rangle$ for $-\pi \leq t \leq \pi$. Then $ds = \sqrt{(-4\sin t)^2 + (4\cos t)^2}\, dt = 4\, dt$. Then the line integral becomes

$$\int_{-\pi}^{\pi} 4\cos t \cdot 4\sin t\, 4\, dt = 64 \int_{-\pi}^{\pi} \cos t \sin t\, dt = 32\sin^2(t)\Big|_{-\pi}^{\pi} = 0$$

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7. Solution. $\dfrac{\partial p}{\partial x} = y^3 \sin(z)$ so $p = xy^3 \sin(z) + h(y, z)$.

$\dfrac{\partial p}{\partial y} = 3xy^2 \sin(z) + \dfrac{\partial h}{\partial y} = 3xy^2 \sin(z)$ so $h(y, z) = g(z)$.

$\dfrac{\partial p}{\partial z} = xy^3 \cos(z) + g'(z) = xy^3 \cos(z)$ so

$p(x, y, z) = xy^3 \sin(z)$. Since $p(0, 0, 0) = 0$ this is the desired function.

Thus $p\left(3, 2, \dfrac{\pi}{2}\right) = 3 \cdot 2^3 \cdot \sin\left(\dfrac{\pi}{2}\right) = 24$.

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8. Solution. $M = 3y - e^{\sin(x^2)}\cos(3x^2)$ and $N = 5x + \arctan(y^2)$, $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 5 - 3 = 2$ so by Green's Theorem, $\oint \left(4y - e^{\sin(x^2)}\cos(3x^2)\right) dx + \left(5x + \arctan(y^2)\right) dy = \iint_D 2\, dA$ where D is the disk of radius 2 so the answer is $2 \cdot \pi 2^2 = 8\pi$.

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9. Solution. We use spherical coordinates. Then the three inequalities defining the solid can be rewritten as

$$\rho^2 \leqslant 9$$
$$\rho^2 \leqslant 2\rho^2 \cos^2\phi \Rightarrow |\cos\phi| \geqslant \frac{\sqrt{2}}{2}$$
$$\rho\cos\phi \geqslant 0 \Rightarrow \cos\phi \geqslant 0$$

This yields

$$0 \leqslant \rho \leqslant 3$$
$$0 \leqslant \phi \leqslant \frac{\pi}{4}$$

and

$$0 \leqslant \theta \leqslant 2\pi$$

Therefore the iterated integrals should be

$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^3 \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta$$

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10. Solution. We integrate $\displaystyle\iint_D \rho \, dA = \int_0^{2\pi} \int_0^1 (1 + r\sin(\theta)) r \, dr d\theta$ which evaluates quite straightforwardly to π.

11. Solution. First find the equation of the plane: $(i+j)\times(i+k) = i-j-k$ so $x-y-z=0$ so $z=x-y$.

$$\begin{aligned}
\iiint_V \cos\left(\frac{\pi}{2}z\right)\,dV &= \int_0^1\int_0^x\int_0^{x-y} \cos\left(\frac{\pi}{2}z\right)\,dz\,dy\,dx \\
&= \int_0^1\int_0^x \frac{2}{\pi}\sin\left(\frac{\pi}{2}z\right)\Big|_0^{x-y}\,dy\,dx \\
&= \frac{2}{\pi}\int_0^1\int_0^x \sin\left(\frac{\pi}{2}(x-y)\right)\,dy\,dx \\
&= \frac{4}{\pi^2}\int_0^1 \cos\left(\frac{\pi}{2}(x-y)\right)\Big|_0^x\,dx \\
&= \frac{4}{\pi^2}\int_0^1 1-\cos\left(\frac{\pi}{2}x\right)\,dx \\
&= \frac{4}{\pi^2}-\frac{8}{\pi^3}\sin\left(\frac{\pi}{2}x\right)\Big|_0^1 \\
&= \frac{4}{\pi^2}-\frac{8}{\pi^3}
\end{aligned}$$

12. Solution. Let $u = x + y$ and $v = 2x - y$. Then the region R in the uv-plane is $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 3$. We can solve for x and y:

$$x = \frac{1}{3}(u+v)$$
$$y = \frac{1}{3}(2u - v).$$

The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3} \cdot \frac{-1}{3} - \frac{1}{3} \cdot \frac{2}{3} = -\frac{1}{3}.$$

Therefore the integral in terms of u, v is

$$\int_0^3 \int_0^1 u^2 \cdot \frac{1}{3}\, du\, dv == \int_0^3 \frac{u^3}{3} \cdot \frac{1}{3}\Big|_0^1 dv = \frac{1}{9} \int_0^3 dv = \frac{1}{3}.$$

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13. Solution. For part (a), by Green's Theorem $\oint_C xy\mu\, dy = \oint_C 0\, dx + xy\mu\, dy = \oint_C \langle 0, xy\mu \rangle \bullet \mathbf{r} = \iint_D \left(\frac{\partial\, xy\mu}{\partial\, x} - \frac{\partial\, 0}{\partial\, y} \right) dA$. Since μ is a function of y, $\frac{\partial\, xy\mu}{\partial\, x} = y\mu$, so $\oint_C xy\mu\, dy = \iint_D y\mu\, dA$ which is the formula for the moment about the x axis for the region D with density μ.

For part (b), the tangent vector to the curve at the point $\mathbf{r}(t)$ is $\mathbf{r}'(t) = \left\langle 3t^2 - 1, 4t^3 + 2t \right\rangle$. Pick any point, say $t = 0$. Then at $\mathbf{r}(0) = \langle 0, 0 \rangle$, $\mathbf{r}'(0) = \langle -1, 0 \rangle$. For Green's Theorem you need the orientation so that as you walk around the curve in the preferred direction to the region is on your left. Hence the preferred direction at $\langle 0, 0 \rangle$ is $< 1, 0 >$ so the orientation given by the parametrization is opposite to the one given by Green's Theorem.

For part (c), we do $\oint_C xy\, dy = -\int_{-1}^1 (t^3 - t)(t^4 + t^2)(4t^3 + 2t)\, dt = -2 \int_{-1}^1 t^4(t^2 - 1)(t^2 + 1)(2t^2 + 1)\, dt = -2 \int_{-1}^1 t^4(t^4 - 1)(2t^2 + 1)\, dt$

If you actually want to know the answer $2 \int_{-1}^1 t^4(2t^6 + t^4 - 2t^2 - 1)\, dt = 2 \int_{-1}^1 (2t^{10} + t^8 - 2t^6 - t^4)\, dt = 2 \left(\frac{2t^{11}}{11} + \frac{t^9}{9} - \frac{2t^7}{7} - \frac{t^5}{5} \right)\Big|_{-1}^1 = 4 \left(\frac{2}{11} + \frac{1}{9} - \frac{2}{7} - \frac{1}{5} \right)$. You were not asked for the answer so there was no credit for just doing the integral.

Some credit would have been given for arguing that the integral is negative but the moment is positive so the sign must be -1 and hence the parametric orientation is opposite to the Green's Theorem orientation.
