1. Sarah the architect is designing a modern building. The base of the building is the region in the xy-plane bounded by $x = 0$, $y = 0$, and $y = 3 - x$. The building itself has a height bounded between $z = 0$ and $z = 9 - x - 2y$. What is the volume of the building?

(a) 27 (b) $\frac{81}{2}$ (c) 0 (d) $\frac{27}{2}$ (e) $\frac{81}{2} - \frac{9x}{2} - 3x^2$

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2. Consider the ellipsoid given by $x^2 + 2y^2 + 3z^2 = 15$ and the parabaloid given by $x^2 + y^2 = z$. What is the equation for the tangent line of the curve of intersection of these two surfaces at the point $P = (1, 1, 2)$?

(a) $\mathbf{r}(t) = \langle 1, 1, 2\rangle + t\langle -28, 26, -4\rangle$

(b) $\mathbf{r}(t) = \langle 1, 1, 2\rangle + t\langle 2, 4, 12\rangle$

(c) $\mathbf{r}(t) = \langle 1, 1, 2\rangle + t\langle 2, 2, -1\rangle$

(d) $\mathbf{r}(t) = \langle 1, 1, 2\rangle + t\langle 2, 3, 15\rangle$

(e) $\mathbf{r}(t) = \langle 2, 4, 12\rangle + t\langle 2, 2, -1\rangle$

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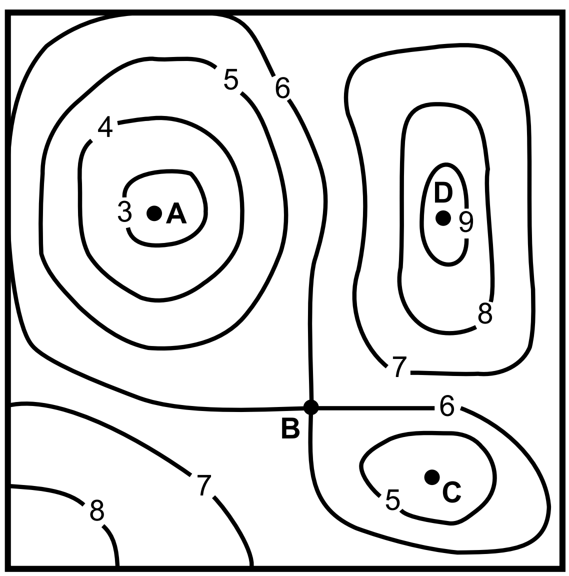
3. Below is a contour plot of a function $f(x, y)$ with labeled f-values on the contours. Points **A**, **B**, **C**, and **D** are critical points for f. Which point(s) are local minimums?



(a) A and C (b) B (c) D

(d) A, C, and D (e) A, B, C, and D

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4. Joe the chef is making a pizza. He starts with a cylinder of dough and spins it, increasing the radius and decreasing the height (while keeping the volume constant). At this instant, the radius is 5 cm, the height is 3 cm, and the radius is increasing at a rate of 2 cm/s. At what rate is the height changing?

(a) $-\frac{12}{5}$ cm/s (b) $-\frac{6}{5}$ cm/s (c) $-\frac{20}{3}$ cm/s (d) -250π cm/s (e) 0 cm/s

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5. Suppose $f(x,y)$ is a function with critical points at $(0,0)$, $(\sqrt{3},-\sqrt{3})$, and $(-\sqrt{3},\sqrt{3})$. Suppose $f_{xx}(x,y)=12x^2$, $f_{yy}(x,y)=12y^2$, and $f_{xy}(x,y)=12$. Determine which points are local maximums, which points are local minimums, and which points are saddle points using the Second Derivative Test.

(a) $(0,0)$ is a saddle point, $(\sqrt{3},-\sqrt{3})$ and $(-\sqrt{3},\sqrt{3})$ are local minimums.

(b) $(0,0)$ is a saddle point, $(\sqrt{3},-\sqrt{3})$ and $(-\sqrt{3},\sqrt{3})$ are local maximums.

(c) $(0,0)$, $(\sqrt{3},-\sqrt{3})$, and $(-\sqrt{3},\sqrt{3})$ are saddle points.

(d) $(0,0)$, $(\sqrt{3},-\sqrt{3})$, and $(-\sqrt{3},\sqrt{3})$ are local maximums.

(e) $(0,0)$ is a saddle point, $(\sqrt{3},-\sqrt{3})$ is a local min, and $(-\sqrt{3},\sqrt{3})$ is a local max.

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6. Suppose you are on a mountain whose height is given by $h(x,y)=12-x^2+y^2$. From the point $(1,1)$, which direction should you head in order to walk up the steepest slope?

(a) $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ (b) $\left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$ (c) $\left\langle -\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}} \right\rangle$

(d) $\left\langle \frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}} \right\rangle$ (e) There are several such directions.

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7. Use Lagrange Multipliers to maximize the function $f(x,y)=3x^2+2y+2$ relative to the constraint $x^2+y^2=1$. The maximum f-value is:

(a) $16/3$ (b) 4 (c) 5 (d) 7 (e) 0

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8. Consider the surface $x+y^2=z^3$. What is the equation for the normal line at the point $(-1,3,2)$?

(a) $\mathbf{r}(t)=\langle -1,3,2\rangle + t\langle 1,6,-12\rangle$ (b) $\mathbf{r}(t)=\langle -1,3,2\rangle + \langle 1,2y,-3z^2\rangle$

(c) $\mathbf{r}(t)=\langle -1,3,2\rangle + t\langle -1,9,-8\rangle$ (d) $\mathbf{r}(t)=\langle 1,6,-12\rangle + t\langle -1,9,-8\rangle$

(e) $\mathbf{r}(t)=\langle -1,3,2\rangle + t\langle 1,0,0\rangle$

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9. Find the critical point of the function $f(x, y) = -4y^2 + yx^2 - 2x + 8$.

(a) $(2, 1/2)$ (b) $(0, 0)$ (c) $(1, 1)$ (d) $(4, 0)$ () $(1/3, 2)$

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10. Find $D_{\mathbf{u}}f(2, \pi)$ where $f(x, y) = x\sin(y)$ where $\mathbf{u}$ is in the direction of $\mathbf{v} = \langle 1, 2\rangle$

(a) $-4/\sqrt{5}$ (b) -4 (c) $1/\sqrt{5}$ (d) 2 (e) $-2/\sqrt{5}$

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11. Use Lagrange Multipliers to maximize the function $f(x, y, z) = x + y + z$ subject to the constraints $g(x, y, z) = x - y - 3z = 0$ and $h(x, y, z) = y^2 + 2z^2 = 1$.

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12. Compute

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4 + 1}\, dxdy.$$

(Hint: You need to reverse the order of integration.)

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13. Compute the integral of $f(x, y) = x^2 + y^2$ over the circular disk $x^2 + y^2 \leq 4$. (Hint: Use polar coordinates.)

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14. Consider the box with one vertex at the origin and another vertex along the surface $z = 6 - x - 2y$, with edges parallel to the axes. What point on the surface yields the maximum volume for the box? Show all of your work, but you do not need to use the Second Derivative test to verify this is actually maximum (Note: $x > 0$, $y > 0$, and $z > 0$).

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1. Solution.

$$\int_0^3 \left[\int_0^{3-x} (9-x-2y)dy\right] dx = \int_0^3 \left[9y - xy - y^2\right]_{y=0}^{y=3-x} dx = \int_0^3 (18-6x)dx =$$
$$18x - 3x^2]_{x=0}^{x=3} = 27.$$

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2. Solution. Write $F(x,y,z) = x^2 + 2y^2 + 3z^2 - 15$ and $G(x,y,z) = x^2 + y^2 - z$. Then, $\nabla F = \langle 2x, 4y, 6z\rangle$ and $\nabla G = \langle 2x, 2y, -1\rangle$. So $\nabla F(1,1,2) = \langle 2,4,12\rangle$ and $\nabla G(1,1,2) = \langle 2,2,-1\rangle$. Therefore, the line heads in the direction of $\langle -28, 26, -4\rangle$.

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3. Solution. A and C are.

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4. Solution. $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} + \frac{dV}{dr}\frac{dr}{dt}$, but $V = \pi r^2 h$ and $\frac{dV}{dt} = 0$, so

$$0 = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt} \quad \mapsto \quad \frac{dh}{dt} = -\frac{2h}{r}\frac{dr}{dt}.$$

In our case, $\frac{dh}{dt} = -\frac{12}{5}$.

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5. Solution. $D = 144x^2y^2 - 144$. Thus, $D(0,0) < 0$ and it is a saddle point. $D(\sqrt{3}, -\sqrt{3}) > 0$, $D(-\sqrt{3}, \sqrt{3}) > 0$, $f_{xx}(\sqrt{3}, -\sqrt{3}) > 0$, and $f_{xx}(-\sqrt{3}, \sqrt{3}) > 0$, so both of these points are local minimums.

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6. Solution. $\nabla h = \langle -2x, 2y \rangle$, so $\nabla h(1,1) = \langle -2, 2 \rangle$. The direction is $\frac{\nabla h(1,1)}{|\nabla h(1,1)|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$.

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7. Solution. $g(x,y) = x^2 + y^2$, $k = 1$, $\nabla f = \langle 6x, 2 \rangle$, $\nabla g = \langle 2x, 2y \rangle$, so we get the system $\{6x = 2\lambda x, 2 = 2\lambda y, x^2 + y^2 = 1\}$. From the first equation, $x = 0$ or $\lambda = 3$. If $x = 0$, $y = \pm 1$. If $\lambda = 3$, $2 = 6y$, so $y = 1/3$, so $x = \pm\sqrt{8}/3$. $f(0,1) = 4$, $f(0,-1) = 0$, $f(\sqrt{8}/3, 1/3) = 16/3$, and $f(-\sqrt{8}/3, 1/3) = 16/3$. Max is $16/3$.

8. Solution. Let $F(x,y,z) = x+y^2-z^3$, so $\nabla F = \langle 1, 2y, -3z^2\rangle$, so $\nabla F(-1,3,2) = \langle 1, 6, -12\rangle$. This gives the direction of the line based at $\langle -1, 3, 2\rangle$, so we get

$$\mathbf{r}(t) = \langle -1, 3, 2\rangle + t\langle 1, 6, -12\rangle.$$

9. Solution. $f_x = 2xy - 2 = 0$ and $f_y = -8y + x^2 = 0$ means $xy = 1$ and $8y = x^2$, so $y = 1/x$, so $8/x = x^2$, so $8 = x^3$, so $x = 2$ and $y = 1/2$.

10. Solution. $\mathbf{u} = \langle 1/\sqrt{5}, 2/\sqrt{5}\rangle$, $\nabla f = \langle \sin(y), x\cos(y)\rangle$, so $\nabla f(2,\pi) = \langle 0, -2\rangle$.

$$D_{\mathbf{u}} f(2,\pi) = \nabla f(2,\pi)\cdot \mathbf{u} = -4/\sqrt{5}.$$

11. Solution. $\nabla f = \langle 1,1,1\rangle$, $\nabla g = \langle 1,-1,-3\rangle$, $\nabla h = \langle 0,2y,4z\rangle$, so we get the system

$$\{1=\lambda, 1=-\lambda+2\mu y, 1=-3\lambda+4\mu z, x-y-3z=0, y^2+2z^2=1\}.$$

The first and second equations give $2=2\mu y$, so $y=1/\mu$. Similarly, the first and third equations give us $4=4\mu z$ so $z=1/\mu$, so $y=z$. The last equation then gives us $3y^2=1$ so $y=z=\pm 1/\sqrt{3}$. Finally, the fourth equation tells us that $x=y+3z=4y=\pm 4/\sqrt{3}$. So the points are $(4/\sqrt{3},1/\sqrt{3},1/\sqrt{3})$ and $(-4/\sqrt{3},-1/\sqrt{3},-1/\sqrt{3})$. The maximum value is $(4/\sqrt{3},1/\sqrt{3},1/\sqrt{3})$ at $8/\sqrt{3}$.

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12. Solution. If $D=\{(x,y)|0\le y\le 8, \sqrt[3]{y}\le x\le 2\}$, then we can rewrite D as

$$D=\{(x,y)|0\le x\le 2, 0\le y\le x^3\}.$$

So the integral becomes

$$\int_0^2\left[\int_0^{x^3}\sqrt{x^4+1}dx\right]dy=\int_0^2\left(x^3\sqrt{x^4+1}\right)dy.$$

Now this is done by substitution $u=x^4+1$, $du=4x^3dx$, so $dx=\dfrac{du}{4x^3}$. When $x=0$, $u=1$ and when $x=2$, $u=17$, so this becomes

$$\frac{1}{4}\int_1^{17}\sqrt{u}du=\frac{1}{6}\left((17)^{3/2}-1\right).$$

Or also written as $\dfrac{17\sqrt{17}-1}{6}$.

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13. Solution. $D = \{(r, \theta)|0 \le r \le 2, 0 \le \theta \le 2\pi\}$. $x^2 + y^2 = r^2$, so we get

$$\int_0^{2\pi} \left[\int_0^2 r^3 dr\right] d\theta = \frac{\pi}{2} r^4 \Big|_{r=0}^{r=2} = 8\pi.$$

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14. Solution. We want to maximize $V = xyz$ subject to the constraint $z = 6 - x - 2y$. Substitution yields $V = f(x, y) = 6xy - x^2y - 2xy^2$. Then $f_x = 6y - 2xy - 2y^2 = 0$ and $f_y = 6x - x^2 - 4xy = 0$ when $\{6 - 2x - 2y = 0, 6 - x - 4y = 0\}$. Add -2 times the first equation and the second to get $-6 + 3x = 0$, so $x = 2$. Thus, $y = 1$ and $z = 2$. So the solution is $(2, 1, 2)$.

OR

$\nabla V = \langle yz, xz, xy\rangle$, $g(x, y, z) = x + 2y + z = 6$, $\nabla g = \langle 1, 2, 1\rangle$ so

$$\langle yz, xz, xy\rangle = \lambda \langle 1, 2, 1\rangle \quad \text{and} \quad x + 2y + z = 6$$

Then $xyz = \lambda x$, $xyz = 2\lambda y$, $xyz = \lambda z$ so

$\lambda x = 2\lambda y = \lambda z$.

If $\lambda \neq 0$, $x = 2y = z$, $3x = 6$, $x = 2$, $y = 1$, $z = 2$.

If $\lambda = 0$ $yz = 0$ so $V = 0$ and these points will not be maxima.
