1.(6pts) Find all the critical points of $f(x, y) = 4xy - x^4 - y^4$.

(a) $(0, 0)$, $(1, 1)$, $(-1, -1)$

(b) $(0, 0)$

(c) $(0, 0)$, $(1, -1)$, $(-1, 1)$

(d) $(1, -1)$, $(1, 1)$, $(-1, 1)$, $(-1, -1)$

(e) $(0, 0)$, $(1, -1)$, $(1, 1)$, $(-1, 1)$, $(-1, -1)$

Solution.

$$\nabla f = \left\langle 4y - 4x^3, 4x - 4y^3 \right\rangle = \langle 0, 0 \rangle$$

so $4y - 4x^3 = 0$, and $4x - 4y^3 = 0$. Hence $y = x^3$ and $x - x^9 = 0$. Hence $x = 0$ or $x = \pm 1$. So the critical points are

$$(0, 0) \quad (1, 1) \quad (-1, -1)$$

2.(6pts) Evaluate $\iint_S (\nabla \times \vec{F}) \bullet d\vec{S}$, where $\vec{F}(x, y, z) = -y\vec{\imath} + x\vec{\jmath} + x^2yz\vec{k}$, S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$, oriented downward. (Hint: Use Stokes' Theorem and be careful with orientations.)

(a) -8π

(b) -4π

(c) 0

(d) 4π

(e) 8π

Solution. The boundary of the surface S is the circle of radius 2 at height 4 and so is parametrized by $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 4 \rangle$, $0 \leqslant t \leqslant 2\pi$. By Stoke's Theorem we can compute

$$\int_0^{2\pi} \vec{F} \bullet \vec{r}\,'(t)\, dt = \int_0^{2\pi} \left\langle -2\sin(t), 2\cos(t), \cos^2(t)\sin(t) \cdot 4 \right\rangle \bullet \langle -2\sin(t), 2\cos(t), 0 \rangle \; dt =$$

$$\int_0^{2\pi} 4\sin^2(t) + 4\cos^2(t)\, dt = 8\pi$$

The orientation on the surface for this orientation on the boundary is the upward orientation so the correct answer is -8π.

3.(6pts) Let S be the bounded surface in space parametrized by the equations

$$x(u,v) = u+v, y(u,v) = u-v, z(u,v) = v$$

where $0 \leqslant u \leqslant 4$ and $0 \leqslant v \leqslant 2$. Then the flux integral of the vector field $\vec{F}(x,y,z) = x\vec{\imath} - \vec{\jmath} - \frac{y}{2}\vec{k}$ over the surface S with downward normal is

(a) 24 (b) 20 (c) 40 (d) 10 (e) 0

Solution. Let $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$. Note that $\vec{r_u} = \langle 1,1,0\rangle$ and $r_v = \langle 1,-1,1\rangle$, $\vec{r_u}\times\vec{r_v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \langle 1,-1,-2\rangle$. Hence $\vec{r_u}\times\vec{r_v}$ is the downward normal and

$$\vec{F}\bullet(\vec{r_u}\times\vec{r_v}) = \left\langle x,-1,-\frac{y}{2}\right\rangle \bullet \langle 1,-1,-2\rangle = x(u,v)+1+y(u,v) = 2u+1.$$

Thus

$$\begin{aligned} \iint_S \vec{F}\bullet d\vec{S} &= \int_0^4\int_0^2 (2u+1)\,dv\,du \\ &= \int_0^4 (2u+1)v\Big|_0^2 du = 2\int_0^4 (2u+1)\,du = 2\left(u^2+u\right)\Big|_0^4 = 2(16+4)-0 = 40 \end{aligned}$$

4.(6pts) A spaceship is traveling along the curve $\vec{r}(t) = \langle \cos t, t, \sin t\rangle$. Starting with $t=0$ how long does the spaceship have to travel to travel a distance of 2π.

(a) $\frac{3}{\sqrt{2}}\pi$ (b) It never goes that far. (c) 2π

(d) $\sqrt{2}\pi$ (e) 2

Solution. Distance traveled is $s(t) = \int_0^t |\vec{r}'(t)|\,dt = \int_0^t |\langle -\sin t, 1, \cos t\rangle|\,dt \int_0^t \sqrt{2}dt = \sqrt{2}t$. Now set $\sqrt{2}t = 2\pi$. So $t = \sqrt{2}\pi$.

5.(6pts) Let D be the region in the first quadrant of the xy-plane bounded by the line $y = x-2$ and the parabola $x = y^2$. Let S be the solid under the plane $z = x$ and above the region D. Which integral below is the iterated integral of the function $f(x, y, z) = z - xy$ over the solid S?

(a) $\int_{-1}^{2}\int_{y^2}^{y+2}\int_{0}^{x} (z - xy)\, dz\, dx\, dy$

(b) $\int_{1}^{4}\int_{x-2}^{\sqrt{x}}\int_{0}^{x} (z - xy)\, dz\, dy\, dx$

(c) $\int_{0}^{2}\int_{y^2}^{y+2}\int_{0}^{x} (z - xy)\, dz\, dx\, dy$

(d) $\int_{0}^{4}\int_{x-2}^{\sqrt{x}}\int_{0}^{x} (z - xy)\, dz\, dy\, dx$

(e) $\int_{y}^{y+2}\int_{x-2}^{\sqrt{x}}\int_{0}^{xy} x\, dz\, dy\, dx$

Solution. The region D is enclosed by the curves $x = y^2$, $x = y + 2$, $y = 0$, and $y = 2$. The curve $x = y^2$ is to the left and $x = y + 2$ is to the right. Since the plane $z = x$ lies above the xy-plane for all points in D, the bound on z is $0 \leqslant z \leqslant x$. Altogether:

$$\int_{0}^{2}\int_{y^2}^{y+2}\int_{0}^{x} z - xy\, dz\, dx\, dy$$

6.(6pts) Given the curve $\vec{r}(t) = \left\langle t, t, \frac{t^2}{2} \right\rangle$, find the unit binormal vector at the point $(2, 2, 2)$.

(a) $\frac{1}{\sqrt{2}}\langle 1, 0, -1\rangle$

(b) There is no unit binormal at this point.

(c) $\frac{1}{\sqrt{2}}\langle 1, 0, 1\rangle$

(d) $\frac{1}{\sqrt{2}}\langle -1, 1, 0\rangle$

(e) $\frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle$

Solution. The particle is at the point $(2, 2, 2)$ when and only when $t = 2$.
$\vec{r}\,'(t) = \langle 1, 1, t\rangle$; $\vec{r}\,'(2) = \langle 1, 1, 2\rangle$.
$\vec{r}\,''(t) = \langle 0, 0, 1\rangle$; $\vec{r}\,''(2) = \langle 0, 0, 1\rangle$

Then $\vec{r}\,'(2)\times\vec{r}\,''(2) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}\vec{\imath} - \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}\vec{k} = \langle 1, -1, 0\rangle$.

Then $\vec{B}(2) = \frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle$.

7.(6pts) Let $\vec{F}$ be the vector field $\langle 3x^2 - yz, xz, y - 2x\rangle$. At which of the following points is the curl of $\vec{F}$ perpendicular to the plane $3x + 6y + 6z = 7$?

(a) $(0, 1, 0)$ (b) $(1, 0, 0)$ (c) $(1, 1, 1)$ (d) $(0, 0, 0)$ (e) $(0, 0, 1)$

Solution. The curl of $\vec{F}$ is

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2 - yz & xz & y - 2x \end{vmatrix} = \langle 1 - x, 2 - y, 2z\rangle$$

The normal vector of the plane is $\langle 3, 6, 6\rangle$, so we want a point where the curl of $\vec{F}$ is parallel to $\langle 3, 6, 6\rangle$. If $z = 0$ then the third coordinate of the curl of $\vec{F}$ is 0, and so the curl is not parallel to $\langle 3, 6, 6\rangle$. This rules out $(0, 1, 0)$, $(1, 0, 0)$, and $(0, 0, 0)$. For the same reason x cannot be 1, which rules out $(1, 1, 1)$. So the answer must be $(0, 0, 1)$. Indeed, the curl of $\vec{F}$ at $(0, 0, 1)$ is $\langle 1, 2, 2\rangle$, which is parallel to $\langle 3, 6, 6\rangle$.

8.(6pts) Suppose $\vec{r}(t)$ is a vector-valued function such that $\vec{r}\,'(t) = \langle e^t, 2t + 4, \pi\cos(\pi t)\rangle$ and $\vec{r}(1) = \langle 0, 7, 1\rangle$. Find $\vec{r}(0)$.

(a) $\langle 1, 0, 0\rangle$ (b) $\langle -e, 1, \pi - 1\rangle$ (c) $\langle 0, 7, 1\rangle$ (d) $\langle 0, 4, \pi\rangle$ (e) $\langle 1 - e, 2, 1\rangle$

Solution. After taking antiderivatives:

$$\vec{r}(t) = \left\langle e^t + c_1, t^2 + 4t + c_2, \sin(\pi t) + c_3\right\rangle$$

for some constants c_1, c_2, c_3. From $\vec{r}(1) = \langle 0, 7, 1\rangle$, it follows that

$$e + c_1 = 0$$
$$5 + c_2 = 7$$
$$0 + c_3 = 1$$

So $\vec{r}(t) = \langle e^t - e, t^2 + 4t + 2, \sin(\pi t) + 1\rangle$, which means $\vec{r}(0) = \langle 1 - e, 2, 1\rangle$.

9.(6pts) Let S be the bounded surface in space parametrized by the equations

$$x(u,v) = u + v, y(u,v) = u - v, z(u,v) = v$$

where $0 \leqslant u \leqslant 4$ and $0 \leqslant v \leqslant 2$. Then the surface integral

$$\iint_S (x+y)z\, dS =$$

(a) -32 (b) 32 (c) $-32\sqrt{6}$ (d) $32\sqrt{6}$ (e) $8\sqrt{6}$

Solution. Let $\vec{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$. Note that $\vec{r}_u = \langle 1,1,0\rangle$ and $\vec{r}_v = \langle 1,-1,1\rangle$, $\vec{r}_u \times \vec{r}_v = \langle 1,-1,-2\rangle$ and $|\vec{r}_u \times \vec{r}_v| = \sqrt{6}$.

$$\begin{aligned}\iint_S (x+y)z\, dS &= \int_0^4 \int_0^2 (x(u,v) + y(u,v))z(u,v)\, |\vec{r}_u \times \vec{r}_v|\; dv\, du \\ &= \int_0^4 \int_0^2 (2u)v\sqrt{6}\, dv\, du \\ &= = 32\sqrt{6}\end{aligned}$$

10.(6pts) Let $p(x,y)$ be a function such that $\nabla p = \langle 2x + y, x + 2y\rangle$ and $p(0,0) = 1$. Find $p(1,1)$.

(a) 4 (b) 0 (c) 2 (d) -1 (e) 3

Solution. Let $p(x,y)$ be a potential function. Then $\dfrac{\partial p}{\partial x} = 2x+y$ so $p(x,y) = x^2+xy+h(y)$. Then $\dfrac{\partial p}{\partial y} = x+2y$ so $p(x,y) = x+h'(y) = x+2y$, so $h'(y) = y$ and $p(x,y) = x^2+xy+y^2+C$. $p(0,0) = C$ so $p(x,y) = x^2 + xy + y^2 + 1$. Then $p(1,1) = 4$.

11.(6pts) Find the projection of the vector $\langle 3, 1, -1\rangle$ onto the vector $\langle 1, 0, -1\rangle$.

(a) $\left\langle \frac{1}{\sqrt{12}}, 0, -\frac{1}{\sqrt{12}} \right\rangle$ (b) $\langle 6, 2, -2\rangle$ (c) $\langle 2, 0, -2\rangle$

(d) $\langle 2, 1, -2\rangle$ (e) $\langle 5, 1, -2\rangle$

Solution. $\text{proj}_{\langle 1,0,-1\rangle}\big(< 3, 1, -1 >\big) = \dfrac{\langle 3, 1, -1\rangle \bullet \langle 1, 0, -1\rangle}{\langle 1, 0, -1\rangle \bullet \langle 1, 0, -1\rangle} \langle 1, 0, -1\rangle = \dfrac{4}{2} \langle 1, 0, -1\rangle =$ $\langle 2, 0, -2\rangle$

12.(6pts) Find $\dfrac{\partial x}{\partial z}$ at the point $(1, 2, 3)$ where x is defined implicitly as a function of y and z by the equation $xyz - 6 = e^{xyz} - e^6$.

(a) $\dfrac{2e^6 - 2}{6 - 6e^6}$ (b) $\dfrac{2e^6 - 2}{6e^6 + 6}$ (c) $\dfrac{2e^6 - 2}{6e^6 - 6}$ (d) $\dfrac{2e^6 + 2}{6 - 6e^6}$ (e) $\dfrac{2e^6 + 2}{6e^6 - 6}$

Solution. $\dfrac{\partial}{\partial z} : xyz = e^{xyz}$.

$$\frac{\partial x}{\partial z} yz + \frac{\partial y}{\partial z} xz + \frac{\partial z}{\partial z} xy = \left(\frac{\partial x}{\partial z} yz + \frac{\partial y}{\partial z} xz + \frac{\partial z}{\partial z} xy \right) e^{xyz}$$

But $\dfrac{\partial y}{\partial z} = 0$ so

$$\frac{\partial x}{\partial z} yz + xy = \left(\frac{\partial x}{\partial z} yz + xy \right) e^{xyz}$$

$$\frac{\partial x}{\partial z} 6 + 2 = \left(\frac{\partial x}{\partial z} 6 + 2 \right) e^6$$

$$\frac{\partial x}{\partial z} \left(6 - 6e^6\right) = 2e^6 - 2$$

$$\frac{\partial x}{\partial z} = \frac{2e^6 - 2}{6 - 6e^6} = \frac{2(e^6 - 1)}{6(1 - e^6)} = -\frac{1}{3}$$

13.(6pts) Evaluate $\int_{\mathcal{C}} \vec{F} \bullet d\vec{r}$ where $\vec{F} = \langle e^x, -xy \rangle$ and $\mathcal{C}$ is the boundary of the square with corners $\{(0,0),(0,2),(2,2),(2,0)\}$ in a clockwise direction.

(a) -4 (b) 4 (c) 0 (d) 6 (e) -6

Solution. Use Green's Theorem to get $\int_{\mathcal{C}} \vec{F} \bullet d\vec{r} = -\iint_R -y\, dA = \iint_R y dA$. Notice that this is the moment about the x-axis. So since the area of R is 4 and center of mass is $(1,1)$, this integral have a value of 4. It is also easy to solve $\int_0^2 \int_0^2 y\, dy dx$.

14.(6pts) The equation of the plane which contains the point $(0,0,0)$, $(1,0,1)$ and $(1,1,0)$

(a) $x + y + z = 0$ (b) $-x + y + z = 2$ (c) $x - y + z = 0$

(d) $-x + y + z = 0$ (e) $x + y + z = 2$

Solution. The vector

$$(\vec{\imath} + \vec{k}) \times (\vec{\imath} + \vec{\jmath}) = -\vec{\imath} + \vec{\jmath} + \vec{k}$$

is normal to the plane. Moreover the plane passes through the origin. So

$$-x + y + z = 0$$

is the equation of the plane.

15.(6pts) At what points does the curve $\vec{r}(t) = \langle t, 0, 2t+3\rangle$ intersect the paraboloid $z = x^2 + y^2$?

(a) $(-3, 0, -3)$ (b) $(3, 0, 9)$, $(-1, 0, 1)$ (c) $(-3, 0, -3)$, $(1, 0, 5)$

(d) $(-1, 0, 1)$ (e) $(3, 0, 9)$

Solution. We need to solve $2t + 3 = t^2 + 0^2$ so $t^2 - 2t - 3 = (t-3)(t+1) = 0$. Solutions are $t = 3$ and hence $(3, 0, 9)$ and $t = 1$ and hence $(-1, 0, 1)$.

16.(6pts) Evaluate $\displaystyle\int_{\mathcal{C}} z\,dx - y\,dy + 3x\,dz$ where $\mathcal{C}$ is defined by $\vec{r}(t) = \langle t^3, t^2, t\rangle$, $0 \leqslant t \leqslant 2$.

(a) 32 (b) 8 (c) 20 (d) -8 (e) 16

Solution. $\displaystyle\int_{\mathcal{C}} z\,dx - y\,dy + 3x\,dz$ where $\mathcal{C} = \int_{\mathcal{C}} \langle z, -y, 3x\rangle \bullet d\vec{r} = \int_0^2 \langle t, -t^2, 3t^3\rangle \bullet \langle 3t^2, 2t, 1\rangle\ dt = \int_0^2 3t^3 - 2t^3 + 3t^3\,dt = \int_0^2 4t^3\,dt = 2^4 = 16$.

17.(6pts) Compute the tangential component of the acceleration of a particle at $t = \dfrac{\pi}{2}$ whose motion is given by $\vec{r}(t) = \left\langle 4\cos(t), 4\sin(t), \dfrac{3}{\pi}t^2 \right\rangle$.

(a) $\dfrac{5}{\pi}$ (b) $\dfrac{18}{5\pi}$ (c) $\dfrac{9}{5\pi}$

(d) $\dfrac{4}{5}\sqrt{25 + \dfrac{18}{\pi^2}}$ (e) 0

Solution. Recall

$$a_T(t) = \frac{\vec{r}'(t) \bullet \vec{r}''(t)}{|\vec{r}'(t)|^2}.$$

Now $\vec{r}'(t) = \left\langle -4\sin(t), 4\cos(t), \dfrac{6}{\pi}t \right\rangle$ and $\vec{r}''(t) = \left\langle -4\cos(t), -4\sin(t), \dfrac{6}{\pi} \right\rangle$. Thus

$$a_T(\pi) = \frac{\langle -4, 0, 3\rangle \bullet \left\langle 0, -4, \frac{6}{\pi} \right\rangle}{\sqrt{0^2 + 4^2 + 3^2}} = \frac{18}{5\pi}.$$

18.(6pts) Let $f(x, y)$ be any function with continuous second order derivatives. Let (a, b) be a critical point such that $f_{xy}(a, b) = 0$, $f_{yy}(a, b) > 0$ and $f_{xx}(a, b) \neq 0$, then,

(a) (a, b) is a local minimum if $f_{xx}(a, b) < 0$.

(b) (a, b) is never a saddle point.

(c) (a, b) is a local minimum if $f_{xx}(a, b) > 0$.

(d) (a, b) is a local maximum if $f_{xx}(a, b) > 0$.

(e) (a, b) is a local maximum if $f_{xx}(a, b) < 0$.

Solution. Since $f_{xy}(a, b) = 0$, we have

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2 = f_{xx}f_{yy}$$

which is clearly nonzero under the assumptions. Since it is given that $f_{xx}(a, b) > 0$ the sign of D is determined by $f_{yy}(a, b)$. Thus we have two cases

1. If $f_{yy}(a, b) < 0$, then $D(a, b) < 0$, then (a, b) is a saddle point. This eliminates the case "(a, b) is never a saddle point ".
2. If $f_{yy}(a, b) > 0$, then $D(a, b) > 0$. Since $f_{xx}(a, b) > 0$, by the Second Derivative Test, (a, b) must be local minimum.

19.(6pts) Find $\displaystyle\iint_D e^{x^2+y^2}\,dA$ where D is the disk centered at the origin of radius a.

(a) $2\pi e^{a^2}$ (b) $a\pi^2$ (c) $\pi\left(e^{a^2}-1\right)$ (d) $ae^{\pi a^2}$ (e) 0

Solution. Convert to iterated polar integral: $\displaystyle\iint_D e^{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^a e^{r^2} r\,dr\,d\theta = \int_0^{2\pi} \left.\frac{e^{r^2}}{2}\right|_0^a d\theta = \int_0^{2\pi} \frac{e^{a^2}-1}{2} d\theta = \pi\left(e^{a^2}-1\right)$

20.(6pts) A thin wire W in space parametrized by the equation

$$\langle x(t), y(t), z(t)\rangle = \langle \cos t, \sin t, t\rangle$$

where $0 \leqslant t \leqslant \pi$, has density

$$\rho(x,y,z) = z$$

at the point (x,y,z) on the wire. Then the mass of the wire is

(a) $\dfrac{\pi^2}{4}$ (b) $\dfrac{\sqrt{2}\pi^2}{2}$ (c) $\dfrac{\sqrt{2}\pi^2}{4}$ (d) $\dfrac{\sqrt{2}\pi^2}{8}$ (e) $\dfrac{\sqrt{2}\pi}{4}$

Solution. The mass $m = \displaystyle\int_W \rho(x,y,z)\,ds$, where s is the arc-length parametrization.

$$\begin{aligned}\int_W \rho(x,y,z)ds &= \int_0^\pi \rho(x(t),y(t),z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt \\ &= \int_0^\pi t\sqrt{2}\,dt \\ &= \sqrt{2}\left.\frac{t^2}{2}\right|_{t=0}^{\pi} \\ &= \frac{\sqrt{2}\pi^2}{2}\end{aligned}$$

21.(6pts) Which one of the followings is the directional derivative of $f(x,y,z) = xe^y + ye^z + ze^x$ at the point $(0,0,1)$ in the direction of the vector $\vec{v} = \langle -1, 2, -2\rangle$?

(a) $-\frac{e}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{-4+2e}{3}$ (d) $\frac{1}{3}$ (e) $-4+2e$

Solution. $\nabla f(x,y,z) = \langle e^y + ze^x, xe^y + e^z, ye^z + e^x\rangle$ so $\nabla f(0,0,1) = \langle 2, e, 1\rangle$ and hence

$$D_{\langle -1,2,-2\rangle} f(0,0,1) = \frac{\langle -1,2,-2\rangle \bullet \nabla f(0,0,1)}{|\langle -1,2,-2\rangle|} = \frac{\langle -1,2,-2\rangle \bullet \langle 2,e,1\rangle}{\sqrt{9}} = \frac{2e-4}{3}$$

22.(6pts) The maximum value of the function

$$f(x,y) = x + 2y$$

on the ellipse on xy-plane given by the equation $\frac{x^2}{2} + y^2 = 1$ is

(a) $\sqrt{\frac{2}{3}}$ (b) 2 (c) 1 (d) $\sqrt{6}$ (e) $\sqrt{\frac{3}{2}}$

Solution. We are trying to maximize $f(x,y) = x + 2y$, under the constraint $g(x,y) = \frac{x^2}{2} + y^2 = 1$. We apply methods of Lagrange Multipliers: So we solve the equations

(1) $f_x(x,y) = \lambda g_x(x,y) \Rightarrow 1 = \lambda x$,
(2) $f_y(x,y) = \lambda g_y(x,y) \Rightarrow 2 = 2\lambda y$, and,
(3) $g(x,y) = \frac{x^2}{2} + y^2 = 1$

simultaneously. Observe that $\lambda \neq 0$, so we can set

$$x = \frac{1}{\lambda}, y = \frac{1}{\lambda}.$$

Therefore $g(x,y) = \frac{1}{2\lambda^2} + \frac{1}{\lambda^2} = 1$. Thus

$$\lambda = \pm\sqrt{\frac{3}{2}}$$

and $f(x,y) = x + 2y = \frac{1}{\lambda} + \frac{2}{\lambda} = \frac{3}{\lambda}$ is maximum when $\lambda = \sqrt{\frac{3}{2}}$ and the maximum value is

$$3\sqrt{\frac{2}{3}} = \sqrt{6}.$$

23.(6pts) Find the points at which the *direction* of fastest change of the function $f(x,y) = x^2 + y^2 - 3x - 4y + 2016$ is parallel to $\vec{\imath} + 2\vec{\jmath}$.

(a) $(2,3)$

(b) All points on the line $y = 2x - 1$

(c) All points on the line $y = x + 1$

(d) $\left(\frac{3}{2} + \frac{1}{2\sqrt{5}}, 2 + \frac{1}{\sqrt{5}}\right)$

(e) $(1,1)$

Solution. At the point (x,y) the direction of fastest increase of f is $\nabla f(x,y) = \langle 2x-3, 2y-4\rangle$. Hence $\langle 2x-3, 2y-4\rangle = \lambda\,\langle 1,2\rangle$ so $2x - 3 = \lambda$ and $2y - 4 = 2\lambda$. Hence $2y - 4 = 2(2x - 3)$ so $y = 2x - 1$.

OR

At the point (x,y) the direction of fastest increase of f is $\nabla f(x,y) = \langle 2x-3, 2y-4\rangle$. The vector $\langle 2x-3, 2y-4\rangle$ is parallel to $\langle 1,2\rangle$ provided $\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2x-3 & 2y-4 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 4x - 6 - (2y - 4) = 0$ or $y = 2x - 1$.

24.(6pts) Let $\vec{F} = \langle x + 2xy + e^{yz}, y - y^2 + \sin(x^2 + z^2), z + xy\rangle$ be a vector field and let E be a solid circular cylinder of radius 2 and height 3. Compute the flux integral $\iint_{\partial E} \vec{F} \bullet d\vec{S}$ with the outward normal.

(a) 12π

(b) 48

(c) 52π

(d) 36π

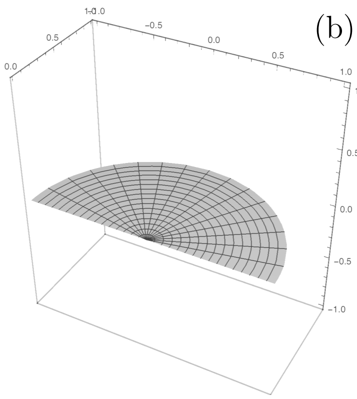
(e) Can not be determined from the given information.

Solution. $\nabla \bullet \vec{F} = \dfrac{\partial(x + 2xy + e^{yz})}{\partial x} + \dfrac{\partial(y - y^2 + \sin(x^2 + z^2))}{\partial y} + \dfrac{\partial(z + xy)}{\partial z} =$ $(1 + 2y) + (1 - 2y) + 1 = 3$. By the Divergence Theorem,

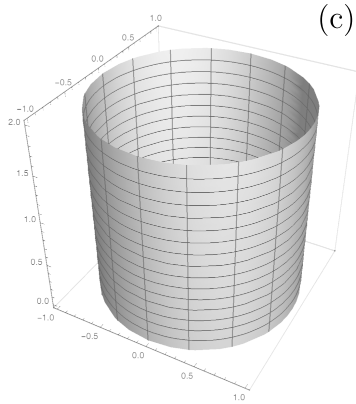
$$\iint_{\partial E} \vec{F} \bullet d\vec{S} = \iiint_E (\nabla \bullet \vec{F})\, dV = 3 \cdot \text{volume}(E) = 3 \cdot \pi \cdot 2^2 \cdot 3 = 36\pi$$

25.(6pts) Identify the parametric surface $\vec{r}(u,v) = \langle \sin(u)\cos(v), \sin(u)\sin(v), \cos(u)\rangle$ where $0 \leqslant u \leqslant \pi$ and $0 \leqslant v \leqslant 2\pi$.

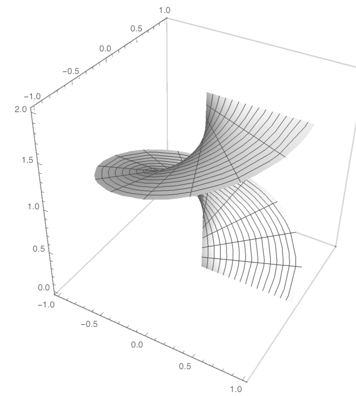
(a)



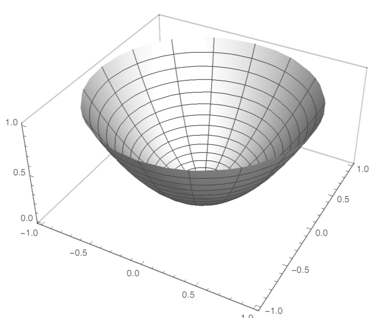
(b)



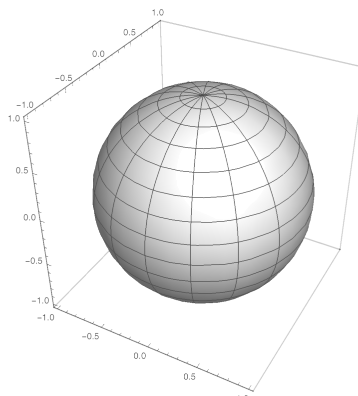
(c)



(d)



(e)



Solution. This is spherical coordinates for $\rho = 1$ $\theta = v$ and $\phi = u$, so this is a sphere of radius 1.