

Let f be a function of a vector variable of n variables, $\vec{x} = (x_1, \dots, x_n)$

$$f(x_1, \dots, x_n) = f(\vec{x})$$

The function is *differentiable* at the point $\vec{a} = (a_1, \dots, a_n)$ provided the partial derivatives $\frac{\partial f}{\partial x_i}(\vec{a})$ exist $i = 1, \dots, n$ and the following formula holds Let $\Delta x = (\Delta x_1, \dots, \Delta x_n)$. For f to be differentiable we additionally require that

$$\lim_{\Delta x \rightarrow \vec{0}} \frac{\left| f(\vec{a} + \Delta x) - \left(f(\vec{a}) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\vec{a}) \Delta x_i \right) \right|}{|\Delta x|} = 0$$

Let $\Delta f(\vec{a}, \Delta x) = f(\vec{a} + \Delta x) - f(\vec{a})$. Then

$$\Delta f(\vec{a}, \Delta x) \approx \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\vec{a}) \Delta x_i$$

provided the partials exist. *Differentiable* means roughly that the right-hand side is the best linear approximation to Δf that there is.

If the partials are continuous in a small ball around \vec{a} then f is differential at the point \vec{a} .

The *gradient* of a function $f(\vec{x})$ is an n -vector provided \vec{x} is an n -vector. If $\vec{x} = (x_1, \dots, x_n)$ then

$$\nabla f(\vec{x}) = \left(\frac{\partial f}{\partial x_1}(\vec{x}), \dots, \frac{\partial f}{\partial x_n}(\vec{x}) \right)$$

The formulas above can be expressed more succinctly using the gradient and the dot product:

$$\Delta f(\vec{a}, \Delta x) \approx \nabla f(\vec{a}) \bullet \Delta x$$

The chain rule can be summarized as follows. If $x_i = x_i(\vec{t})$, $i = 1, \dots, n$ where $\vec{t} = (t_1, \dots, t_m)$ then let $g(t_1, \dots, t_m) = f(x_1(\vec{t}), \dots, x_n(\vec{t}))$. Fix all the t_i 's but one, say t_k . Let

$$\vec{r}_k(t) = (x_1(t_1, \dots, t, \dots, t_m), \dots, x_n(t_1, \dots, t, \dots, t_m))$$

where t is in the k^{th} position. Then

$$\vec{r}'_k(t) = \left(\frac{\partial x_1}{\partial t_k}(t_1, \dots, t_m), \dots, \frac{\partial x_n}{\partial t_k}(t_1, \dots, t_m) \right)$$

$$\frac{\partial g}{\partial t_k}(t_1, \dots, t_m) = \nabla f(\vec{r}_k(t)) \bullet \vec{r}'_k(t)$$

One class of problems studies how $f(\vec{x})$ varies as \vec{x} does. The derivative of f at the point \vec{a} in the direction of \vec{v} is

$$D_{\vec{v}}f(\vec{a}) = \frac{1}{|\vec{v}|} \nabla f(\vec{a}) \bullet \vec{v}$$

The direction of maximal increase of the function f at the point \vec{a} is the direction of $\nabla f(\vec{a})$ and the value of this rate of increase is $|\nabla f(\vec{a})|$. The direction of maximal decrease is the direction of $-\nabla f(\vec{a})$ and the value is $-|\nabla f(\vec{a})|$. If $\nabla f(\vec{a}) = \vec{0}$ there are no such directions.

A second class of problems studies a fixed surface $f(\vec{x}) = c$. To complicate matters still further, these two kinds of problems overlap. One way to study how $f(\vec{x})$ changes is to study the level surfaces, $f(\vec{x}) = c$.

If a point \vec{a} is fixed on $f(\vec{x}) = c$ then $f(\vec{a}) = c$. Suppose $\vec{r}(t)$ is a curve with $\vec{r}(t_0) = \vec{a}$ and suppose moreover that $\vec{r}(t)$ lies on $f(\vec{x}) = c$. This means $f(\vec{r}(t)) = c$. By the Chain Rule,

$$0 = \nabla f(\vec{a}) \bullet \vec{r}'(t_0)$$

so the tangent vector to the curve is normal to the gradient.

The tangent line, plane . . . , is the object containing the point \vec{a} and perpendicular to $\nabla f(\vec{a})$. If $\vec{a} = (a_1, \dots, a_n)$ and if $\nabla f(\vec{a}) = (b_1, \dots, b_n)$ then the equation for the tangent object is

$$b_1(x_1 - a_1) + \dots + b_n(x_n - a_n) = 0$$

or

$$\nabla f(\vec{a}) \bullet (\vec{x} - \vec{a}) = 0$$

If $\frac{\partial f}{\partial x_n}(\vec{a}) \neq 0$ (and all the partials are continuous near \vec{a}) then there is a theorem called the *Implicit Function Theorem* which says there exists a differentiable function $g(x_1, \dots, x_{n-1})$ defined in a neighborhood of (a_1, \dots, a_{n-1}) such that

$$f(x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1})) = c$$

One can ask which direction on $f(\vec{x}) = c$ is the direction of maximal increase on the curve, surface, This is the same question as asking for the direction of maximal increase of g and so we need to compute the gradient of g .

By the Chain Rule, for each i , $1 \leq i \leq n-1$

$$0 = \nabla f(\vec{x}) \bullet \langle 0, \dots, 1, \dots, 0, \frac{\partial g}{\partial x_i} \rangle$$

where the 1 is in the i^{th} position. Hence

$$\frac{\partial g}{\partial x_i} = -\frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_n}}$$

and so

$$\nabla g = -\left\langle \frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_n}}, \dots, \frac{\frac{\partial f}{\partial x_{n-1}}}{\frac{\partial f}{\partial x_n}} \right\rangle$$